# INC 341 Feedback Control Systems: Lecture 4 Linearization

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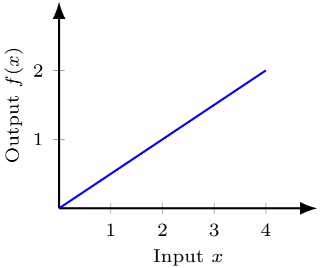
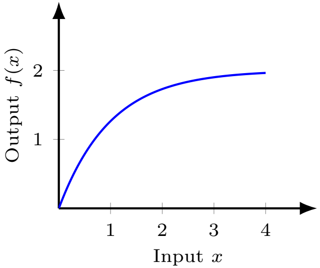
# Linear system vs Nonlinear system

- Linear systems possesses two properties: *Superposition* and *homogeneity*.
- Nonlinear systems do not.



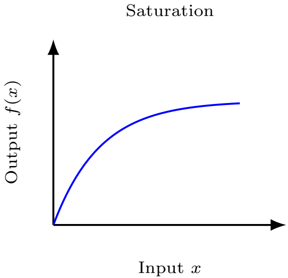



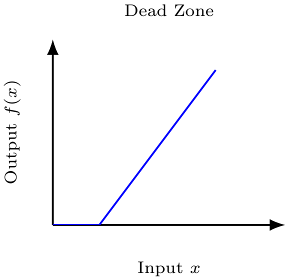

- Linear function $f(\alpha x) = \alpha f(x)$
- Nonlinear function $f(\alpha x) \neq \alpha f(x)$

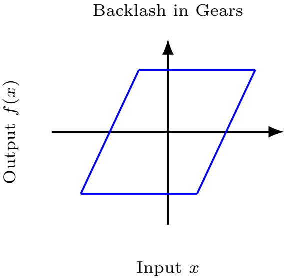
# Physical nonlinearities

Some examples of physical nonlinearities:

- An electronic amplifier is linear over a specific range by exhibits the nonlinearity called *saturation* at high input voltages.
- A motor that does not respond at very low input voltages due to frictional forces exhibits a nonlinearity called *dead zone*.
- Gears that do not fit tightly exhibit a nonlinearity called *backlash*.



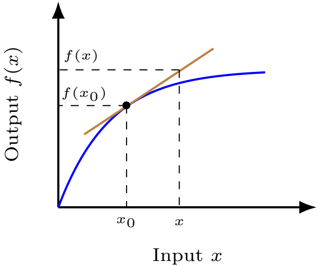



# Linearization

The systems that we consider so far assumed to be linear. However, if there are some nonlinear components are presented, we have to linearize the systems before finding the transfer functions. The linearization steps are:

- Check where is the nonlinear component and write the nonlinear differential equation of the system.
- Linearize the nonlinear part using a first order Taylor series approximation at the equilibrium point.

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x_0} (x - x_0) + \text{ high order term } .$$




$$f(x) - f(x_0) \approx m_a(x - x_0) \quad \Rightarrow \quad \delta f(x) \approx m_a \delta x$$

and

$$f(x) \approx f(x_0) + m_a(x - x_0) \approx f(x_0) + m_a \delta_x$$

where $m_a$ is a slope at the tangent point.

# Linearization

## Example

**Problem:** Linearize $f(x) = 5\cos x$ about $x = \pi/2$.

**Solution:** Start with find the slope at $x = x_0 = \pi/2$.

$$m = \frac{df}{dx} = -5\sin x\bigg|_{x=\frac{\pi}{2}} = -5$$

Since $f(x_0) = 5\cos\frac{\pi}{2} = 0$, then

$$f(x) = f(x_0) + m\delta x = -5\delta x.$$

This system can be represented as

$$f(x) = -5\delta x,$$

for small excursion of $x$.

**Problem:** Linearize the system below for small excursions about $x = \pi/4$.

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \cos x = 0.$$

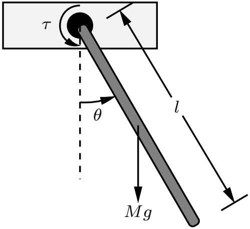
**Solution:** the nonlinear term $\cos x$ can be linearized as follow:

$$\cos x = \cos\frac{\pi}{4} + \left.\frac{d\cos x}{dx}\right|_{x=\frac{\pi}{4}} \delta x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\delta x$$

Substituting $x = x_0 + \delta x$ and the linearized of $\cos x$ to the system, we have

$$\frac{d^2\left(\frac{\pi}{4} + \delta x\right)}{dt^2} + 2\frac{d\left(\frac{\pi}{4} + \delta x\right)}{dt} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\delta x = 0$$

$$\frac{d^2\delta x}{dt^2} + 2\frac{d\delta x}{dt} - \frac{\sqrt{2}}{2}\delta x = -\frac{\sqrt{2}}{2}$$




The motion equation:

$$J\frac{d^2\theta}{dt^2} + \frac{Mgl}{2}\sin\theta = \tau(t),$$

where $\tau(t)$ is a small-signal force.

Since $\theta_0 = 0$, the nonlinear term $\sin\theta$ can be linearized as follow:

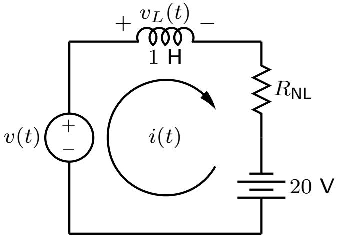
$$\sin\theta = \sin\theta_0 + \left.\frac{d\sin\theta}{d\theta}\right|_{\theta=0}(\theta - \theta_0) = \cos\theta_0\delta\theta = \delta\theta$$

Then,

$$J\frac{d^2(\theta+\delta\theta)}{dt^2} + \frac{Mgl}{2}\delta\theta = \tau(t) \Rightarrow \left(Js^2 + \frac{Mgl}{2}\right)\delta\Theta(s) = \hat{\tau}(s)$$

$$\frac{\delta\Theta(s)}{\hat{\tau}(s)} = \frac{1/J}{s^2 + \frac{Mgl}{2J}}$$

# Transfer Function – Nonlinear Electrical Network




Find the transfer function, $V_L(s)/V(s)$, for the electrical network shown in Fig., which contains a nonlinear resistor whose voltage-current relationship is defined by $i_r = 2e^{0.1v_r}$, where $i_r$ and $v_r$ are the resistor current and voltage, respectively. Also, $v(t)$ in Fig is a small-signal source.

Applying Kirchhoff's voltage law around the loop, where $i_r = i(t)$ yields

$$L\frac{di}{dt} + 10\ln\frac{1}{2}i(t) - 20 = v(t)$$

Set the small-signal source, $v(t)$, equal to zero. We have $di/dt = 0$ and $10\ln\frac{1}{2}i = 20$. Then we have $i_0 = i_r = 14.78$ A.

$$\ln\frac{1}{2}i(t) = \ln\frac{1}{2}i_0 + \left.\frac{d(\ln\frac{1}{2}i)}{di}\right|_{i=i_0}(i - i_0) = \ln\frac{1}{2}i_0 + \left.\frac{1}{i}\right|_{i=i_0}\delta i = \ln\frac{1}{2}i_0 + \frac{1}{i_0}\delta i$$

# Transfer Function – Nonlinear Electrical Network

Substituting the result into the system equation, the linearized equation becomes

$$L\frac{d\delta i}{dt} + 10\left(\ln\frac{1}{2}i_0 + \frac{1}{i_0}\delta i\right) - 20 = v(t)$$

$$\frac{d\delta i}{dt} + 10\,(2 + 0.0677\delta i) - 20 = v(t)$$

$$\frac{d\delta i}{dt} + 0.677\delta i = v(t).$$

Taking Laplace transform, yield

$$\delta I(s) = \frac{V(s)}{s + 0.677}$$

Since $v_L(t) = Ldi/dt$, we have $v_L(t) = d\delta i/dt$ and $V_L(s) = s\delta I(s)$. Finally, we obtain

$$\frac{V_L(s)}{V(s)} = \frac{s}{s + 0.667},$$

for small excursions about $i = 14.78$ A or, equivalently, about $v(t) = 0$.

# Reference


1. Norman S. Nise, "*Control Systems Engineering*,$6^{th}$ edition, Wiley, 2011
2. Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini, "*Feedback Control of Dyanmic Systems*",$4^{th}$ edition, Prentice Hall, 2002