# Lecture 6: Fourier Transform III

**Dr.-Ing. Sudchai Boonto**
**Assistant Professor**

Department of Control System and Instrumentation Engineering
King Mongkut's Unniversity of Technology Thonburi
Thailand

# Signal Energy

The signal energy $E_f$ of a signal $f(t)$ is defined by

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

Signal energy can be related to the signal spectrum $F(\omega)$ by

$$E_f = \int_{-\infty}^{\infty} f(t) f^*(t) dt = \int_{-\infty}^{\infty} f(t) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{-j\omega t} d\omega \right] dt,$$

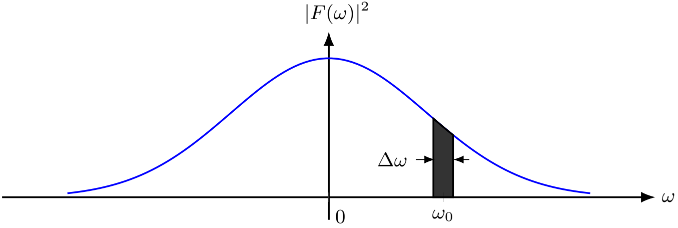
where $f^*(t)$ is the conjugate of $f(t)$. Interchanging the order of integration yields

$$\begin{aligned} E_f &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \left[ \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) F^*(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \end{aligned}$$

The **Parseval's theorem**

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

# Signal Energy




- The energy of a signal $f(t)$ results from energies contributed by all the spectral components of the signal $f(t)$. The total signal energy is the area under $|F(\omega)|^2$ (divided by $2\pi$).
- If we consider only a small band $\Delta\omega$ ($\Delta\omega \to 0$), the energy $\Delta E_f$ of the spectral components in this ban is the area of $|F(\omega)|^2$ under this band (divided by $2\pi$):

$$\Delta E_f = \frac{1}{2\pi}|F(\omega)|^2\Delta\omega = |F(\omega)|^2\Delta\mathcal{F}, \qquad \frac{\Delta\omega}{2\pi} = \Delta\mathcal{F} \text{ Hz}$$

# Signal Energy

- Therefore, the energy contributed by the components in this band of $\Delta\mathcal{F}$ (in hertz) is $|F(\omega)|^2\Delta\mathcal{F}$.
- The total signal energy is the sum of energies of all such bands and is indicated by the area under $|F(\omega)|^2$ is the **energy spectral density** (per unit bandwidth in hertz).
- For real signals, $F(\omega)$ and $F(-\omega)$ are conjugates, and $|F(\omega)|^2$ is an even function of $\omega$ because

$$|F(\omega)|^2 = F(\omega)F^*(\omega) = F(\omega)F(-\omega)$$

- The energy can be expressed as

$$E_f = \frac{1}{\pi}\int_0^\infty |F(\omega)|^2 d\omega$$

- It follows that the energy contributed by spectral components of frequencies between $\omega_1$ and $\omega_2$ is

$$\Delta E_f = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega$$

# Signal Energy

## Example I

Find the energy of signal $f(t) = e^{-at}u(t)$. Determine the frequency $W$ (rad/s) so that the energ contributed by the spectral components of all the frequencies below $W$ is $95\%$ of the signal energy $E_f$.

**Solution:** We have

$$E_f = \int_{-\infty}^{\infty} f^2(t)dt = \int_0^{\infty} e^{-2at}dt = \frac{1}{2a}$$

For this signal

$$F(\omega) = \frac{1}{j\omega + a}$$

and

$$E_f = \frac{1}{\pi}\int_0^{\infty} |F(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^{\infty} \frac{1}{\omega^2 + a^2} d\omega = \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\Big|_0^{\infty} = \frac{1}{2a}$$

The band $\omega = 0$ to $\omega = W$ contains $95\%$ of the signal energy, that is, $0.95/2a$.

# Signal Energy

## Example I

Therefore, with $\omega_1 = 0$ and $\omega_2 = W$, we obtain

$$\frac{0.95}{2a} = \frac{1}{\pi}\int_0^W \frac{d\omega}{\omega^2 + a^2} = \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\bigg|_0^W = \frac{1}{\pi a}\tan^{-1}\frac{W}{a}$$

$$\frac{0.95\pi}{2} = \tan^{-1}\frac{W}{a} \Rightarrow W = 12.706a \text{ rad/s}$$

This result indicates that the spectral components of $f(t)$ in the band from $0$ (dc) to $12.706a$ rad/s ($2.02a$ Hz) contribute $95\%$ of the total signal energy; all the remaining spectral components (in the band from $12.706a$ rad/s to $\infty$ ) contribute only $5\%$ of the signal energy.

# Application of Fourier Transform

## System Analysis

Suppose a system is represented by a second-order differential equation with constant coefficients:

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t)$$

and that the initial conditions are zero. Let $x(t) = \delta(t)$. Find $y(t)$.
**Solution:** Computing the Fourier transform of the system, we get

$$\left[(j\omega)^2 + 3j\omega + 2\right] Y(\omega) = X(\omega)$$

Then

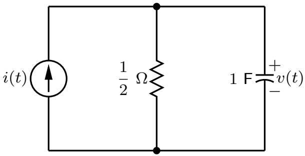
$$\begin{aligned} H(\omega) &= \frac{1}{(j\omega)^2 + 3j\omega + 2} = \frac{1}{(j\omega+1)(j\omega+2)} \\ &= \frac{1}{j\omega+1} - \frac{1}{j\omega+2} \end{aligned}$$

Then $Y(\omega) = H(\omega)X(\omega) = H(\omega)$ and the inverse Fourier transform of $Y(\omega)$ is

$$y(t) = \left[e^{-t} - e^{-2t}\right] u(t)$$

# Application of Fourier Transform
## System Analysis

Given the linear circuit shown below, use Fourier transforms to find the time-domain steady-state response $v(t)$ to the input function $i(t) = 10\sin(9t + 30^\circ)$ .




**Solution:** The transfer function is

$$\frac{V(\omega)}{I(\omega)} = H(\omega) = \frac{0.5(1/j\omega)}{0.5 + 1/j\omega} = \frac{1}{2 + j\omega}$$

Since $i(t) = 10\sin(9t + 30^\circ) = 10\sin(9t + \frac{\pi}{6}) = 10\sin(9(t + \frac{\pi}{54}))$, we have

$$\mathcal{F}\{i(t)\} = I(\omega) = j10\pi[\delta(\omega + 9) - \delta(\omega - 9)]e^{j\pi/54}$$

$$V(\omega) = H(\omega)I(\omega) = \frac{j10\pi}{2 + j\omega}[\delta(\omega + 9) - \delta(\omega - 9)]e^{j\omega\pi/54}$$

Then

$$\begin{aligned}
v(t) &= \mathcal{F}^{-1}\{V(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)e^{j\omega t}d\omega \\
&= j5\left[\int_{-\infty}^{\infty}\delta(\omega+9)\frac{e^{j\omega\pi/54}}{2+j\omega}e^{j\omega t}d\omega - \int_{-\infty}^{\infty}\delta(\omega-9)\frac{e^{j\omega\pi/54}}{2+j\omega}e^{j\omega t}d\omega\right] \\
&= j5\left[\frac{e^{-j\pi/6}}{2-j9}e^{-j9t} - \frac{e^{j\pi/6}}{2+j9}e^{j9t}\right] = 5\left[H(-9)e^{-j(9t+\pi/6)} - H(9)e^{j(9t+\pi/6)}\right] \\
&= j5\left[|H(9)|\left(e^{-j(9t+\pi/6-\angle H(9))} - e^{j(9t+\pi/6-\angle H(9))}\right)\right] \\
&= 10|H(9)|\sin(9t+\frac{\pi}{6}-\angle H(9)) = 1.08\sin(9t+30^\circ-77.5^\circ) \\
&= 1.08\sin(9t-47.5^\circ) = 1.08\cos(9t-137.5^\circ)
\end{aligned}$$

Using phasors is easily with $I = 10\angle 30^\circ - 90^\circ = 10\angle -60^\circ$ and $Z = \frac{1}{2+j9} = 0.108\angle -77.5^\circ$ so that

$$V = IZ = 1.08\angle -137.5^\circ \text{ or } v(t) = 1.08\cos(9t-137.5^\circ)$$

# Application to Communication Systems

The application of the Fourier transform in communications is clear. The representation of signals in the frequency domain and the concept of modulation are basic in communications. The basic applications are:

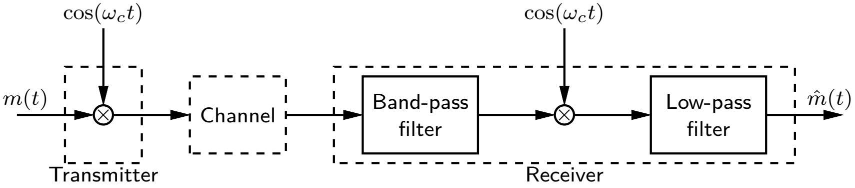
- Amplitude (linear) modulation
- Angle (nonlinear) modulation

Given the low-pass nature of most message signals, it is necessary to shift in frequency the spectrum of the message to avoid using a very large antenna. This can be attained by means of modulation, which is done by changing either the magnitude or the phase of a carrier:

$$A(t)\cos(\omega_c t + \theta_c(t))$$

- When $A(t)$ is proportional to the message, for constant phase, we have **amplitude modulation (AM)**.
- If we let $\theta(t)$ change with the message, keeping the amplitude constant, we then have **frequency modulation (FM)** or **phase modulation (PM)** which are called **angle modulations**.

# AM with Suppressed Carrier




In amplitude modulation, we have

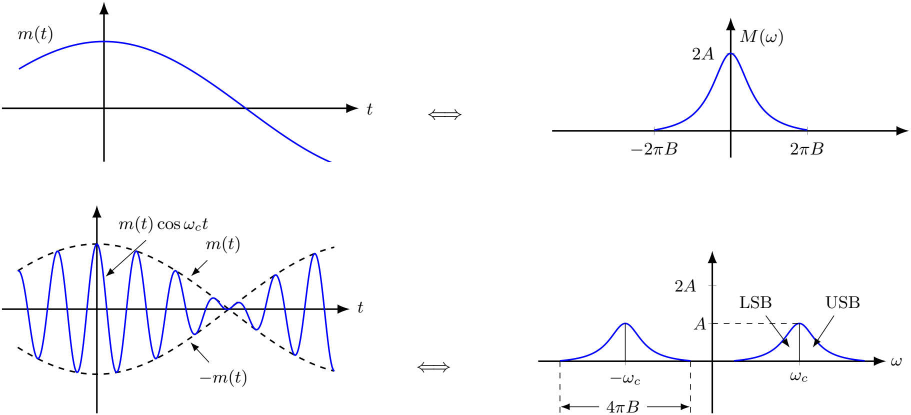
$$s(t) = m(t)\cos(\omega_c t),$$

where $m(t)$ is a message signal (e.g., voice or music or both). The $\cos(\omega_c t)$ is a carrier signal with frequency $\omega_c \gg \omega_0$, where $\omega_0 = 2\pi f_0$ is the maximum frequency in the message (for music $f_0$ is about 22 kHz). the signal $s(t)$ is called **amplitude modulated with suppressed carrier (AM-SC)**. From Fourier transform, we have

$$m(t) \Longleftrightarrow M(\omega)$$

$$m(t)\cos(\omega_c t) \Longleftrightarrow \frac{1}{2}\left[M(\omega+\omega_c) + M(\omega-\omega_c)\right]$$

# AM with Suppressed Carrier



# AM with Suppressed Carrier

- The process of modulation shifts the spectrum of the modulating signal to the left and the right by $\omega_c$.
- If the bandwidth of $m(t)$ is $B$ Hz, then the bandwidth of the modulated signal is $2B$ Hz.
- The modulated signal spectrum centered at $\omega_c$ is composed of two parts: a portion tht lies above $\omega_c$, known as the **upper sideband (USB)**, and a portion that lies below $\omega_c$, known as the **lower sideband (LSB)**. This is also the same as the spectrum centered at $-\omega_c$.
- This form of modulation is called **double sideband (DSB)** modulation.

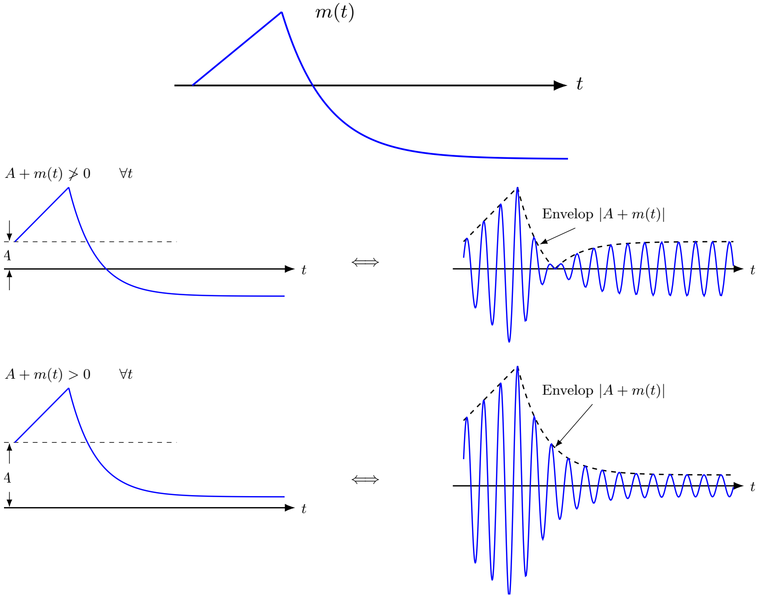
# Amplitude Modulation (AM)

- For the suppressed carrier scheme, a receiver must generate a carrier in frequency and phase synchronism with the carrier at the transmitter that may be located hundreds or thousands of miles away. This method could be costly.
- The other alternative is for the transmitter to transmit a carrier $A \cos \omega_c t$ (along with the modulated signal $m(t) \cos \omega_c t$] so that there is no need to generate a carrier at the receiver. In this case the transmitter needs to transmit much larger power, a rather expensive procedure.
- The second option (transmitting a carrier along with the modulated signal) is the obvious choice. This is the so-called **AM (amplitude modulation)**, in which the transmitted signal $\varphi_{\text{AM}}(t)$ is given by

$$
\begin{aligned}
\varphi_{\text{AM}}(t) &= \underbrace{A \cos \omega_c t}_{\text{carrier}} + \underbrace{m(t) \cos \omega_c t}_{\text{modulation message}} \\
&= [A + m(t)] \cos \omega_c t, \ \text{where } A + m(t) \geq 0
\end{aligned}
$$

- The AM signal is identical to the DSB-SC signal with $A + m(t)$ as the modulating signal.

# Amplitude Modulation (AM)



# Amplitude Modulation (AM)

- If $A$ is large enough so that $A + m(t) \geq 0$ (in nonnegative) for all values of $t$. The envelope has the same shape as $m(t)$ (although riding on a dc of magnitude $A$)
- If $A$ is not large enough so that $A + m(t) \ngtr 0$ for all values of $t$. The envelope shape is not $m(t)$ , for some parts get rectified.
- If $A$ is large enough, we can detect the desired signal $m(t)$ by detecting the envelope. We shall see that the envelope detection is an extremely simple and inexpensive operation, which does not require generation of a local carrier for the demodulation. However the envelope of AM has the information about $m(t)$ only if the AM signal $[A + m(t)]\cos\omega_c t$ satisfies the condition $A + m(t) > 0$ for all $t$.
- If $m_p$ is the peak amplitude (positive or negative) of $m(t)$, the $m(t) \geq -m_p$. Hence, the condition is equivalent to

$$A \geq m_p$$

- We define the modulation index $\mu$ as

$$\mu = \frac{m_p}{A}, \qquad 0 \leq \mu \leq 1$$

Sketch $\varphi_{\text{AM}}(t)$ for modulation indices of $\mu = 0.5$ $(50\%$ modulation) and $\mu = 1$ $(100\%$ modulation), when $m(t) = B\cos\omega_m t$.

**Solution:** In this case, $m_p = B$ and the modulation index is

$$\mu = \frac{B}{A}$$

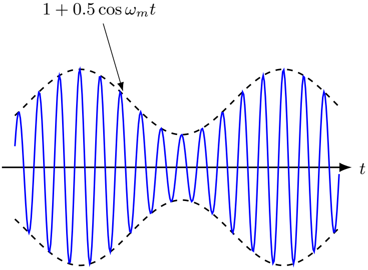
Hence, $B = \mu A$ and

$$m(t) = B\cos\omega_m t = \mu A\cos\omega_m t$$

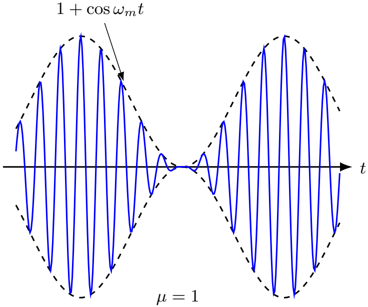
Therefore

$$\varphi_{\text{AM}} = [A + m(t)]\cos\omega_c t = A\,[1 + \mu\cos\omega_m t]\cos\omega_c t$$

# Amplitude Modulation (AM)



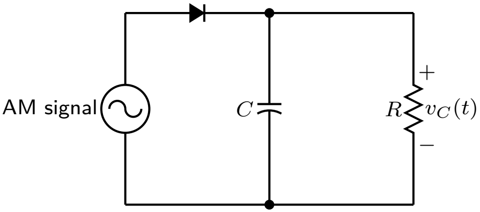

$\mu = 0.5$




$\mu = 1$

# Amplitude Modulation (AM)

## Demodulation of AM: The Envelope Detector




- In practice, we use the noncoherent methods of AM demodulation, the **envelope detection.**
- The Figure shows the simple circuit to do this.

# Reference


1. Naresh, K. Sinha, *Linear Systems*, John Wiley & Sons, Inc., 1991.
2. Lathi, B. P., *Signal Processing & Linear Systems*, Berkeley-Cambridge Press, 1998.
3. Watcharapong Khovidhungij, *Signals, Systems, and Control*, Chulalongkorn University Press, 2016