# Lecture 7: Frequency Response and Continuous-Time Filters II

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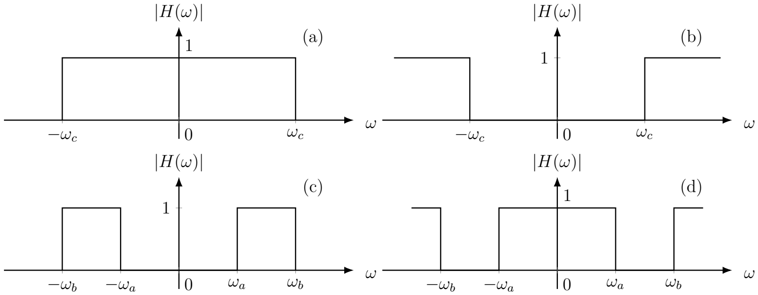
# Introduction to Filters

- An electric filter can be defined as a network or system that transforms an input signal in some specified way to yield an output signal with desired characteristics.
- Filter is a frequency-selective device, and its frequency response has significant values only in certain bands along the frequency axis.
- Filters are used extensively in electronic devices. For instance, telephone, radio, telegraph, television, radar, sonar, and space vehicles utilize filters in one form or another.
- Filters may be either **analog** or **digital**. Here we consider only analog filters. The components of an analog filter may be passive circuit elements like RLC circuits. Alternatively, it may be an active one; RLC circuit with operational amplifiers.
- The earliest work in filter theroy was done by Campbell in the United States and Wagner in Germany in 1915.

# Filter Classifications

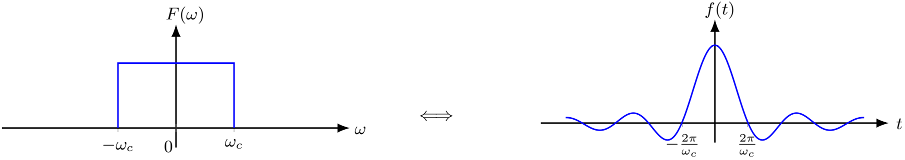
Filters can be classified on the basis of their frequency response: low-pass(a), high-pass(b), band-pass(c), band-eliminating(notch)(d), and all-pass filters.

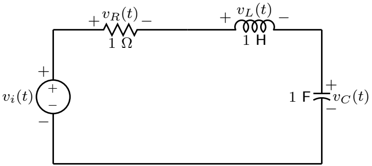



- These are **ideal filter**
- There are several way to show that the ideal filter is not realizable.
- One way, we can show that we cannot make $|H(j\omega)| = -\infty$ dB at any frequency.
- $|H(j\omega)| = 1 \times 10^{-5} \Rightarrow -100$ dB only.

# Filter Classifications

- It may also show that the inverse transform of the rectangular pulse in frequency-domain is the $\mathrm{sinc}$ function in time-domain. For example, the impulse response of the ideal low-pass filter will be the $\mathrm{sinc}$ function, which is noncausal and therefore physically unrealizable.
- It is not really necessary to requeire the ideal frequency response characteristics.






- **Low-pass filter**

$$V_C(s) = \frac{V_i(s)/s}{1+s+1/s} = \frac{V_i(s)}{s^2+s+1}$$

$$H_{\mathsf{lp}}(s) = \frac{1}{s^2+s+1} = \frac{1}{(j\omega/1)^2 + j2\zeta(\omega/1)+1}$$

at low frequency $\omega = 0$ (dc), the inductor is short circuit and the capacitor is an open circuit, so that the voltage in the capacitor is equal to the voltage in the source. On the other hand, if the frequency of the input source is very high, then the inductor is an open circuit ($j\infty$) and the capacitor a short circuit ($1/j\infty$) so that the capacitor voltage is zero.

- **High-pass filter** Now we let the output be the voltage across the inductor. We have

$$H_{\mathrm{hp}}(s) = \frac{V_L(s)}{V_i(s)} = \frac{s^2}{s^2 + s + 1}$$

  At high frequency the impedance of inductor is zero, so that the inductor voltage is zero, and for very high frequency the impedance of the inductor is very large so that it can be considered open circuit and the voltage in the inductor equals that of the source.

- **Band-pass filter** Letting the output be the voltage across the resistor, its transfer function is

$$H_{\mathrm{bp}}(s) = \frac{V_R(s)}{V_i(s)} = \frac{s}{s^2 + s + 1}$$

  For zero frequency, the capacitor is an open circuit so the voltage across the resistor is zero. At the very high frequency the impedance of the inductor is very large, or an open circuit and the voltage across the resistor is zero. For some middle frequency the serial of the inductor and the capacitor resonates and will have zero impedance. We have the maximum voltage across the resistor.

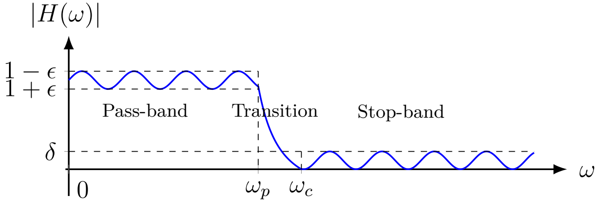
# Filter Classifications

## Simple RLC circuit

- **Band-stop filter** Suppose we consider as output the voltage across the connection of the inductor and the capacitor. At low and high frequencies, the impedance of the LC connection is very high, or open circuit, and so the output voltage is the input voltage. At the resonance frequency $\omega = 1$ the impedance of the LC connection is zero, so the output voltage is zero. The band-stop filter is

$$H_{\mathsf{bs}}(s) = \frac{s^2 + 1}{s^2 + s + 1}$$

# Approximation Filter




- Generally, a certain amount of tolerance is permissible, both in the pass-band and the stop-band, so that the frequency response of a low-pass filter can be of the form shown in Figure above. Where the frequency response has been shown for only positive values of $\omega$.
- We have a transition band between the pass-band and the stop-band. The frequency range from $\omega_c$ to $\omega_p$.
- In the pass-band it is required that the gain be within the range $1 \pm \epsilon$, where $\epsilon$ is a specified tolerance.
- In the stop-band the gain must not exceed another specified tolerance $\delta$.
- By reducing the values of the tolerances and the width of the transition region, we obtain a closer approximation to the ideal filter.

# Butterworth Filter

The amplitude response $|H(j\omega)|$ of an $n$th order Butterworth low-pass filter is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

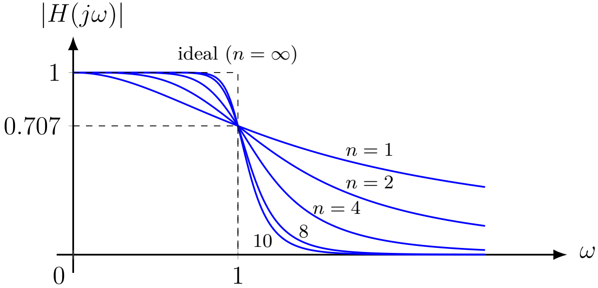
At $\omega = \omega_c$, the gain $|H(j\omega)| = 1/\sqrt{2}$ or $-3$ dB.

- Because the power is proportional to the amplitude squared, the power ratio (output power to input power) drops by a factor $2$ at $\omega = \omega_c$
- The $\omega_c$ is called the **half-power frequency** (the amplitude ratio of $\sqrt{2}$ is $3$ dB).
- It is convenient to consider a normalized filter $\mathcal{H}(s)$, whose half-power frequency is $1$ rad/s ($\omega_c = 1$). The amplitude characteristic reduces to

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2n}}}$$

- We can obtain the desired transfer function $H(s)$ for any value of $\omega_c$ by simple frequency scaling, where we replace $s$ by $s/\omega_c$ in the normalized $\mathcal{H}(s)$.

# Butterworth Filter




- The Butterworth amplitude response decreases monotonically. Moreover, the first $2n - 1$ derivatives of the amplitude response are zero at $\omega = 0$. For this reason this characteristic is called **maximally flat** at $\omega = 0$.
- The filter gain is $1(0\text{dB})$ at $\omega = 0$ and $0.707(-3\text{dB})$ at $\omega = 1$ for all $n$. Therefore, the $3-$dB (or half power) bandwidth is $1$ rad/s for all $n$.
- The last $n$, the amplitude response approaches the ideal characteristic.

# Butterworth Filter

To determine the corresponding transfer function $\mathcal{H}(s)$, recall that $\mathcal{H}(-j\omega)$ is the complex conjugate of $\mathcal{H}(j\omega)$. Therefore

$$\mathcal{H}(j\omega)\mathcal{H}(-j\omega) = |\mathcal{H}(j\omega)|^2 = \frac{1}{1+\omega^{2n}}$$

Substituting $s = j\omega$ in this equation, we obtain

$$\mathcal{H}(s)\mathcal{H}(-s) = \frac{1}{1+(s/j)^{2n}}$$

The poles of $\mathcal{H}(s)\mathcal{H}(-s)$ are given by

$$s^{2n} = -(j)^{2n}$$

In this result we use the fact that $-1 = e^{j\pi(2k-1)}$ for integral values of $k$, and $j = e^{j\pi/2}$ to obtain

$$s^{2n} = e^{j\pi(2k-1+n)}, \qquad k \text{ integer}$$

# Butterworth Filter

This equation yields the poles of $\mathcal{H}(s)(-s)$ as

$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)}, \qquad k = 1, 2, 3, \ldots, 2n$$

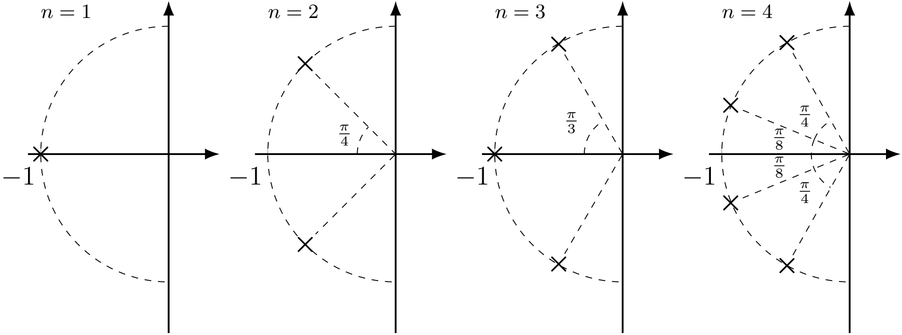
- We can see that all poles have a unit magnitude; that is, they are located on a unit circle in the $s$-plane separated by angel $\pi/n$.
- Since $\mathcal{H}(s)$ is stable and causal, its poles must lie in the LHP, which are

$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)} = \cos\frac{\pi}{2n}(2k+n-1) + j\sin\frac{\pi}{2n}(2k+n-1),$$

where $k = 1, 2, 3, \ldots, n$ and $\mathcal{H}(s)$ is given by

$$\mathcal{H}(s) = \frac{1}{(s-s_1)(s-s_2)\cdots(s-s_n)}$$

# Butterworth Filter




We find the poles of $\mathcal{H}(s)$ for $n = 4$ to be at angles $5\pi/8$, $7\pi/8$, $9\pi/8$, and $11\pi/8$. These lie on the unit circle and are given by $-0.3827 \pm j0.9239$, $-0.9239 \pm j0.3827$. Hence, $\mathcal{H}(s)$ is

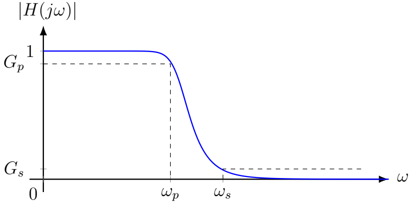
$$\mathcal{H}(s) = \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.478s + 1)}$$

# Butterworth Filter

## Determination of $n$, the Filter Order

If $\hat{G}_x$ is the gain of a low-pass Butterworth filter in dB units at $\omega = \omega_x$, then

$$\hat{G}_x = 20\log|H(j\omega_x)| = -20\log\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}} = -10\log\left[1+\left(\frac{\omega_x}{\omega_c}\right)^{2n}\right]$$




We have

$$\hat{G}_p = -10\log\left[1+\left(\frac{\omega_p}{\omega_c}\right)^{2n}\right]$$

$$\hat{G}_s = -10\log\left[1+\left(\frac{\omega_s}{\omega_c}\right)^{2n}\right]$$

Or

$$\left(\frac{\omega_p}{\omega_c}\right)^{2n} = 10^{-\hat{G}_p/10} - 1, \qquad \left(\frac{\omega_s}{\omega_c}\right)^{2n} = 10^{-\hat{G}_s/10} - 1$$

# Butterworth Filter

## Determination of $n$, the Filter Order

Dividing both equations, we obtain

$$\left(\frac{\omega_s}{\omega_p}\right)^{2n} = \left[\frac{10^{-\hat{G}_s/10} - 1}{10^{-\hat{G}_p/10} - 1}\right]$$

and

$$n = \frac{\log\left[\left(10^{-\hat{G}_s/10} - 1\right) / \left(10^{-\hat{G}_p/10} - 1\right)\right]}{2\log(\omega_s/\omega_p)}$$

Also we have

$$\omega_c = \frac{\omega_p}{\left[10^{-\hat{G}_p/10} - 1\right]^{1/2n}}$$

$$\omega_c = \frac{\omega_s}{\left[10^{-\hat{G}_s/10} - 1\right]^{1/2n}}$$

Design a Butterworth low-pass filter to meet the specifications:

1. Pass-band gain to lie between 1 and $G_p = 0.794(\hat{G}_p = -2$ dB) for $0 \leq \omega < 10$.
2. Stop-band gain not to exceed $G_s = 0.1$ ($\hat{G}_s = -20$ dB) for $\omega \geq 20$.

**Solution:**

1. Determine $n$. Here $\omega_p = 10$, $\omega_s = 20$, $\hat{G}_p = -2$ dB, and $\hat{G}_s = -20$ dB.

$$n = \frac{\log\left[\left(10^{-\hat{G}_s/10} - 1\right) / \left(10^{-\hat{G}_p/10} - 1\right)\right]}{2\log(\omega_s/\omega_p)} = 3.701 \approx 4$$

2. Determine $\omega_c$. From

$$\omega_c = \frac{\omega_p}{\left[10^{-\hat{G}_p/10} - 1\right]^{1/2n}} = 10.693 \Rightarrow G_p = 0.794$$

3. The normalized transfer function $\mathcal{H}(s)$ for $n = 4$ is

$$\mathcal{H}(s) = \frac{1}{s^4 + 2.6131s^3 + 3.4142s^2 + 2.613s + 1}$$

# Butterworth Filter

## Determination of $n$, the Filter Order: Example

4. The final filter transfer function $H(s)$ is with $\omega_c = 10.693$, so we have

$$H(s) = \frac{1}{\left(\frac{s}{10.693}\right)^4 + 2.6131\left(\frac{s}{10.693}\right)^3 + 3.4142\left(\frac{s}{10.693}\right)^2 + 2.6131\left(\frac{s}{10.693}\right) + 1}$$
$$= \frac{13073.7}{s^4 + 27.942s^3 + 390.4s^2 + 3194.88s + 13073.7}$$

# Chebyschev Filter

The frequency response of the Butterworth filter is flat near the frequency $\omega = 0$ , but its cutoff rate is not high. A better cutoff rate is obtained by using **Chebyschev filters** with the magnitude response

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 C_n^2(\omega)}},$$

where $C_n(\omega)$ , the $n$th-order Chebyshev polynomial, is given by

$$C_n(\omega) = \cos(n \cos^{-1} \omega)$$

$C_n(\omega)$ is also expressible in polynomial form as follow:

- $C_0(\omega) = \cos(0) = 1$ and $C_1(\omega) = \cos(\cos^{-1}(\omega)) = \omega$
- Let us define $\omega = \cos\phi$ so that $C_n(\omega) = \cos n\phi$. By using the trigonometric identity

$$\cos[(n+1)\phi] + \cos[(n-1)\phi] = 2\cos n\phi \cos\phi$$

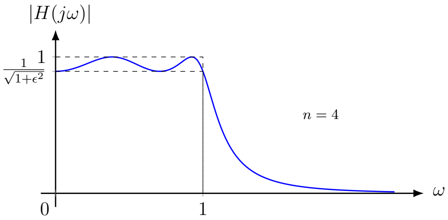
Then we have

$$C_{n+1}(\omega) + C_{n-1}(\omega) = 2C_n(\omega)\omega \Rightarrow C_{n+1}(\omega) = 2\omega C_n(\omega) - C_{n-1}(\omega)$$

# Chebyschev Filter

The following are easily established:

$$
\begin{aligned}
&C_0(\omega) = 1, \qquad C_1(\omega) = \omega \\
&C_2(\omega) = 2\omega C_1(\omega) - C_0(\omega) = 2\omega^2 - 1 \\
&C_3(\omega) = 2\omega C_2(\omega) - C_1(\omega) = 4\omega^3 - 3\omega \\
&C_4(\omega) = 2\omega C_3(\omega) - C_2(\omega) = 8\omega^4 - 8\omega^2 + 1 \\
&C_5(\omega) = 2\omega C_4(\omega) - C_3(\omega) = 16\omega^5 - 20\omega^3 + 5\omega \\
&C_6(\omega) = 2\omega C_5(\omega) - C_4(\omega) = 32\omega^6 - 48\omega^4 + 18\omega^2 - 1 \\
&C_7(\omega) = 2\omega C_6(\omega) - C_5(\omega) = 64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega
\end{aligned}
$$



# Chebyschev Filter

- The Chebyshev amplitude response has ripoles in the passband and is smooth (monotonic) in the stop-band. The pass-band is $0 \leq \omega \leq 1$, and there is a total of $n$ maxima and minima over the pass-band $0 \leq \omega \leq 1$
- We observe that

$$C_n^2(0) = \begin{cases} 0, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$$

  Therefore, the dc gain is

$$|\mathcal{H}(0)| = \begin{cases} 1, & n \text{ odd} \\ \frac{1}{\sqrt{1+\epsilon^2}}, & n \text{ even} \end{cases}$$

- The parameter $\epsilon$ controls the height of ripples. In the pass-band, $r$, the ratio of the maximum gain to the minimum gain is $r = \sqrt{1+\epsilon^2}$. The ratio $r$ is

$$\hat{r} = 20 \log \sqrt{1+\epsilon^2} = 10 \log(1+\epsilon^2) \Longrightarrow \epsilon^2 = 10^{\hat{r}/10} - 1$$

# Chebyschev Filter

- The ripple is present only over the pass-band $0 \leq \omega \leq 1$. At $\omega = 1$, the amplitude response is $1/\sqrt{1+\epsilon^2} = 1/r$. For $\omega > 1$, the gain decreases monotonically. Because all the ripples in the pass-band are of equal height, the Chebyshev polynomials $C_n(\omega)$ are known as **equal-ripple functions**.
- If we reduce the ripple, the pass-band behavior improves, but it does so at the cost of stope-band behavior. As $\epsilon$ is reduced, the gain in the stop-band increases, and vice-versa.
- The transfer function $H(s)$ corresponding to the function $|H(j\omega)|$ can be obtained by noting that

$$|H(j\omega)|^2 = H(j\omega)H(-j\omega)$$

It follows that the poles of $H(s)$ can be obtained from the roots in the left half of the of the complex plant of the equation

$$1 + \epsilon^2 C_n^2(\omega) = 0$$

# Reference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