

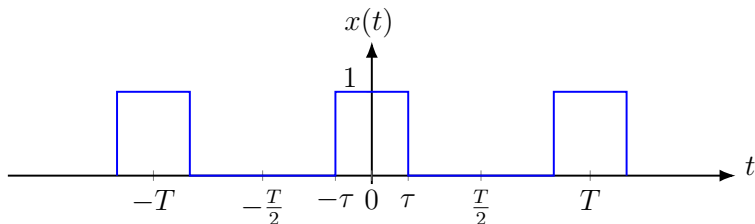
Lecture 6: Fourier Transform I

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Introduction



Find the exponential Fourier coefficients of a periodic rectangular pulse train with period T_0 . During the period $-\frac{T_0}{2} < t < \frac{T_0}{2}$ the pulse is given by

$$x(t) = \begin{cases} 1, & |t| < \tau \\ 0, & \tau < |t| < \frac{T_0}{2} \end{cases}$$

From

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

Introduction

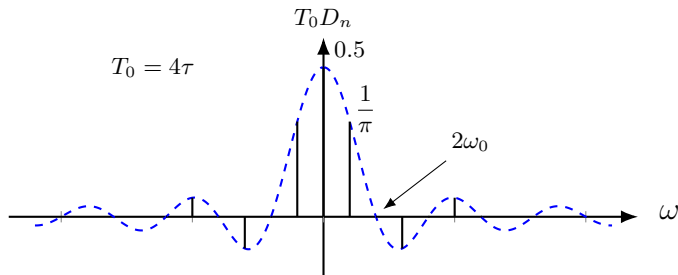
Using an even function property of $x(t)$, we have $b_n = 0$

$$\begin{aligned}a_0 &= \frac{1}{T_0} \int_{-\tau}^{\tau} dt = \frac{1}{T_0} 2\tau = \frac{2\tau}{T_0} \\D_n &= \frac{1}{T_0} \int_{-\tau}^{\tau} e^{-jn\omega_0 t} dt = \frac{-1}{jn\omega_0 T_0} e^{-jn\omega_0 t} \Big|_{-\tau}^{\tau} \\&= \frac{-1}{jn\omega_0 T_0} \left[\frac{e^{-jn\omega_0 \tau} - e^{jn\omega_0 \tau}}{2j} \right] = \frac{2}{n\omega_0 T_0} \sin n\omega_0 \tau \\&= \frac{\sin 2n\pi \frac{\tau}{T_0}}{n\pi}, \quad n \neq 0 \text{ and } \omega_0 T_0 = 2\pi\end{aligned}$$

If we fix τ and vary T_0 . For example, $T_0 = 4\tau$

$$\begin{aligned}D_n &= \frac{2 \sin n\omega_0 \tau}{n\omega_0 T_0} = \frac{\sin n\pi/2}{n\pi} \\D_1 &= D_{-1} = \frac{1}{\pi}, \quad D_2 = D_{-2} = 0, \quad D_3 = D_{-3} = \frac{-1}{3\pi}, \quad D_4 = D_{-4} = 0 \\D_5 &= D_{-5} = \frac{1}{5\pi}, \quad D_6 = D_{-6} = 0, \quad D_7 = D_{-7} = \frac{-1}{7\pi}, \quad D_8 = D_{-8} = 0\end{aligned}$$

Introduction

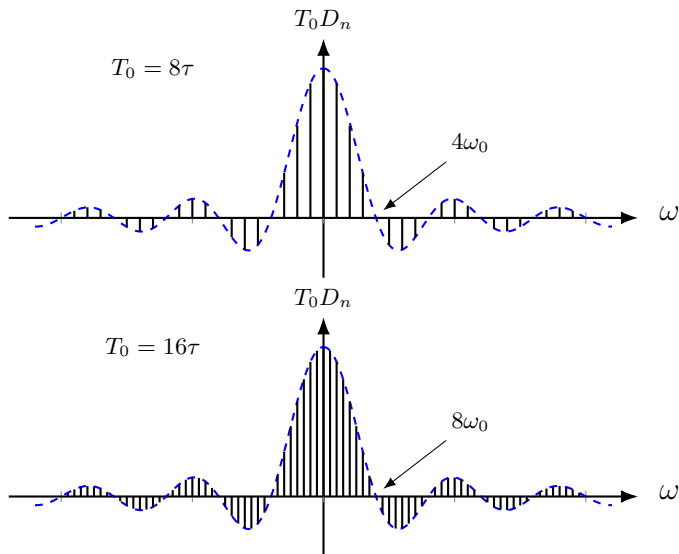


Consider that

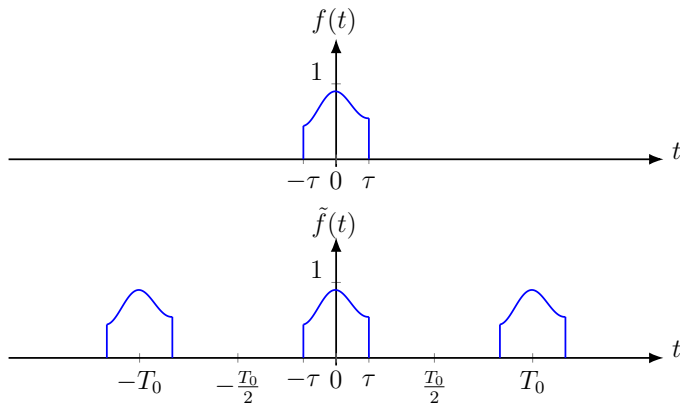
$$D_n = \frac{2 \sin n\omega_0\tau}{n\omega_0 T_0}$$
$$T_0 D_n = \frac{2 \sin n\omega_0\tau}{n\omega_0} = \left. \frac{2 \sin \omega\tau}{\omega} \right|_{\omega=n\omega_0}$$

We can think of that ω is a continuous variable. The function $\frac{2 \sin \omega\tau}{\omega}$ represents the envelope of $T_0 D_n$ and these coefficients are simply equally spaced samples of this envelope.

Introduction



Aperiodic Signal Representation by Fourier Integral



- We can construct a periodic signal $\tilde{f}(t)$ formed by repeating the signal $f(t)$ at intervals of T_0 second.
- The period T_0 is made long enough to avoid overlap between the repeating pulses.

Aperiodic Signal Representation by Fourier Integral

- The periodic signal $\tilde{f}(t)$ can be represented by an exponential Fourier series.
- If we let $T_0 \rightarrow \infty$, the pulses in the periodic signal repeat after an infinite interval and, therefore

$$\lim_{T_0 \rightarrow \infty} \tilde{f}(t) = f(t)$$

- Thus, the Fourier series representing $\tilde{f}(t)$ will also represent $f(t)$ in the limit $T_0 \rightarrow \infty$.
- The exponential Fourier series for $\tilde{f}(t)$ is given by

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$
$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{f}(t) e^{-jn\omega_0 t} dt, \quad \omega_0 = \frac{2\pi}{T_0}$$

Aperiodic Signal Representation by Fourier Integral

- Since $\tilde{f}(t) = f(t)$ for $|t| < T_0/2$ and also $f(t) = 0$ outside this interval, then the above equation can be rewritten as

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_{-\infty}^{\infty} f(t) e^{-jn\omega_0 t} dt$$

- The envelope of $T_0 D_n$, the function of $\omega = n\omega_0$ can be defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt, \quad D_n = \frac{1}{T_0} F(n\omega_0)$$

- Since

$$\begin{aligned} \tilde{f}(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T_0} F(n\omega_0) e^{jn\omega_0 t} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n\omega_0) e^{jn\omega_0 t} \omega_0, \quad \omega_0 = \frac{2\pi}{T_0} \end{aligned}$$

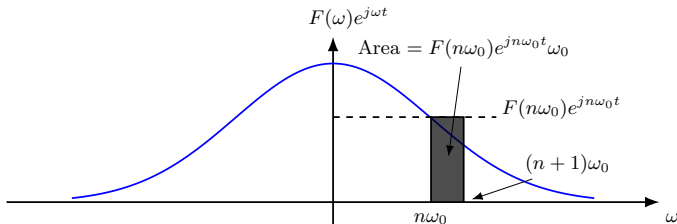
Aperiodic Signal Representation by Fourier Integral

- By replacing ω_0 with $\Delta\omega_0$ and since $T_0 \rightarrow \infty$, $\Delta\omega \rightarrow 0$ and $\tilde{f}(t) \rightarrow f(t)$. Therefore

$$f(t) = \lim_{T_0 \rightarrow \infty} \tilde{f}(t) = \lim_{\Delta\omega \rightarrow 0} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n\Delta\omega) e^{jn\Delta\omega t} \Delta\omega$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

- Each term in the summation on the right hand side of the above equation is the area of a rectangle of height $F(n\omega_0) e^{jn\omega_0 t}$ and width ω_0



Aperiodic Signal Representation by Fourier Integral

- We can $F(\omega)$ the **direct** Fourier transform of $f(t)$, and $f(t)$ the **inverse** Fourier transform of $F(\omega)$. We call $f(t)$ and $F(\omega)$ a Fourier transform pair. Symbolically, this statement is expressed as

$$F(\omega) = \mathcal{F}[f(t)] \quad \text{and} \quad f(t) = \mathcal{F}^{-1}[F(\omega)]$$

or

$$f(t) \Leftrightarrow F(\omega)$$

and

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega$$

Aperiodic Signal Representation by Fourier Integral

Example I

Find the Fourier transform of $e^{-at}u(t)$.

Solution: By definition,

$$F(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-(a+j\omega)t}dt = \left. \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \right|_0^{\infty}$$

But $|e^{-j\omega t}| = 1$. Therefore, as $t \rightarrow \infty$, $e^{-(a+j\omega)t} = e^{-at}e^{-j\omega t} = 0$ if $a > 0$. Therefore

$$F(\omega) = \frac{1}{a+j\omega}, \quad a > 0$$

Expressing $a+j\omega$ in the polar form as $\sqrt{a^2+\omega^2}e^{j \tan^{-1}(\frac{\omega}{a})}$, we obtain

$$F(\omega) = \frac{1}{\sqrt{a^2+\omega^2}} e^{-j \tan^{-1}(\frac{\omega}{a})}$$

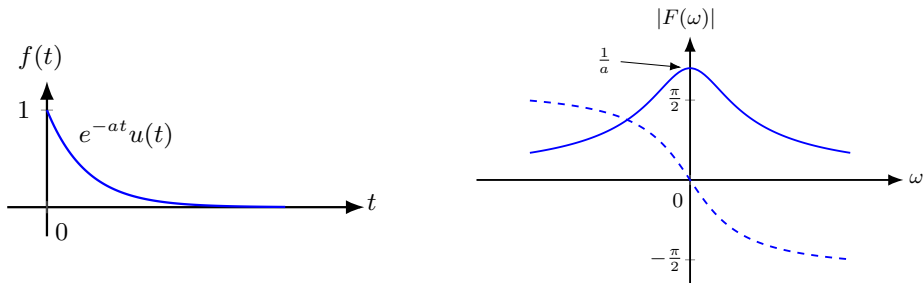
Therefore

$$|F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \quad \text{and} \quad \angle F(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Aperiodic Signal Representation by Fourier Integral

Example I cont.

The amplitude spectrum $|F(\omega)|$ and the phase spectrum $\angle F(\omega)$ are depicted as follow:



- $|F(\omega)|$ is an even function of ω
- $\angle F(\omega)$ is an odd function of ω

Existence of the Fourier Transform

- The existence of the Fourier transform is assured for any $f(t)$ satisfying the Dirichlet condition. The condition is

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

Because $|e^{-j\omega t}| = 1$ we obtain

$$|F(\omega)| \leq \int_{-\infty}^{\infty} |f(t)| dt$$

- However this is just a sufficient condition.
- The Fourier Transform is linear since

$$f_1(t) \Longleftrightarrow F_1(\omega) \quad \text{and} \quad f_2(t) \Longleftrightarrow F_2(\omega)$$

then

$$a_1 f_1(t) + a_2 f_2(t) \Longleftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$$

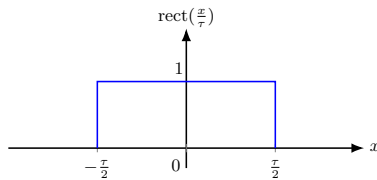
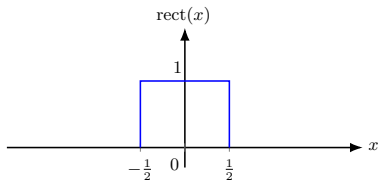
Some Useful Functions

Unit Gate Function

Unit Gate Function:

We define a unit gate function $\text{rect}(x)$ as a gate pulse of unit height and unit width, centered at the origin,

$$\text{rect}(x) = \begin{cases} 0, & |x| > \frac{1}{2} \\ \frac{1}{2}, & |x| = \frac{1}{2} \\ 1, & |x| < \frac{1}{2} \end{cases}$$



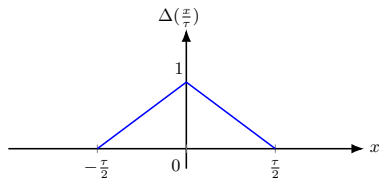
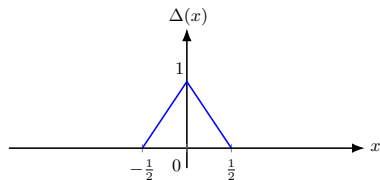
Some Useful Functions

Unit Triangle Function

Unit Triangle Function:

We define a unit triangle function $\Delta(x)$ as a triangular pulse of unit height and unit width, centered at the origin,

$$\Delta(x) = \begin{cases} 0, & |x| > \frac{1}{2} \\ 1 - 2|x|, & |x| < \frac{1}{2} \end{cases}$$



Some Useful Functions

Interpolation Function $\text{sinc}(x)$

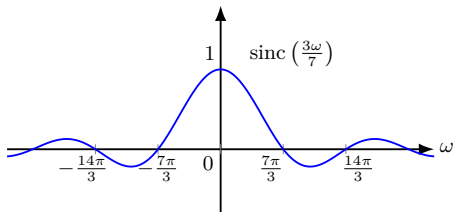
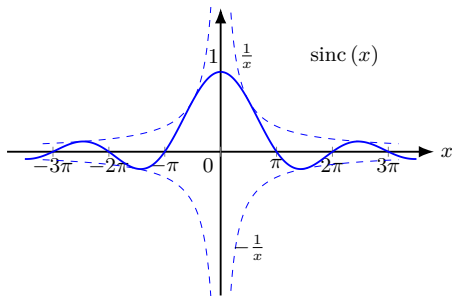
The function $\sin x/x$ is the “sine over argument” function denoted by $\text{sinc}(x)$. This function is very important in signal processing. It is also known as the **filtering or interpolation function**. We define

$$\text{sinc}(x) = \frac{\sin x}{x}$$

- $\text{sinc}(x)$ is an even function of x
- $\text{sinc}(x) = 0$ when $\sin x = 0$ except at $x = 0$, where it appears indeterminate. This means that $\text{sinc}(x) = 0$ for $x = \pm\pi, \pm2\pi, \pm3\pi, \dots$
- Using L'Hôpital's rule, we find $\text{sinc } 0 = 1$.
- $\text{sinc}(x)$ is the product of an oscillating signal $\sin x$ (of period 2π) and a monotonically decreasing function $1/x$
- Consider $\text{sinc}(3\omega/7)$. Argument $\frac{3\omega}{7} = \pi$ when $\omega = 7\pi/3$. Therefore, the first zero of this function occurs at $\omega = 7\pi/3$.

Some Useful Functions

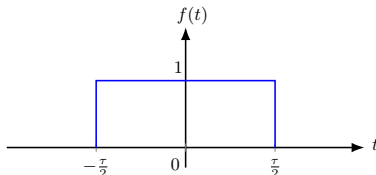
Interpolation Function $\text{sinc}(x)$



Some Useful Functions

Example II

Find the Fourier transform of $f(t) = \text{rect } \frac{t}{\tau}$



Solution:

$$F(\omega) = \int_{-\infty}^{\infty} \text{rect } \frac{t}{\tau} e^{-j\omega t} dt$$

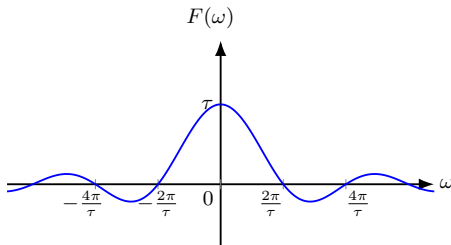
Since $\text{rect } \frac{t}{\tau} = 1$ for $|t| < \frac{\tau}{2}$ and since it is zero for $|t| > \frac{\tau}{2}$,

$$F(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

Some Useful Functions

Example II

$$\begin{aligned} F(\omega) &= \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt \\ &= -\frac{1}{j\omega} \left(e^{-j\omega\tau/2} - e^{j\omega\tau/2} \right) = \frac{2 \sin\left(\frac{\omega\tau}{2}\right)}{\omega} \\ &= \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right) \end{aligned}$$



Fourier Transform

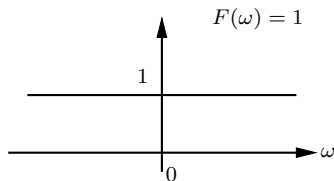
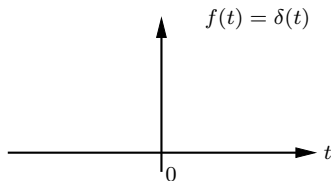
unit impulse

Find the Fourier transform of the unit impulse $\delta(t)$.

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

or

$$\delta(t) \Longleftrightarrow 1$$



Fourier Transform

unit impulse

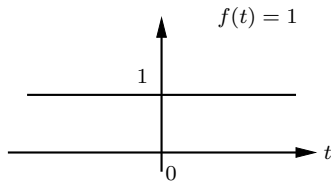
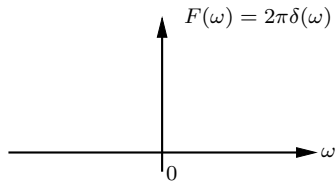
Fine the inverse Fourier transform of $\delta(\omega)$

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}$$

Therefore

$$\frac{1}{2\pi} \Longleftrightarrow \delta(\omega)$$

$$1 \Longleftrightarrow 2\pi\delta(\omega)$$



Fourier Transform

unit impulse

Find the inverse Fourier transform of $\delta(\omega - \omega_0)$.

Solution:

Using the sampling property of the impulse function, we obtain

$$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

Therefore

$$\begin{aligned} \frac{1}{2\pi} e^{j\omega_0 t} &\Longleftrightarrow \delta(\omega - \omega_0) \\ e^{j\omega_0 t} &\Longleftrightarrow 2\pi \delta(\omega - \omega_0) \end{aligned}$$

- To represent the everlasting exponential $e^{j\omega_0 t}$, we need a single everlasting exponential $e^{j\omega t}$ with $\omega = \omega_0$.
- We also have

$$e^{-j\omega_0 t} \Longleftrightarrow 2\pi \delta(\omega + \omega_0)$$

Fourier Transform

Sinusoidal

Find the Fourier transforms of the everlasting sinusoid $\cos \omega_0 t$

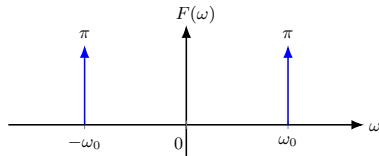
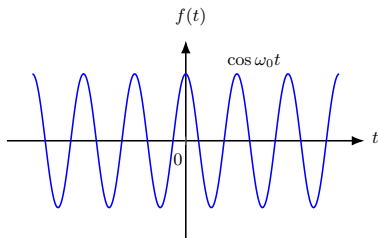
Solution:

Recall the Euler formula

$$\cos \omega_0 t = \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t})$$

Using the previous result, we have

$$\cos \omega_0 t \iff \pi [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$



Fourier Transform

Sinusoidal

Find the Fourier transforms of the everlasting sinusoid $\cos \omega_0 t$

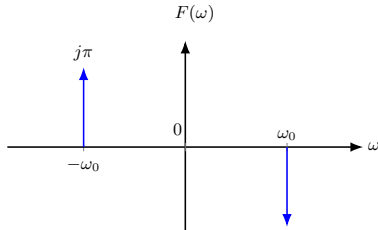
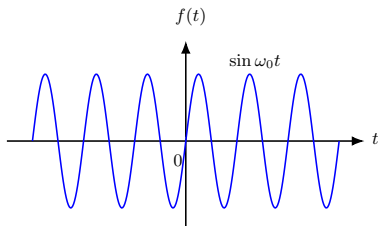
Solution:

Recall the Euler formula

$$\sin \omega_0 t = \frac{1}{2j} (e^{j\omega_0 t} - e^{-j\omega_0 t})$$

Using the previous result, we have

$$\sin \omega_0 t \Longleftrightarrow j\pi [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$



Fourier Transform

Unit step

Find the Fourier transform of the unit step function $u(t)$.

Solution: Direct integration

$$U(\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t} dt = \int_0^{\infty} e^{-j\omega t} dt = \left. \frac{-1}{j\omega} e^{-j\omega t} \right|_0^{\infty}$$

The upper limit of $e^{-j\omega t}$ as $t \rightarrow \infty$ yields an indeterminate answer. ($e^{-j\omega\infty}$) We can consider $u(t)$ as a decaying exponential $e^{-at}u(t)$ in the limit as $a \rightarrow 0$. Thus

$$u(t) = \lim_{a \rightarrow 0} e^{-at}u(t)$$

and

$$U(\omega) = \lim_{a \rightarrow 0} \mathcal{F}\{e^{-at}u(t)\} = \lim_{a \rightarrow 0} \frac{1}{a + j\omega}$$

Fourier Transform

Unit step

Separate real and imaginary part yields

$$\begin{aligned}U(\omega) &= \lim_{a \rightarrow 0} \left[\frac{a}{a^2 + \omega^2} - j \frac{\omega}{a^2 + \omega^2} \right] \\&= \lim_{a \rightarrow 0} \left[\frac{a}{a^2 + \omega^2} \right] + \frac{1}{j\omega}\end{aligned}$$

The function $a/(a^2 + \omega^2)$ has interesting properties. First, the area under this function is π regardless of the value of a

$$\int_{-\infty}^{\infty} \frac{a}{a^2 + \omega^2} d\omega = \tan^{-1} \frac{\omega}{a} \Big|_{-\infty}^{\infty} = \pi$$

Second, when $a \rightarrow 0$, this function approaches zero for all $\omega \neq 0$, and all its area (π) is concentrated at a single point $\omega = 0$. Clearly, as $a \rightarrow 0$, this function approaches an impulse of strength π . Thus

$$U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

1. Naresh, K. Sinha, *Linear Systems*, John Wiley & Sons, Inc., 1991.
2. Lathi, B. P., *Signal Processing & Linear Systems*, Berkeley-Cambridge Press, 1998.
3. Watcharapong Khovidhungij, *Signals, Systems, and Control*, Chulalongkorn University Press, 2016