INC 122 : AC Power Analysis

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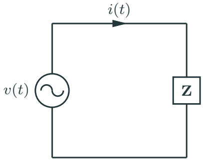
Learning Outcomes

Students should be able to:

- Calculate the average power dissipated in and AC circuit.
- Design an AC circuit for maximum power transfer.
- Calculate real, reactive, complex, and apparent power.
- State the purpose of power-factor correction, and design the correction needed for a given combination load.
- Explain how power can be masured.

Instantaneous Power



The steady-state voltage and current for the network can be written as

$$v(t) = V_M \cos(\omega t + \theta_v)$$
$$i(t) = I_M \cos(\omega t + \theta_i)$$

The instantaneous power is then

$$p(t) = v(t)i(t) = V_M I_M \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

Since $\cos A \cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$, the instantaneous power can be written as

$$p(t) = \frac{V_M I_M}{2}\left[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)\right]$$

There are two terms; the first term is a constant (i.e., it is time independent), and the second term is a cosine wave of twice the excitation frequency (2ω).

Instantaneous Power

The circuit in the previous slide has the following parameters: $v(t) = 4\cos(\omega t + 60^\circ)$ and $\mathbf{Z} = 2\underline{/30^\circ}\ \Omega$. We wish to determine equations for the current and the instantaneous power as a function of time, and plot these functions with the voltage on a single graph for comparison.

Since

$$\mathbf{I} = \frac{4\underline{/60^\circ}}{2\underline{/30^\circ}} = 2\underline{/30^\circ} \text{ A}$$

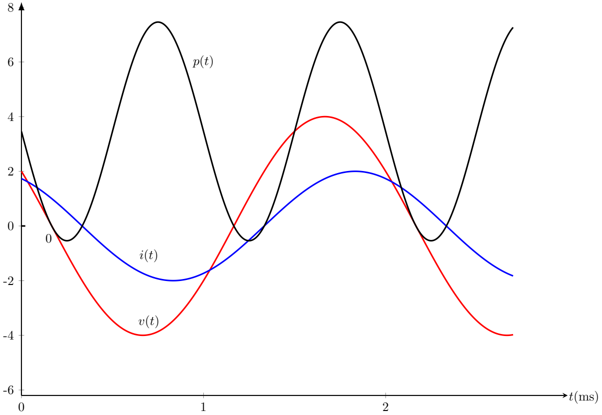
then

$$i(t) = 2\cos(\omega t + 30^\circ) \text{ A}$$

and

$$\begin{aligned} p(t) &= \frac{V_M I_M}{2}\left(\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)\right) \\ p(t) &= 4\left[\cos(30^\circ) + \cos(2\omega t + 90^\circ)\right] \\ &= 3.46 + 4\cos(2\omega t + 90^\circ) \text{ W} \end{aligned}$$

Instantaneous Power



The plots of $v(t), i(t)$, and $p(t)$ for the circuit using $f = 50$ Hz.

Average Power

The average value of any periodic waveform (e.g., a sinusoidal function) can be computed by integrating the function over a complete period and dividing this result by the period. **Note** In one circuit we have only one frequency.
The average power is

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} V_M I_M \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)dt,$$

where t_o is arbitrary, $T = 2\pi/\omega$ is the period of the voltage or current, and P is measured in watts. The above integration becomes

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{V_M I_M}{2}\left[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)\right] dt$$

As we know the average value of a cosine wave over one complete period is zero. Therefore

$$P = \frac{1}{2} V_M I_M \cos(\theta_v - \theta_i)$$

Average Power

For purely resistive circuit, the difference between θ_v and θ_i is zero. Then

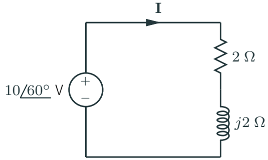
$$P = \frac{1}{2} V_M I_M$$

For purely reactive circuit, the difference between θ_v and θ_i is $\pm 90°$. Then

$$P = \frac{1}{2} V_M I_M \cos(90°) = 0 \text{ W}$$

The purely reactive impedances absorb no average power, they are often called lossless elements. The purely reactive network operates in a mode in which it stores energy over one part of the period and releases it over another.

Average Power



Determine the average power absorbed by the impedance shown in Fig.

Using voltage divider approach,

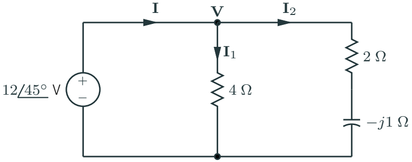
$$V_R = \frac{10\angle 60^\circ (2)}{2 + j2} = \frac{20\angle 60^\circ}{2\sqrt{2}\angle 45^\circ} = 7.07\angle 15^\circ \text{ V}$$

and therefore,

$$P = \frac{1}{2} V_M I_M = \frac{1}{2}(7.07)\frac{7.07}{2}$$

$$P = \frac{1}{2}\frac{(7.07)^2}{2} = 12.5 \text{ W}$$

Note: We can use $P = \frac{1}{2} V_M I_M$, and $P = \frac{1}{2} I_M^2 R$ as well.



Determine both the total average power absorbed and the total average power supplied.

From the circuit we note that

$$\mathbf{I}_1 = \frac{12\angle 45^\circ}{4} = 3\angle 45^\circ \text{ A}, \qquad \mathbf{I}_2 = \frac{12\angle 45^\circ}{2 - j1} = \frac{12\angle 45^\circ}{2.24\angle -26.57^\circ} = 5.37\angle 71.57^\circ \text{ A}$$

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 3\angle 45^\circ + 5.37\angle 71.57^\circ = (2.12 + j2.12) + (1.70 + j5.1)$$
$$= 3.82 + j7.22 = 8.16\angle 62.1^\circ$$

The average power absorbed in the $4\ \Omega$ resistor is

$$P_4 = \frac{1}{/} V_M I_M = \frac{1}{2}(12)(3) = 18 \text{ W}$$

Average Power

The average power absorbed in the $2\ \Omega$ resistor is

$$P_2 = \frac{1}{2} I_{2M}^2 R = \frac{1}{2}(5.37)^2(2) = 28.8 \text{ W}$$

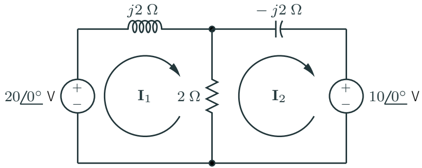
Therefore, the total average power absorbed is

$$P_A = 18 + 28.8 = 46.8 \text{ W}$$

The total average power supplied by the source is

$$\begin{aligned} P_S &= \frac{1}{2} V_M I_M \cos(\theta_v - \theta_i) = \frac{1}{2}(12)(8.16)\cos(45^\circ - 62.1^\circ) \\ &= 46.8 \text{ W} \end{aligned}$$

Average Power



Find the average power absorbed by each of the three passive elements in the circuit, as well as the average power supplied by each source. (Note: in this case we cannot use superposition technique to find power, because the power function is not linear. However we can use it to find currents and voltages in the circuit.)

Without analyzing the circuit, we already know that the average power absorbed by the two reactive elements is zeros. Using Mesh analysis, we have

$$(2 + j2)\mathbf{I}_1 - 2\mathbf{I}_2 = 20\angle 0°$$
$$-2\mathbf{I}_1 + (2 - j2)\mathbf{I}_2 = -10\angle 0°$$

The values of $\mathbf{I}_1$ and $\mathbf{I}_2$ are

$$\mathbf{I}_1 = 5 - j10 = 11.18\angle -63.43° \text{ A}$$
$$\mathbf{I}_2 = 5 - j5 = 7.071\angle -45° \text{ A}$$

Average Power

The downward current through the $2\ \Omega$ resistor is

$$\mathbf{I} = \mathbf{I}_1 - \mathbf{I}_2 = -j5 = 5\underline{/-90^\circ} \text{ A}$$
$$i(t) = 5\cos(\omega t - 90^\circ)$$

Then $I_M = 5$, and the average power absorbed by the resistor is

$$P_R = \frac{1}{2} I_M^2 R = \frac{1}{2} 5^2 (2) = 25 \text{ W}$$

The voltage $20\underline{/0^\circ}$ and associated current $\mathbf{I}_1 = 11.18\underline{/-63.43^\circ}$ A satisfy the active sign, and the power delivered by the source is

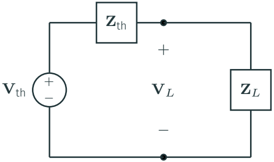
$$P_{\text{left}} = \frac{1}{2}(20)(11.18)\cos(0^\circ - (-63.43^\circ)) = 50 \text{ W}$$

Using the active sign convention, we find the power absorbed by the right source is

$$P_{\text{right}} = \frac{1}{2} 10(7.071)\cos(0^\circ - (-45^\circ)) = 25 \text{ W and}$$
$$P_{\text{left}} = P_{\text{right}} + P_R = 50 \text{ W}$$

Maximum Average Power Transfer

- The maximum power transfer to a resistive load. We knew that if the network excluding the load was represented by a Thévenin equivalent circuit, the maximum power transfer would result if the value of the load resistor was equal to the Thévenin equivalent resistance (i.e., $R_L = R_{th}$).
- Here we consider the maximum average power being absorbed by the load impedance $\mathbf{Z}_L$.



The average power at the load is

$$P_L = \frac{1}{2} V_L I_L \cos(\theta_{v_L} - \theta_{i_L})$$

$$\mathbf{Z}_{\text{th}} = R_{th} + jX_{th}$$

$$\mathbf{Z}_L = R_L + jX_L$$

Maximum Average Power Transfer

We have

$$\mathbf{I}_L = \frac{\mathbf{V}_{\text{th}}}{\mathbf{Z}_{\text{th}} + \mathbf{Z}_L} = \frac{\mathbf{V}_{\text{th}}}{R_{th} + jX_{\text{th}} + R_L + jX_L} = \frac{\mathbf{V}_{\text{th}}}{R_{\text{th}} + R_L + j(X_{\text{th}} + X_L)}$$

$$\mathbf{V}_L = \mathbf{V}_{\text{th}} \frac{\mathbf{Z}_L}{\mathbf{Z}_{\text{th}} + \mathbf{Z}_L} = \mathbf{V}_{th} \frac{R_L + jX_L}{R_{th} + jX_{\text{th}} + R_L + jX_L} = \mathbf{V}_{th} \frac{R_L + jX_L}{R_{th} + R_L + j(X_{th} + X_L)}$$

The magnitude of $\mathbf{I}_L$ and the phase angle are

$$|\mathbf{I}_L| = \frac{|\mathbf{V}_{\text{th}}|}{\sqrt{(R_{\text{th}} + R_L)^2 + (X_{\text{th}} + X_L)^2}}, \qquad \angle \mathbf{I}_L = \angle \mathbf{V}_{\text{th}} - \tan^{-1}\left(\frac{X_{\text{th}} + X_L}{R_{\text{th}} + R_L}\right)$$

Similarly, the magnitude of $\mathbf{V}_L$ and the phase angle are

$$|\mathbf{V}_L| = \frac{|\mathbf{V}_{\text{th}}|\sqrt{R_L^2 + X_L^2}}{\sqrt{(R_{\text{th}} + R_L)^2 + (X_{\text{th}} + X_L)^2}},$$

$$\angle \mathbf{V}_L = \angle \mathbf{V}_{th} + \tan^{-1}\left(\frac{X_L}{R_L}\right) - \tan^{-1}\left(\frac{X_{\text{th}} + X_L}{R_{th} + R_L}\right)$$

Maximum Average Power Transfer

In short, we can have

$$\mathbf{I}_L = \frac{\mathbf{V}_{\text{th}}}{\sqrt{(R_{\text{th}} + R_L)^2 + (X_{\text{th}} + X_L)^2}}, \quad \mathbf{V}_L = \frac{\mathbf{V}_{\text{th}}\sqrt{(R_L^2 + X_L^2)^2}}{\sqrt{R_{\text{th}} + R_L)^2 + (X_{\text{th}} + X_L}}$$

Since $P_L = \frac{1}{2} V_L I_L \cos(\theta_{v_L} - \theta_{i_L})$, and the phase angle $\theta_{v_L} - \theta_{i_L} = \theta_{\mathbf{Z}_L}$ and

$$\cos\theta_{\mathbf{Z}_L} = \frac{R_L}{\sqrt{R_L^2 + X_L^2}}$$

We have

$$P_L = \frac{1}{2}\frac{|\mathbf{V}_{\text{th}}|^2\sqrt{R_L^2 + X_L^2}\cos(\theta_{\mathbf{Z}_L})}{(R_{\text{th}} + R_L)^2 + (X_{\text{th}} + X_L)^2} = \frac{1}{2}\frac{|\mathbf{V}_{\text{th}}|^2 R_L}{(R_{\text{th}} + R_L)^2 + (X_{\text{th}} + X_L)^2}$$

- If $|\mathbf{V}_{\text{th}}|$ is constant, then the quantity $(X_{\text{th}} + X_L)$ which absorbs no power, and any nonzero value of this quantity only serves to reduce P_L. Hence to have the maximum power transfer we need to have $X_L = -X_{\text{th}}$

Maximum Average Power Transfer

- The maximized problem becomes the maximize

$$P_L = \frac{1}{2}\frac{|\mathbf{V}_{\text{th}}|^2 R_L}{(R_L + R_{\text{th}})^2}$$

By selecting $R_L = R_{\text{th}}$, the maximum average power transfer to the load should be chosen so that

$$\mathbf{Z}_L = R_L + jX_L = R_{\text{th}} - jX_t = \mathbf{Z}_{\text{th}}^*$$

- If the load impedance is purely resistive (i.e., $X_L = 0$), the condition for maximum average power transfer can be derived via the expression

$$\begin{aligned}
\frac{dP_L}{dR_L} &= \frac{d}{dR_L}\left(\frac{1}{2}\frac{|\mathbf{V}_{\text{th}}|^2 R_L}{(R_{\text{th}} + R_L)^2 + (X_{\text{th}} + X_L)^2}\right) = 0 \\
&= \frac{d}{dR_L}\left(\frac{R_L}{(R_{\text{th}} + R_L)^2 + X_{\text{th}}^2}\right) = \frac{(R_{\text{th}} + R_L)^2 + X_{\text{th}}^2 - 2R_{\text{th}}R_L - 2R_L^2}{((R_{\text{th}} + R_L)^2 + X_{\text{th}}^2)^2} \\
&= \frac{R_{\text{th}}^2 + X_{\text{th}}^2 - R_L^2}{((R_{\text{th}} + R_L)^2 + X_{\text{th}}^2)^2} = 0
\end{aligned}$$

Maximum Average Power Transfer

- To make the left hand side is equal zero, we need

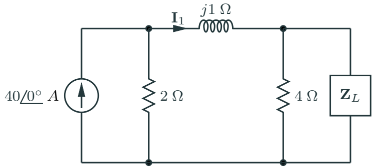
$$R_{\text{th}}^2 + X_{\text{th}}^2 - R_L^2 = 0$$

- Then the value of R_L that maximizes P_L under the condition $X_L = 0$ is $R_L = \sqrt{R_{\text{th}}^2 + X_{\text{th}}^2}$.

Strategy to solve the Maximum Power Transfer Problem:

1. Remove the load $\mathbf{Z}_L$ and find the Thévenin equivalent for the remainder of the circuit.
2. Construct the Thévenin circuit with load $\mathbf{Z}_L$.
3. Select $\mathbf{Z}_L = \mathbf{Z}_{\text{th}}^* = R_{\text{th}} - jX_{\text{th}}$, and then $\mathbf{I}_L = \mathbf{V}_{\text{th}}/2R_{\text{th}}$ and the maximum average power transfer

Maximum Average Power Transfer: Example



The circuit in Fig, we wish to find the value of $\mathbf{Z}_L$ for maximum average power transfer. In addition, we wish to find the value of the maximum average power delivered to the load.

Removing the load and using a current divider, then multiply the current with a resistor $4\ \Omega$, we have

$$\mathbf{V}_{\text{th}} = \mathbf{V}_{\text{oc}} = \mathbf{I}_1(4\ \Omega) = \frac{40\underline{/0^\circ}(2)}{6 + j1}(4) = 5.28\underline{/-9.46^\circ}\ \text{V}$$

Open the current source and see the $\mathbf{Z}_{\text{th}}$ from the $\mathbf{Z}_L$ terminal, we obtain

$$\mathbf{Z}_{\text{th}} = 4||(2 + j1) = \frac{4(2 + j1)}{6 + j1} = 1.40 + j0.43\ \Omega$$

Therefore, $\mathbf{Z}_L$ for the maximum average power transfer is $\mathbf{Z}_L = 1.40 - j0.43\ \Omega$

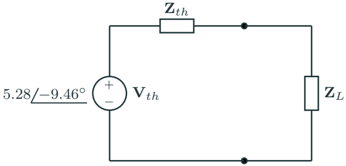
Maximum Average Power Transfer: Example

With $\mathbf{Z}_L$ as given previously, the current in the load is

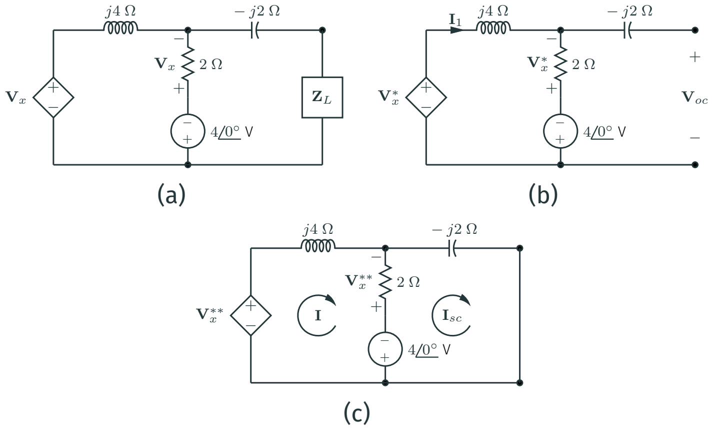
$$\mathbf{I}_L = \frac{5.28\underline{/-9.46^\circ}}{2.8} = 1.89\underline{/-9.46^\circ} \text{ A}$$

Therefore, the maximum average power transferred to the load is

$$P_L = \frac{1}{2} I_M^2 R_L = \frac{1}{2}(1.89)^2(1.4) = 2.50 \text{ W}$$



Maximum Average Power Transfer: Hard Example



From Fig (a), we want to find the value of $\mathbf{Z}_L$ for the maximum average power transfer. In addition, let us determine the value of the maximum average power delivered to the load.

Maximum Average Power Transfer: Hard Example

From figure (b), the Mesh equations for the circuit are

$$-\mathbf{V}_x^* + (2 + j4)\mathbf{I}_1 - 4\underline{/0^\circ} = 0$$

$$\mathbf{V}_x^* + 4 = (2 + j4)\mathbf{I}_1$$

$$\mathbf{V}_x^* = -2\mathbf{I}_1 \implies \mathbf{I}_1 = \frac{1\underline{/-45^\circ}}{\sqrt{2}}$$

The open circuit is then

$$\mathbf{V}_{oc} = 2\mathbf{I}_1 - 4\underline{/0^\circ} = \sqrt{2}\underline{/-45^\circ} - 4\underline{/0^\circ} = -3 - j1 = -3.16\underline{/18.43^\circ} \text{ V}$$

The short circuit current can be derived from Fig (C). The equation for this circuit are

$$-2\mathbf{I}_1 + 4 = (2 + j4)\mathbf{I}_1$$

$$\mathbf{V}_x^{**} + 4 = (2 + j4)\mathbf{I} - 2\mathbf{I}_{sc}$$

$$-4 = -2\mathbf{I} + (2 - j2)\mathbf{I}_{sc}$$

$$\mathbf{V}_x^{**} = -2(\mathbf{I} - \mathbf{I}_{sc}) \implies \mathbf{I}_{sc} = -(1 + j2) \text{ A}$$

The thévenin equivalent impedance is then

$$\mathbf{Z}_{\text{th}} = \frac{\mathbf{V}_{oc}}{\mathbf{I}_{sc}} = \frac{3 + j1}{1 + j2} = 1 - j1 \ \Omega$$

Therefore, for the maximum average power transfer the load impedance should be $\mathbf{Z}_L = 1 + j1 \ \Omega$. The current in this load $\mathbf{Z}_L$ is then

$$\mathbf{I}_L = \frac{\mathbf{V}_{oc}}{\mathbf{Z}_{\text{th}} + \mathbf{Z}_L} = \frac{-3 - j1}{2} = -1.58\underline{/18.43^\circ} \text{ A}$$

Hence, the maximum average power transferred to the load is

$$P_L = \frac{1}{2}(1.58)^2(1) = 1.25 \text{ W}$$

Effective or RMS Power

- The average power absorbed by a resistive load is directly dependent on the type, or types, of sources that are delivering power to the load. For example, If the source was DC, the average power absorbed was I^2R, and if the source was sinusoidal, the average power was $\frac{1}{2}I_M^2R$.
- Both types of waveforms are very important, however they are by no means the only waveforms we will encounter in circuit analysis.
- We need to compare the **effectiveness** of different sources in delivering power to a resistive load.
- We define the **effective value of a periodic waveform**, representing either voltage or current.
- Define the effective value of a periodic current as a constant or DC value, which as current would deliver the same average power to a resistor R. Let us call the constant current I_{eff}.
- The average power delivered to a resistor as a result of I_{eff} and the average power delivered to a resistor by a periodic current $i(t)$ are

$$P = I_{\text{eff}}^2 R \text{ and } P = \frac{1}{T}\int_{t_0}^{t_0+T} i^2(t) R dt$$

Effective or RMS Power

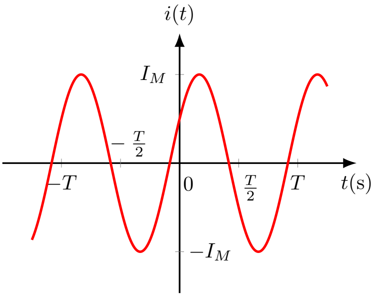
Both power must be equal, so we have

$$I_{\text{eff}}^2 R = \frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)Rdt$$

$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)dt}$$

Since the effective value is found by first determining the square of the current, then computing the average or mean value, and finally taking the square root. We are determining the root mean square, which we abbreviate as rms, and therefore I_{eff} is called I_{rms}.

Effective or RMS Power



What is the rms value of
$i(t) = I_M \cos(\omega t - \theta)$

$$I_{\text{rms}} = \left[\frac{1}{T}\int_0^T i^2(t)dt\right]^{1/2}$$

We have

$$I_{\text{rms}} = \left[\frac{1}{T}\int_0^T I_M^2 \cos^2(\omega t - \theta)dt\right]^{1/2} = \left[\frac{I_M^2}{T}\int_0^T \frac{1}{2} + \frac{1}{2}\cos(2(\omega t - \theta))dt\right]^{1/2}$$

$$= \frac{I_M}{\sqrt{2}}$$ It is dependent of the phase shift.

The second integrant is zero because it is one-period integration.

Effective or RMS Power

On using the rms values for voltage and current,

$$I_{\text{rms}} = \frac{I_M}{\sqrt{2}} \text{ and } V_{\text{rms}} = \frac{V_M}{\sqrt{2}}$$

then the average power can be written as

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i).$$

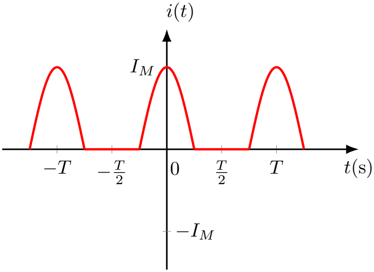
The power absorbed by a resistor R is

$$P = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}^2}{R}$$

In household,

- the nomal 220 V AC electrical outlets have an rms value of 220 V,
- and average value of 0 V,
- and a maximum value of $220\sqrt{2} = 311.127$ V peak.

Effective or RMS Power



A half-rectified current

$$i(t) = \begin{cases} I_M \cos(\omega t), & -\frac{T}{4} \leq t \leq \frac{T}{4} \\ 0, & \frac{T}{4} < t < \frac{3T}{4} \end{cases}$$

We have

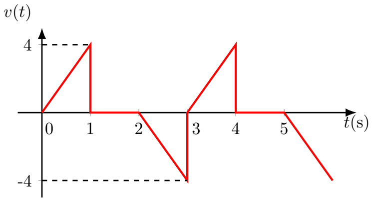
$$I_{\text{rms}} = \left[\frac{1}{T}\int_{-T/4}^{T/4} I_M^2 \cos^2(\omega t)\right]^{1/2} = \left[\frac{I_M^2}{T}\left(\frac{1}{2}t + \frac{1}{2}\cos(2\omega t)\right)\Big|_{-T/4}^{T/4}\right]^{1/2}$$

$$= \left[\frac{I_M^2}{T}\left[\frac{T}{8}+\frac{T}{8}\right] - \frac{I_M^2}{T}\left[\frac{T}{8\pi}\sin(\frac{4\pi}{T}\frac{T}{4}) - \frac{T}{8\pi}\sin(\frac{4\pi}{T}\frac{-T}{4})\right]\right]^{1/2}$$

$$= \frac{I_M}{2} \text{ A}$$

Its rms value is half of a normal sinusoidal current.

Effective or RMS Power



The wave form is periodic with period $T = 3$s.

$$v(t) = \begin{cases} 4t \text{ V}, & 0 < t \leq 1 \text{ s} \\ 0 \text{ V}, & 1 < t \leq 2 \text{ s} \\ -4t + 8 \text{ V}, & 2 < t \leq 3 \text{ s} \end{cases}$$

The rms value is

$$\begin{aligned} V_{\text{rms}} &= \left[\frac{1}{3}\left(\int_0^1 (4t)^2 dt + \int_1^2 (0)^2 dt + \int_2^3 (8-4t)^2 dt\right)\right]^{1/2} \\ &= \left[\frac{1}{3}\left(\left.\right] \frac{16t^3}{3}\bigg|_0^1 + \left(64t - \frac{64t^2}{2} + \frac{16t^3}{3}\right)\bigg|_2^3\right)\right]^{1/2} \\ &= 1.89 \text{ V} \end{aligned}$$

The Power Factor

The power factor is vary important quantity. Its importance stems in part from the economic impact. It has on industrial users of large amounts of power.

- We showed that a load operating in the AC steady state is delivered an average power of

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

- The product $V_{\text{rms}} I_{\text{rms}}$ is referred to as the **apparent power**. The unit of apparent power is volt-amperes (VA) to distinguish it from average power.
- We define the **the power factor (pf)** as the ratio of the average power to the apparent power; that is

$$\text{pf} = \frac{P}{V_{\text{rms}} I_{\text{rms}}} = \cos(\theta_v - \theta_i),$$

where

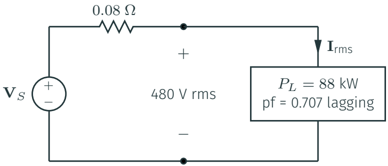
$$\cos(\theta_v - \theta_i) = \cos \theta_{\mathbf{Z}}$$

The Power Factor

- The angle $\theta_v - \theta_i = \theta_{\mathbf{Z}}$ is the phase angle of the load impedance and is often referred to as the **power factor angle**.
- The two extreme positions for the angele correspond to a purely resistive load where $\theta_{\mathbf{Z}} = 0$ and the pf is 1, and the purely reactive load where $\theta_{\mathbf{Z}_L} = \pm 90^\circ$ and the pf is 0.
- If the load is an equivalent RC combination, then the pf angle lies between the limits $-90^\circ < \theta_{\mathbf{Z}_L} < 0^\circ$. On the other hand if the load is an equivalent RL combination, then the pf angle lies between the limits $0 < \theta_{\mathbf{Z}_L} < 90^\circ$
- Since $\cos(\theta_{\mathbf{Z}_L}) = \cos(-\theta_{\mathbf{Z}_L})$, the pf is said to be either **leading** or **lagging** where **these two terms refer to the phase of the current with respect to the voltage**.
- The current leads the voltage in and RC load, the load has a leading pf. In a similar manner, and RL load has a lagging pf.
- Load impedances of $\mathbf{Z}_L = 1 - j1$ and $\mathbf{Z}_L = 2 + j1$ has power factors of $\cos(-45^\circ) = 0.707$ leading and $\cos(26.57^\circ) = 0.894$ lagging, respectively.
- If a load impedance is $10\underline{/30^\circ}$, the power factor is calculated from pf $= \cos(30^\circ) = \frac{3}{2}$ lagging.

The Power Factor: Example

An industrial load consumes 88 kW at a pf of 0.707 lagging from a 480 V rms line. The transmission line resistance from the power company's transformer to the plant is 0.08 Ω. Let us determine the power that must be supplied by the power company (a) under present conditions and (b) if the pf is somehow changed to 0.90 lagging.



(a) We need to find I_{rms} to find P_L. From

$$P_L = I_{\text{rms}} V_{\text{rms}} \cos(\theta_v - \theta_i) \implies I_{\text{rms}} = \frac{P_L}{(\text{pf})(V_{\text{rms}})} = \frac{88 \times 10^3}{(0.707)(480)} = 259.3 \text{ A rms}$$

The power that must be supplied by the power company is

$$P_S = P_L + (0.08) I_{\text{rms}}^2 = 88,000 + (0.08)(259.3)^2 = 93.38 \text{ kW}$$

The Power Factor : Example

(b) Now the pf is changed to 0.90 lagging (closer to 1.0 than in (a)). The rms load current for this conditions is

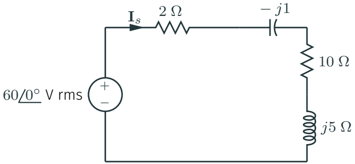
$$I_{\text{rms}} = \frac{P_L}{(\text{pf})(V_{\text{rms}})} = \frac{(88)(10^3)}{(0.90)(480)} = 203.7 \text{ A rms}$$

Under these conditions the power company must generate

$$P_S = P_L + (0.08)I_{\text{rms}}^2 = 88,000 + (0.08)(203.7)^2 = 91.32 \text{ k W}$$

- The first case the power company must generate 93.38 kW in order to supply the plant with 88 kW of power because the low power factor means that the line losses will be high (5.38 kW).
- The second case the power company need only generate 91.32 kW in order to supply the plant with its requird power, and the corresponding line losses are only 3.32 kW.
- The factories will get charge if there pf is low.

The Power Factor : Example



For the circuit of Figure, determine the power factor of the combined loads.

We have

$$P_L = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

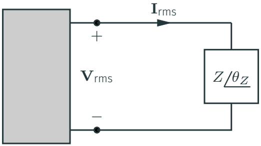
Sine

$$\begin{aligned}
\mathbf{Z}_{eq} &= 2 - j1 + 10 + j5 = 12 + j4 \\
&= \sqrt{12^2 + 4^2}\underline{/\tan^{-1}(4/12)} \\
&= 12.65\underline{/18.43^\circ} \\
\cos(18.43^\circ) &= 0.9487 \text{ lagging}
\end{aligned}$$

Then the pf is 0.9487 lagging.

Complex Power

To study AC steady-state power, we introduce a quantity, called complex power:



$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}},$$

where $\mathbf{I}_{\text{rms}} = I_{\text{rms}}\angle\theta_i = I_R + jI_Z$ and $\mathbf{I}^*_{\text{rms}} = I_{\text{rms}}\angle-\theta_i = I_R - jI_Z$.

Complex power is then

$$\mathbf{S} = V_{\text{rms}}\angle\theta_v I_{\text{rms}}\angle-\theta_i = V_{\text{rms}}I_{\text{rms}}\angle\theta_v - \theta_i$$

or

$$\mathbf{S} = V_{\text{rms}}I_{\text{rms}}\cos(\theta_v - \theta_i) + jV_{\text{rms}}I_{\text{rms}}\sin(\theta_v - \theta_i), \text{ where } \theta_v - \theta_i = \theta_{\mathbf{Z}}$$

- The real part of $\mathbf{S}$ is the real or average power.
- The imaginary part of $\mathbf{S}$ is the reactive or quadrature power.

Complex Power

The complex power can be expressed in the form

$$\mathbf{S} = P + jQ,$$

where

$$P = \text{Re}\{\mathbf{S}\} = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$
$$Q = \text{Im}\{\mathbf{S}\} = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

- The magnitude of the complex power is called the **apparent power**
- The phase angle for complex power is simply the power factor angle.
- The complex power, like apparent power, is measured in volt-amperes, (VA).
- The real power is measured in watts (W) and Q is measured in volt-amperes reactive or (var).

Complex Power

For R, L, C basic circuit elements:

- A resistor $\theta_v - \theta_i = 0°$, $\cos(\theta_v - \theta_i) = 1$ and $\sin(\theta_v - \theta_i) = 0$ As a result, a resistor absorbs real power ($P > 0$) but does not absorb any reactive power ($Q = 0$).
- For an inductor $\theta_v - \theta_i = 90°$

$$P = V_{\text{rms}} I_{\text{rms}} \cos(90°) = 0$$

$$Q = V_{\text{rms}} I_{\text{rms}} \sin(90°) > 0$$

  An inductor absorbs reactive power but does not absorb real power.

- A capacitor, we have $\theta_v - \theta_i = -90°$ and

$$P = V_{\text{rms}} I_{\text{rms}} \cos(-90°) = 0$$

$$Q = V_{\text{rms}} I_{\text{rms}} \sin(-90°) < 0$$

  A capacitor does not absorb any real power. The reactive power is now negative, then the capacitor must be supplying reactive power.

- The reactive power is related to energy storage in capacitors and inductors.

Complex Power

- Since $\mathbf{V}_{\text{rms}} = \mathbf{I}_{\text{rms}}\mathbf{Z}$, then

$$\begin{aligned}\mathbf{S} &= \mathbf{I}_{\text{rms}}\mathbf{Z}(\mathbf{I}^*_{\text{rms}}) = \mathbf{I}_{\text{rms}}\mathbf{I}^*_{\text{rms}}\mathbf{Z} = (I_{\text{rms}}\angle\theta_i I_{\text{rms}}\angle{-\theta_i})\mathbf{Z} \\ &= I^2_{\text{rms}}\angle 0^\circ\mathbf{Z} = I^2_{\text{rms}}\mathbf{Z} = I^2_{\text{rms}}(R + jX) = P + jQ\end{aligned}$$

- On the other hand, we have $\mathbf{I}_{\text{rms}} = \mathbf{V}_{\text{rms}}/\mathbf{Z}$, then

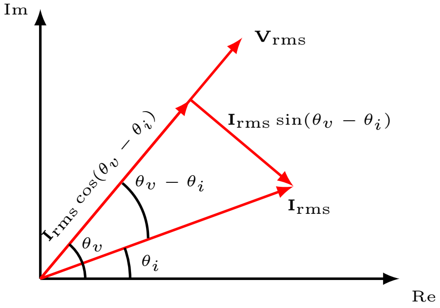
$$\begin{aligned}\mathbf{S} &= \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = V_{\text{rms}}\left(\frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}}\right)^* = \frac{V^2_{\text{rms}}}{\mathbf{Z}^*} = V^2_{\text{rms}}\mathbf{Y}^* \\ &= V^2_{\text{rms}}(G + jB)^* = P + jQ\end{aligned}$$

- If $\mathbf{Z}$ is a capacitor, the admitance for a capacitor is $j\omega C$. We have

$$\mathbf{S} = V^2_{\text{rms}}(j\omega C)^* = -j\omega C V^2_{\text{rms}}$$

- It is negative sign on the complex power. This agrees with our previous statement that a capacitor does not absorb real power but is a source of reactive power.

Complex Power



The phasor current can be split into two components:

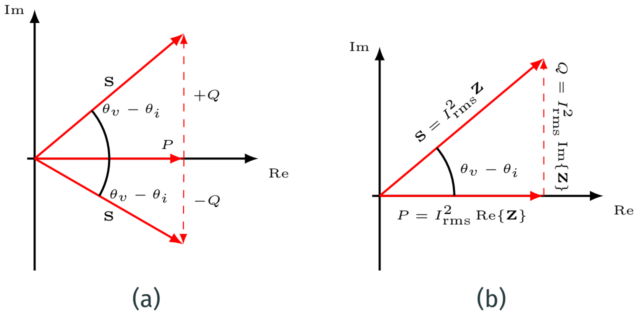
- one that is in phase with $\mathbf{V}_{\text{rms}}$
- and one is $90°$ out of phase with $\mathbf{V}_{\text{rms}}$
- The in-phase component produces the real power, and the $90°$ component, called the **quadrature component**, produces the reactive or quadature.

We have

$$\tan(\theta_v - \theta_i) = \frac{Q}{P}$$

which related the pf angle to P and Q in what is called the **power triangle.**

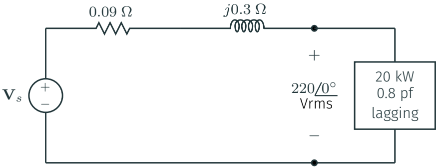
Complex Power



- In Fig (a) If Q is positive, the load is inductive, the power factor is lagging, and the complex number $\mathbf{S}$ lines in the first quadrant.
- If Q is negative, the load is capacitive, the power factor is leading, and the complex nuber $\mathbf{S}$ lines in the fourth quadrant.
- If Q is zero, the load is resistive, the power factor is unity, and the complex number $\mathbf{S}$ lines along the positive real axis.
- Fig (b) illustrates the relationships for an inductive load.

Complex Power: Example

A load operates at 20 kW, 0.8 pf lagging,. The load voltage is $220\angle\theta^0$ Vrms at 60 Hz. The impedance of the line is 0.09 + j0.3 Ω. We wish to determine the voltage and power factor at the input to the line.



We have

$$P = |\mathbf{S}|\cos(\theta)$$

$$|\mathbf{S}| = \frac{P}{\cos\theta} = \frac{P}{\text{pf}} = \frac{20,000}{0.8} = 25,000 \text{ VA}$$

Therefore, at the load

$$\mathbf{S}_{\text{load}} = 25,000\angle\theta = 25,000\angle\cos^{-1}(0.8) = 25,000\angle 36.87^\circ = 20,000 + j15,000 \text{ VA}$$

Since $\mathbf{S}_{\text{load}} = \mathbf{V}_{\text{load}}\mathbf{I}^*_{\text{load}}$

$$\mathbf{I}_{\text{load}} = \left[\frac{25,000\angle 36.87^\circ}{220\angle 0^\circ}\right]^* = 113.64\angle -36.87^\circ$$

Complex Power: Example

The complex power losses in the line are

$$\mathbf{S}_{\text{line}} = I_{\text{load}}^2 \mathbf{Z}_{\text{line}} = (113.64)^2(0.09 + j0.3) = 1162.26 + j3874.21 \text{ VA}$$

The complex power is conserved, therefore the complex power at the generator is

$$\mathbf{S}_S = \mathbf{S}_{\text{load}} + \mathbf{S}_{\text{line}} = 21,162.26 + j18,874.21 = 28,356.25\underline{/41.73^\circ} \text{ VA}$$

Hence, the generator voltage is

$$\mathbf{V}_s = \frac{\mathbf{S}_S}{\mathbf{I}_{\text{load}}^*} = \frac{28,356.25\underline{/41.73^\circ}}{113.64\underline{/36.87^\circ}} = 249.53\underline{/4.86^\circ} \text{ Vrms}$$

and the generator power factor is

$$\cos(\theta_v - \theta_i) = \cos(4.86 - (-36.87)) = \cos(41.73^\circ) = 0.75 \text{ lagging}$$

Note: $\mathbf{V}_s = 249.53\underline{/4.86^\circ}$ and $\mathbf{I}_{load} = 113.64\underline{/-36.87^\circ}$.

Complex Power: Example

Second method: using KVL, we calculated the load current as

$$\mathbf{I}_{\text{load}} = 113.64\underline{/-36.87^\circ}$$

The voltage drop in the transmission line is

$$\mathbf{V}_{\text{line}} = (113.64\underline{/-36.87^\circ})(0.09 + j0.3) = 35.59\underline{/36.43^\circ}$$

Therefore the generator voltage is

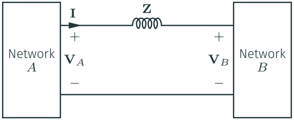
$$\mathbf{V}_S = 220\underline{/0^\circ} + 35.59\underline{/36.43^\circ} = 249.53\underline{/4.86^\circ} \text{ Vrms}$$

Hence, the generator voltage is 249.53 Vrms. In addition,

$$\theta_v - \theta_i = 4.86^\circ - (-36.87^\circ) = 41.73^\circ, \text{ and pf} = \cos(42.73^\circ) = 0.75 \text{ lagging}$$

Complex Power: Example II

Two networks A and B are connected by two conductors having a net impedance of $\mathbf{Z} = 0 + j1\ \Omega$, as shown in Fig. The voltages at the terminals of the networks are $\mathbf{V}_A = 120\angle 30^\circ$ Vrms and $\mathbf{V}_B = 120\angle 0^\circ$ Vrms. We wish to determine the average powr flow between the networks and identify which is the source and which is the load.



$$\mathbf{I} = \frac{\mathbf{V}_A - \mathbf{V}_B}{\mathbf{Z}} = \frac{120\angle 30^\circ - 120\angle 0^\circ}{j1} = 62.12\angle 15^\circ \text{ Arms}$$

The power delivered by network A is

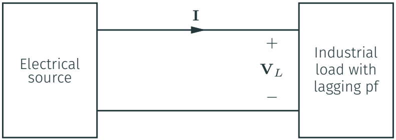
$$P_A = |\mathbf{V}_A||\mathbf{I}|\cos(\theta_{\mathbf{V}_A} - \theta_{\mathbf{I}}) = (120)(62.12)\cos(30^\circ - 15^\circ) = 7200.4 \text{ W}$$

The power absorbed by network B is

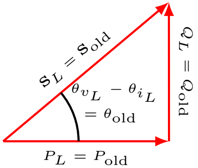
$$P_B = |\mathbf{V}_B||\mathbf{I}|\cos(\theta_{\mathbf{V}_B} - \theta_{\mathbf{I}}) = (120)(62.12)\cos(0^\circ - 15^\circ) = 7200.4 \text{ W}$$

Power Factor Correction

- Industrial plants that require large amounts of power have a wide variety of loads. However by nature the loads normally have a lagging power factor.
- Is there any convenient technique for raising the power factor of a load? Since a typical load may be a bank of induction motors or other expensive machinery, the technique for raising the pf should be an economical one to be feasible.



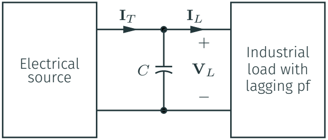
(a)



(b)

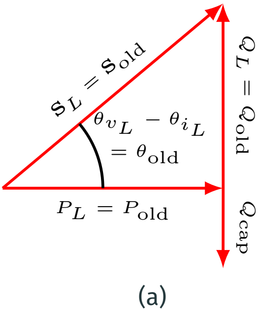
- We could decrease the angle (pf) by increasing P. This is not an economically attractive solution because our increased power consumption would increase the monthly bill from the electric utility.
- We can decrease Q by using capacitors. Capacitors is a source of reactive power and does not asorb real power.

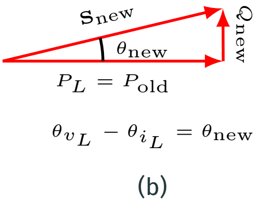
Power Factor Correction



We can connect a capacitor in parallel with our industrial load as shown in Fig.

The corresponing power triangles is shown below:





Power Factor Correction

Let's define

$$\mathbf{S}_{\text{old}} = P_{\text{old}} + jQ_{\text{old}} = |\mathbf{S}_{\text{old}}|\underline{/\theta_{\text{old}}} \qquad \text{and} \qquad \mathbf{S}_{\text{new}} = P_{\text{old}} + jQ_{\text{new}} = |\mathbf{S}_{\text{new}}|\underline{/\theta_{\text{new}}}$$

Then with the addition of the capacitor,

$$\mathbf{S}_{\text{new}} = \mathbf{S}_{\text{old}} + \mathbf{S}_{\text{cap}}$$

Therefore

$$\begin{aligned}\mathbf{S}_{\text{cap}} = \mathbf{S}_{\text{new}} - \mathbf{S}_{\text{old}} &= (P_{\text{new}} - jQ_{\text{new}}) - (P_{\text{old}} + jQ_{\text{old}}) \\ &= j(Q_{\text{new}} - q_{\text{old}}) = j(Q_{\text{cap}})\end{aligned}$$

In general

$$\mathbf{S} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*} \text{ and for a capacitor } \mathbf{Z}^* = -\frac{1}{j\omega C}$$

So that

$$\mathbf{S}_{\text{cap}} = jQ_{\text{cap}} = -j\omega C V_{\text{rms}}^2$$

- The equation can be used to find the required value of C in order to achieve the new specified power factor defined by the new power factor angle as shown in the complex power triangle.
- We can obtain a particular power factor for the total load (industrial load and capacitor) simply by selecting a capacitor and placing it in parallel with the original load.
- We want the power factor to be large, and therefore the power factor angle must be small [i.e., the larger the desired power factor, the smaller the angle $(\theta_{v_L} - \theta_{i_L})$.
- The electrical energy provider sends us a bill for the amount of electrical energy that we hae consumed. The rate is often expressed in baht per kWh and consists of at least two components: (1) the demand charge, which covers the cost of lines, poles, transformers, and so on, and (2) the energy charge, which covers the cost to produce electric energy at power plants.

Power Factor Correction: Example

It is common for an industrial facility operating at a poor power factor to be charged more by the electric utility providing electrical service.

- Let's suppose that our industrial facility operates at 220 Vrms and requires 500 kW at a power factor of 0.75 lagging.
- Assume an energy charge of 5 baht per kWh and a demand charge of zero baht per kW per month if the power factor is between 0.9 lagging and unity and 56.07 baht per kvar per month if the power factor is less than 0.85 lagging.
- Monthly energy charge is $500 \times 24 \times 30 \times 5 = 1,800,000$ baht. Let's calculate the monthly demand charge with the 0.75 lagging power factor. The complex power absorbed by the industrial facility is

$$\mathbf{S}_{\text{old}} = |\mathbf{S}_{\text{old}}| \angle \cos^{-1}(0.75) = \frac{500}{0.75} \angle 41.4^\circ = 666.67 \angle 41.4^\circ = 500 + j441 \text{ kVA}$$

The monthly demand charge is $441 \times 56.07 = 24,726.87$ baht. The total bill is 1,824,726.87 baht.

Power Factor Correction: Example

We can install a capacitor bank as shown in the Fig. to correct th power factor and reduce our demand charge. The demand charge is such that we only need to correct the power factor to 0.85 lagging.

The complex power absorbed by the industrial facility and capacitor bank will be

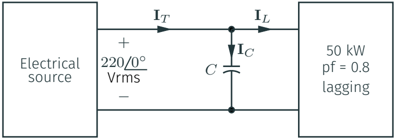
$$\mathbf{S}_{\text{new}} = \frac{500}{0.85} \angle \cos^{-1}(0.85) = 588.24\angle 31.79^{\circ} = 500 + j310 \text{ kVA}$$

With this complex power, the monthly demand charge will be 1,800,000 baht. The kvars of capacitance that we need to correct the power factor to 0.85 lagging is

$$\mathbf{S}_{\text{new}} - \mathbf{S}_{\text{old}} = \mathbf{S}_{\text{cap}} = (500 + j310) - (500 + j441) = -j131 \text{ kvar}$$

Power Factor Correction: Example II

Induction motors used to spin the molds consume 50 kW at a pf of 0.8 lagging from a $220\angle 0^\circ$ Vrms, 50 Hz line. We wish to raise the pf to 0.95 lagging by placing a bank of capacitors in parallel with the load. Find the capacitance using in this work.



From triangle complex power, we have $Q = \tan(\theta)$, then

$$\begin{aligned} Q_{\text{old}} &= P_{\text{old}} \tan(\cos^{-1}(\text{pf})) = (50 \times 10^3) \tan(36.7^\circ) = 37.5 \text{ kvar} \\ \mathbf{S}_{\text{old}} &= P_{\text{old}} + jQ_{\text{old}} = 50,000 + j37,500 \text{ VA} \\ \mathbf{S}_{\text{cap}} &= 0 + jQ_{\text{cap}} \end{aligned}$$

Since the required power factor is 0.95,

$$\theta_{\text{new}} = \cos^{-1}(\text{pf}_{\text{new}}) = \cos^{-1}(0.95) = 18.19^\circ$$

Power Factor Correction: Example II

Then

$$Q_{\text{new}} = P_{\text{old}} \tan(\theta_{\text{new}} = 50,000 \tan(18.19^\circ) = 16,430 \text{ var}$$

Hence

$$Q_{\text{new}} - Q_{\text{old}} = Q_{\text{cap}}$$
$$16,430 - 37,500 = -21,070 \text{ var}$$

Since $\mathbf{S}_{\text{cap}} = V_{\text{rms}}^2 / \mathbf{Z}^*$, then

$$\mathbf{S}_{\text{cap}} = jQ_{\text{cap}} = -j\omega C V_{\text{rms}}^2$$

We have

$$C = \frac{Q_{\text{cap}}}{-\omega V_{\text{rms}}^2} = \frac{-21,070}{-2\pi(50)(220^2)} = 1386\ \mu\text{F}$$

It is difficult to find the exact value of capacitance. We can use an nearest capacitor bank.

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