Lecture 5a: Time-Domain Analysis of Continuous-Time Systems with MATLAB and Maxima

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October 21, 2021

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Contents

Zero-Input Response $y_0(t)$

To solve a differential equation, you can use a command **dsolve** to solve the equation.
For an continuous-time LTI system specified by the differential equation

$$(D^2 + 4D + k)y(t) = (3D + 5)f(t)$$

determine the zero-input component of the response if the initial conditions are $y_0(0) = 3$, and $\dot{y}_0(0) = -7$ for two values of k: (a) 3 (b) 4 (c) 40.

```
syms y(t) t
Dy = diff(y, 1);
cond1 = y(0) == 3; cond2 = Dy(0) == -7; conds = [cond1; cond2];

sys1 = diff(y,2) + 4*diff(y,1) + 3*y == 0
```

sys1(t) = $\frac{\partial^2}{\partial t^2}\, y\,(t) + 4\,\frac{\partial}{\partial t}\, y\,(t) + 3\,y\,(t) = 0$

```
sys2 = diff(y,2) + 4*diff(y,1) + 4*y == 0
```

sys2(t) = $\frac{\partial^2}{\partial t^2}\, y\,(t) + 4\,\frac{\partial}{\partial t}\, y\,(t) + 4\,y\,(t) = 0$

Zero-Input Response $y_0(t)$

```
sys3 = diff(y,2) + 4*diff(y,1) + 40*y == 0
```

sys3(t) = $$\frac{\partial^2}{\partial t^2}\, y(t) + 4\,\frac{\partial}{\partial t}\, y(t) + 40\, y(t) = 0$$

```
y1 = dsolve(sys1, conds)
```

y1 = $$\mathrm{e}^{-t} + 2\,\mathrm{e}^{-3\,t}$$

```
y2 = dsolve(sys2, conds)
```

y2 = $$-\mathrm{e}^{-2\,t}\,(t-3)$$

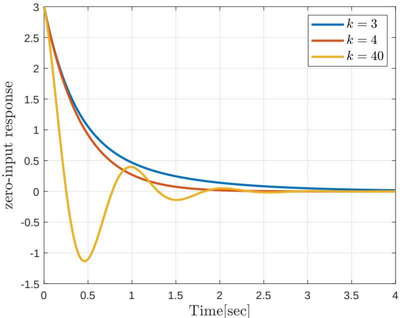
```
y3 = dsolve(sys3, conds)
```

y3 = $$\frac{\mathrm{e}^{-2\,t}\,(18\,\cos(6\,t) - \sin(6\,t))}{6}$$

Zero-Input Response $y_0(t)$

To plot y_0 respect to t, we can use a command eval as follow:

```
ts = -1:0.01:10;
y1c = y1*heaviside(t);
y2c = y2*heaviside(t);
y3c = y3*heaviside(t);
y1p = subs(y1c,t,ts); y2p = subs(y2c,t,ts); y3p = subs(y3c,t,ts);
plot(ts,y1p,ts,y2p,ts,y3p, 'linewidth', 2)
h = legend('k =3', 'k=4', 'k=40');
```



Zero-State Response $y_s(t)$

For an LTI system specified by the differential equation

$$(D^2 + 3D + 2)y(t) = Df(t)$$

To calculate the zero-state response, we can use MATLAB to calculate as follow:

$$h(t) = b_n\delta + P(D)y_n(t)u(t)$$

In this case $b_n = 0$ and the initial values of $y_n(t)$ are $y_n(0^-) = 0$ and $\dot{y}(0^-) = 1$.

```
syms y(t) t
Dy = diff(y,1);
cond1 = y(0) == 0; cond2 = Dy(0) == 1;
conds = [cond1; cond2]; sys4 = diff(y,2) + 3*diff(y,1) + 2*y(t) == 0
y_n = dsolve(sys4, conds)
```

y_n = $\mathrm{e}^{-t} - \mathrm{e}^{-2\,t}$

```
Dy_n = diff(y_n)
```

Dy_n = $2\,\mathrm{e}^{-2\,t} - \mathrm{e}^{-t}$

Zero-State Response $y_s(t)$

Therefore

$$h(t) = 0 + (2e^{-2t} - e^{-t})\mathbb{1}(t)$$

Zero-State Response $y_s(t)$

If $f(t) = 10e^{-3t}$ we have

```
syms y(t) t tau
Dy = diff(y, 1);
ft = 10*exp(-3*t);
sys5 = diff(y,2) + 3*diff(y,1) + 2*y == 0
```

sys5(t) = $\frac{\partial^2}{\partial t^2}\, y(t) + 3\, \frac{\partial}{\partial t}\, y(t) + 2\, y(t) = 0$

```
cond1 = y(0) == 0; cond2 = Dy(0) == 1;
conds = [cond1; cond2];
y_n = dsolve(sys4, conds);
ht = diff(y_n)
```

ht = $2\,\mathrm{e}^{-2\,t} - \mathrm{e}^{-t}$

```
% convolution of f(t) and h(t)
ys = int(subs(ft, tau)*subs(ht, t-tau), tau, 0, t)
```

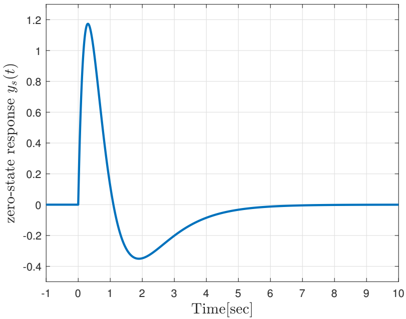
ys = $-5\,\mathrm{e}^{-3\,t}\left(\mathrm{e}^{2\,t} - 4\,\mathrm{e}^{t} + 3\right)$

Zero-State Response $y_s(t)$

```
ysc = ys*heaviside(t)
```

ysc = $-5\,e^{-3\,t}\,\text{heaviside}(t)\,\left(e^{2\,t} - 4\,e^{t} + 3\right)$

```
ts = -1:0.01:10;
ysp = subs(ysc, t, ts);
plot(ts,ysp, 'linewidth', 2)
axis([-1 10 -0.5 1.3]); grid;
```



Classical Method

Solve the differential equation

$$(D^2 + 3D + 2)y(t) = Df(t)$$

for the input $f(t) = 5t + 3$ if $y(0^+) = 2$ and $\dot{y}(0^+) = 3$.

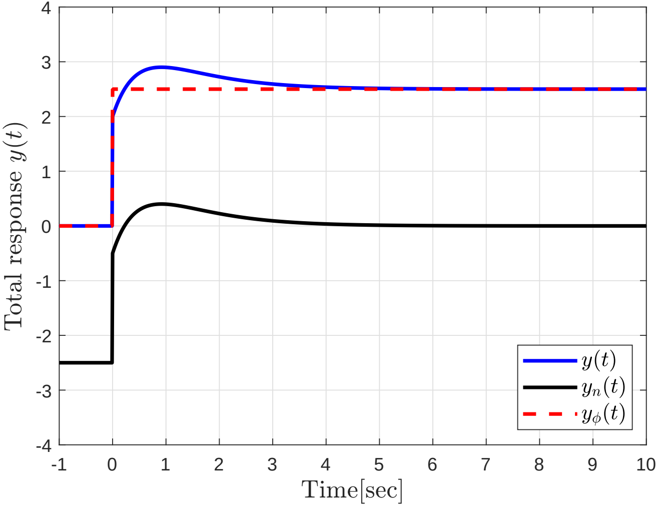
```
syms y(t) t
Dy = diff(y,1)

f = 5*t + 3;
sys1 = diff(y,2) + 3*diff(y,1) + 2*y == diff(f,t)
```

sys1(t) = $\frac{\partial^2}{\partial t^2}\, y(t) + 3\, \frac{\partial}{\partial t}\, y(t) + 2\, y(t) = 5$

```
cond1 = y(0) == 2; cond2 = Dy(0) == 3;
conds = [cond1; cond2];
y = dsolve(sys1, conds)
```

y = $2\, \mathrm{e}^{-t} - \frac{5\, \mathrm{e}^{-2\, t}}{2} + \frac{5}{2}$


4
3
2
1
0
-1
-2
-3
-4
-1
0
1
2
3
4
5
6
7
8
9
10
Total response $y(t)$
Time[sec]
$y(t)$
$y_n(t)$
$y_\phi(t)$

Contents

Maxima

- Maxima Codes: The goodness of Maxima is the symbolic based computation with pretty given solutions. Moreover, it is an opensource software.
- You can find Maxima at http://maxima.sourceforge.net/

Zero-Input Response $y_0(t)$

To solve a differential equation, you can use a command ode2 to solve the equation. In Maxima, we use Ctrl+Enter to get a solution. For an continuous-time LTI system specified by the differential equation $(D^2 + 4D + k)y(t) = (3D + 5)f(t)$ determine the zero-input component of the response if the initial conditions are $y_0(0) = 3$, and $\dot{y}_0(0) = -7$ for two values of k: (a) 3 (b) 4 (c) 40.

Solution: part (a) $k = 3$

```
eq1: 'diff(y,t,2) + 4*'diff(y,t,1) + 3*y = 0;
```

$$\frac{d^2}{dt^2} y + 4\left(\frac{d}{dt} y\right) + 3y = 0$$

```
sol1: ode2(eq1,y,t);
```

$$y = \%k1\ \%e^{-t} + \%k2\ \%e^{-3t}$$

```
ps1: ic2(sol1,t=0,y=3,'diff(y,t)=-7);
```

$$y = \%e^{-t} + 2\%e^{-3t}$$

Zero-Input Response $y_0(t)$

part (b) $k = 4$

```
eq2: 'diff(y,t,2) + 4*'diff(y,t,1) + 4*y = 0;
```

$$\frac{d^2}{dt^2} y + 4 \left(\frac{d}{dt} y \right) + 4y = 0$$

```
sol2: ode2(eq2,y,t);
```

$$y = (\%k2 t + \%k1)\ \%e^{-2t}$$

```
ps2: ic2(sol2,t=0,y=3,'diff(y,t)=-7);
```

$$y = (3 - t)\ \%e^{-2t}$$

Zero-Input Response $y_0(t)$

part (c) $k = 40$

```
eq3: 'diff(y,t,2) + 4*'diff(y,t,1) + 40*y = 0;
```

$$\frac{d^2}{dt^2} y + 4 \left(\frac{d}{dt} y \right) + 40y = 0$$

```
sol3: ode2(eq3,y,t);
```

$$y = \%e^{-2t} \ (\%k1 \ \sin(6t) + \%k2 \ \cos(6t))$$

```
ps3: ic2(sol3,t=0,y=3,'diff(y,t)=-7);
```

$$y = \%e^{-2t} \ \left(3\cos(6t) - \frac{\sin(6t)}{6} \right)$$

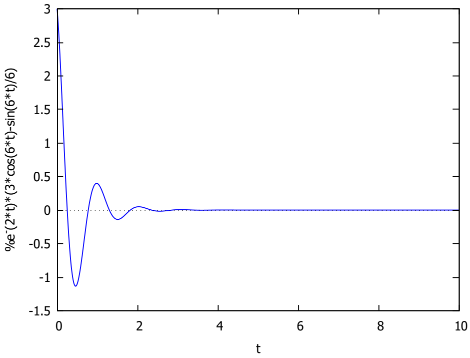
```
ratsimp(\%);
```

$$y = -\frac{\%e^{-2t} \ (\sin(6t) - 18\cos(6t))}{6}$$

Zero-Input Response $y_0(t)$

To plot y respect to t, we can use following commands:

```
/* define a variable to obtain a left hand side value */
/* use ps3 as an example */
us: rhs(ps3)
plot2d(us,[t,0,10]);
```



Zero-State Response $y_s(t)$

For an LTI system specified by the differential equation

$$(D^2 + 3D + 2)y(t) = Df(t)$$

To calculate the zero-state response, we can use Maxima command set listed below:

$$h(t) = b_n\delta + P(D)y_n(t)u(t)$$

In this case $b_n = 0$ and the initial values of $y_n(t)$ are $y_n(0^-) = 0$ and $\dot{y}(0^-) = 1$.

```
eq4: 'diff(y,t,2) + 3*'diff(y,t,1)+2*y=0;
gs4: ode2(eq4,y,t);
ps4: ic2(gs4,t=0,y=0,'diff(y,t)=1);
Dy: diff(ps4,t);
```

To obtain $\frac{d}{dt}y = 2\%e^{-2t} - \%e^{-t}$. Therefore

$$h(t) = 0 + (2e^{-2t} - e^{-t})u(t)$$

Zero-State Response $y_s(t)$

If $f(t) = 10e^{-3t}$ we have

```
eq1: 'diff(y,t,2) + 3*'diff(y,t,1)+2*y=0;
gs1: ode2(eq1,y,t);
ps1: ic2(gs1,t=0,y=0,'diff(y,t)=1);
y1: diff(ps1,t);
ytau: subst(tau,t,rhs(y1));
ft : 10*exp(-3*(t-tau))

/* convolution of f(t) and y_s(t) */
assume(t>0);
result1: integrate(ytau*ft,tau,0,t);
/* Use command 'ratsimp' to simplify the equation */
```

The last line will result

$$y_s(t) = -5e^{-t} + 20e^{-2t} - 15e^{-3t},\ t \geq 0$$

Finally, the total response is $y(t) = y_0(t) + y_s(t)$.

Classical Method

Solve the differential equation

$$(D^2 + 3D + 2)y(t) = Df(t)$$

for the input $f(t) = 5t + 3$ if $y(0^+) = 2$ and $\dot{y}(0^+) = 3$.

```
Df: diff(5*t+3,t);
eq5: 'diff(y,t,2) + 3*'diff(y,t,1)+2*y =Df;
gs5: ode2(eq5,y,t);
ps5: ic2(gs5,t=0,y=2,'diff(y,t)=3);
```

The last line will show

$$y(t) = 2\%e^{-t} - \frac{5\%e^{-2t}}{2} + \frac{5}{2}$$

Reference

1. Xie, W.-C., *Differential Equations for Engineers*, Cambridge University Press, 2010.
2. Goodwine, B., *Engineering Differential Equations: Theory and Applications*, Springer, 2011.
3. Kreyszig, E., *Advanced Engineering Mathematics*, 9th edition, John Wiley & Sons, Inc., 1999.
4. Lathi, B. P., *Signal Processing & Linear Systems*, Berkeley-Cambridge Press, 1998.