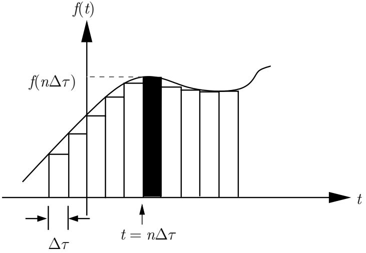
Lecture 5: Fourier Series I

Dr.-Ing. Sudchai Boonto
Assistant Professor

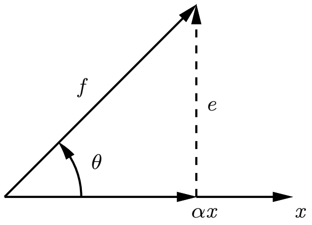
Department of Control System and Instrumentation Engineering
King Mongkut's Unniversity of Technology Thonburi
Thailand

Motivation

- An arbitrary input $f(t)$ can be expressed as a sum of its impulse components.



- There are infinite possible ways of expressing an input $f(t)$ in terms of other signals.
- This chapter addresses the Fourier series method.



x and f are vectors with magnitudes $|x|$ and $|f|$, respectively. The dot (inner or scalar) product of these two vector is

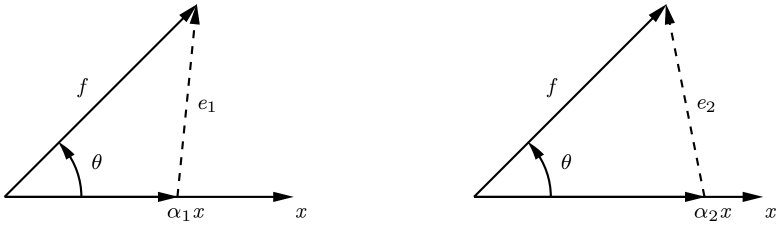
$$f \cdot x = |f||x| \cos\theta,$$

where θ is the angle between these vectors.

- The vector f can be expressed in terms of vector x as

$$f = cx + e$$

- this is not the only way to express f in terms of x .



- The figure show two of the infinite other possibilities.

$$f = \alpha_1 x + e_1 = \alpha_2 x + e_2$$

- Three representations f is represented in terms of x plus another vector called the **error vector**
- If $f \simeq \alpha x$, the error in the approximation is the vector $e = f - \alpha x$.
- Mathematically, we α such that e is minimum.

- The length of the component of f along x is

$$|f|\cos\theta = \alpha|x|$$

- Multiplying both sides by $|x|$ yields

$$\alpha|x|^2 = |f||x|\cos\theta = f\cdot x$$

Therefore

$$\alpha = \frac{f\cdot x}{x\cdot x} = \frac{1}{|x|^2} f\cdot x$$

- When f and x are perpendicular (orthogonal), then f has a zero component along x or $\alpha = 0$
- We define f and x to be **orthogonal** if the inner (scalar or dot) product of the two vectors is zero, if $f\cdot x = 0$.

Signals and Vectors

Component of a Signal

We could use the concept of a vector component and orthogonality with signals.

- Consider the problem of approximating a real signal $f(t)$ in terms of another real signal $x(t)$ over the interval $[t_1,\ t_2]$:

$$f(t) \simeq \alpha x(t), \qquad t_1 \leq t \leq t_2$$

- the error $e(t)$ in this approximation is

$$e(t) = \begin{cases} f(t) - \alpha x(t), & t_1 \leq t \leq t_2 \\ 0, & \text{otherwise} \end{cases}$$

- We select the signal energy as a measured tool. The best approximation, we need to minimize the error signal–that is, minimize its size, which is its energy E_e over the interval $[t_1,\ t_2]$ given by

$$E_e = \int_{t_1}^{t_2} e^2(t)\,dt = \int_{t_1}^{t_2} [f(t) - \alpha x(t)]^2\,dt$$

To minimize E_e the necessary condition is

$$\frac{dE_e}{d\alpha} = 0$$

$$\frac{d}{d\alpha}\left[\int_{t_1}^{t_2} [f(t) - \alpha x(t)]^2 \, dt\right] = 0$$

Expanding the squared term, we obtain

$$\frac{d}{d\alpha}\left[\int_{t_1}^{t_2} f^2(t)\, dt\right] - \frac{d}{d\alpha}\left[2\alpha \int_{t_1}^{t_2} f(t)x(t)\, dt\right] + \frac{d}{d\alpha}\left[\alpha^2 \int_{t_1}^{t_2} x^2(t)\, dt\right] = 0$$

$$-2\int_{t_1}^{t_2} f(t)x(t)\, dt + 2\alpha \int_{t_1}^{t_2} x^2(t)\, dt = 0$$

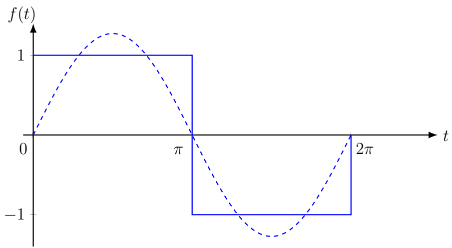
$$\alpha = \frac{\int_{t_1}^{t_2} f(t)x(t)\, dt}{\int_{t_1}^{t_2} x^2(t)\, dt} = \frac{1}{E_x}\int_{t_1}^{t_2} f(t)x(t)\, dt$$

- This is behavior is similar to the behavior of vectors.
- The area under the product of two signals corresponds to the inner product of two vectors.
- The inner product of $f(t)$ and $x(t)$ is defined by the area under the product of $f(t)$ and $x(t)$ and denoted by $\langle f, x \rangle$.
- energy of a signal is the inner product of a signal with itself, for instant $\langle x, x \rangle$.
- The E_e is minimum if the signals $f(t)$ and $x(t)$ are orthogonal over the interval $[t_1,\ t_2]$
- The real signals $f(t)$ and $x(t)$ to be orthogonal over the interval $[t_1,\ t_2]$ if

$$\int_{t_1}^{t_2} f(t)x(t)\,dt = 0$$

For the square signal $f(t)$ shown in Fig. below find the component in $f(t)$ of the form $\sin t$. In other words, approximate $f(t)$ in terms of $\sin t$.

$$f(t) \simeq \alpha \sin t, \qquad 0 \leq t \leq 2\pi$$



Find α that minimize the energy of the error signal.

$$x(t) = \sin t \text{ and } E_x = \int_0^{2\pi} \sin^2(t)\, dt = \pi$$

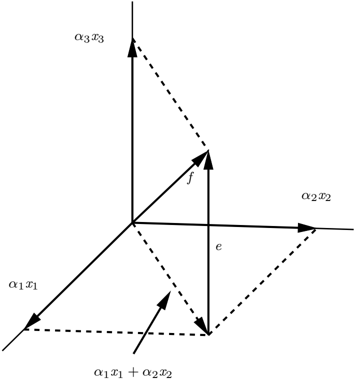
$$\alpha = \frac{1}{\pi}\int_0^{2\pi} f(t) \sin t dt$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin t \; dt + \int_{\pi}^{2\pi} -\sin t \; dt\right] = \frac{4}{\pi}$$

Thus

$$f(t) \simeq \frac{4}{\pi} \sin t$$

represents the best approximation of $f(t)$ by the function $\sin t$, which will minimize the error energy.



$$f \simeq \alpha_1 x_1 + \alpha_2 x_2$$

The error e in this approximation is

$$e = f - (\alpha_1 x_1 + \alpha_2 x_2)$$
$$f = \alpha_1 x_1 + \alpha_2 x_2 + e$$

We can observer that the error vector is orthogonal to both the vector x_1 and x_2

- if we approximate f with three mutually orthogonal vector: x_1, x_2, and x_3
- the vectors x_1, x_2, and x_3 represent a *complete set* of orthogonal vectors in three-dimensional space.
- $\alpha_i = \dfrac{f \cdot x_i}{x_i \cdot x_i} = \dfrac{1}{|x_i|^2} f \cdot x_i,\ i = 1, 2, 3$

Define orthogonality of a real signal set $x_1(t)$, $x_2(t)$, $\cdots$, $x_N(t)$ over interval $[t_1,\ t_2]$ as

$$\int_{t_1}^{t_2} x_m(t)x_n(t)\,dt = \begin{cases} 0, & m \neq n \\ E_n & m = n \end{cases}$$

If the energies $E_n = 1$ for all n, then the set is normalized and is called an **orthonormal set**. An orthogonal set can always be normalized by dividing $x_n(t)$ by $\sqrt{E_n}$ for all n.

- The approximation of the signal $f(t)$ over the interval $[t_1,\ t_2]$ is a set of N real, mutually orthogonal signals $x_1(t)$, $x_2(t)$, $\ldots$, $x_N(t)$ as

$$\begin{aligned} f(t) &\simeq \alpha_1 x_1(t) + \alpha_2 x_2(t) + \cdots + \alpha_N x_N(t) \\ &= \sum_{n=1}^{N} \alpha_n x_n(t) \end{aligned}$$

- The error $e(t)$ in the approximation is

$$e(t) = f(t) - \sum_{n=1}^{N} \alpha_n x_n(t)$$

The error signal $e(t)$ is minimized if we choose

$$\alpha_n = \frac{\displaystyle\int_{t_1}^{t_2} f(t)x_n(t)\,dt}{\displaystyle\int_{t_1}^{t_2} x_n^2(t)\,dt} = \frac{1}{E_n}\int_{t_1}^{t_2} f(t)x_n(t)\,dt \qquad n = 1, 2, \ldots, N$$

The error signal energy E_e is given by

$$E_e = \int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{N} \alpha_n^2 E_n$$

The error energy E_e generally decreases as N is increased because the term $\alpha_k^2 E_k$ is nonnegative.

$$\begin{aligned} f(t) &= \alpha_1 x_1(t) + \alpha_2 x_2(t) + \cdots + \alpha_n x_n(t) + \cdots \\ &= \sum_{n=1}^{\infty} \alpha_n x_n(t) \qquad t_1 \le t \le t_2 \end{aligned}$$

Signal representation by Orthogonal Signal Set

Orthogonal Signal Space

- The series on the right-hand side is called the **generalized Fourier series** of $f(t)$ with respect to the set $\{x_n(t)\}$
- the error energy $E_e \to 0$ as $N \to \infty$ for every member of some particular class.
- We call that the set $\{x_n(t)\}$ is complete on $[t_1,\ t_2]$ for that class of $f(t)$ and the set $\{x_n(t)\}$ is called a set of **basis functions** or **basis signals**.

Trigonometric Fourier Series

Consider a signal set:

$$\{1, \cos\omega_0 t, \cos 2\omega t, \ldots, \cos n\omega_0 t, \ldots;\ \sin\omega_0 t, \sin 2\omega_0 t, \ldots, \sin n\omega_0 t, \ldots\}$$

- A sinusoid of frequency $n\omega_0$ is called the nth **harmonic** of the sinusoid of frequency ω_0 when n is an integer.
- In this set the sinusoid of frequency ω_0, called the **fundamental**.
- This set is orthogonal over any interval of duration $T_0 = 2\pi/\omega_0$, which is the period of the fundamental.
- we can show that

$$\int_{T_0} \cos n\omega_0 t \cos m\omega_0 t dt = \begin{cases} 0, & n \neq m \\ \frac{T_0}{2}, & m = n \neq 0 \end{cases}$$

$$\int_{T_0} \sin n\omega_0 t \sin m\omega_0 t dt = \begin{cases} 0, & n \neq m \\ \frac{T_0}{2}, & n = m \neq 0 \end{cases}$$

$$\int_{T_0} \sin n\omega_0 t \cos m\omega_0 t dt = 0 \ \forall n \text{ and } m$$

Trigonometric Fourier Series

We can express a signal $f(t)$ by a trigonometric Fourier series over any interval of duration T_0 seconds as

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right), \qquad t_1 \leq t \leq t_1 + T_0,$$

where $\omega_0 = 2\pi/T_0$. Using the orthogonality of signal we can determine the Fourier coefficients a_0, a_n and b_n as

$$a_n = \frac{\displaystyle\int_{t_1}^{t_1+T_0} f(t) \cos n\omega_0 t dt}{\displaystyle\int_{t_1}^{t_1+T_0} \cos^2 n\omega_0 t dt}$$

In denominator, the integrand is $T_0/2$ when $n \neq 0$ (with $m = n$). For $n = 0$ the denominator is T_0. Hence

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) dt, \qquad a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \cos n\omega_0 t dt, \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \sin n\omega_0 t dt, \qquad n = 1, 2, 3, \ldots$$

Since in each frequency, we have

$$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = C_n \cos(n\omega_0 t + \theta_n),$$

where

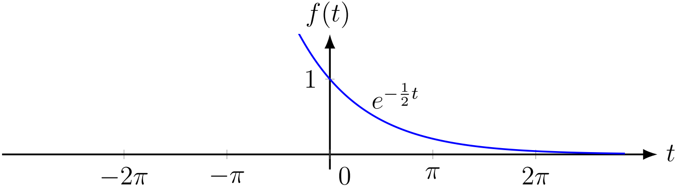
$$C_n = \sqrt{a_n^2 + b_n^2}$$
$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

We denote the dc term a_0 by C_0, that is

$$C_0 = a_0$$

The **compact form** of the trigonometric Fourier series is

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n), \qquad t_1 \leq t \leq t_1 + T_0$$



Find the compact trigonometric Fourier series for the exponential $e^{-t/2}$ over the shaded interval $0 \leq t \leq \pi$.
Since $T_0 = \pi$, then the fundamental frequency is $\omega_0 = \frac{2\pi}{T_0} = 2$. Therefore

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos 2nt + b_n \sin 2nt\right), \qquad 0 \leq t \leq \pi$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}\, dt = 0.504$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2} \cos 2nt dt = 0.504 \left(\frac{2}{1+16n^2}\right)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\sin 2nt dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

Therefore

$$f(t) = 0.504\left[1+\sum_{n=1}^{\infty}\frac{2}{1+16n^2}\left(\cos 2nt + 4n\sin 2nt\right)\right], \qquad 0 \le t \le \pi$$

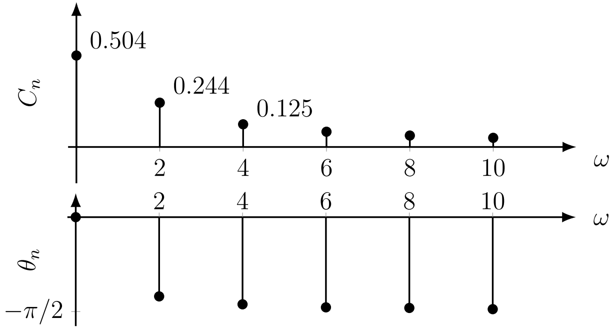
We can find the compact Fourier series as follow:

$$C_0 = a_0 = 0.504$$

$$C_n = \sqrt{a_n^2+b_n^2} = 0.504\sqrt{\frac{4}{(1+16n^2)^2}+\frac{64n^2}{(1+16n^2)^2}} = 0.504\left(\frac{2}{\sqrt{1+16n^2}}\right)$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \tan^{-1}(-4n) = -\tan^{-1} 4n$$

n	0	1	2	3	4	5	6	7
C_n	0.504	0.244	0.125	0.084	0.063	0.0504	0.042	0.036
θ_n	0	−75.96	−82.87	−85.24	−86.42	−87.14	−87.61	−87.95

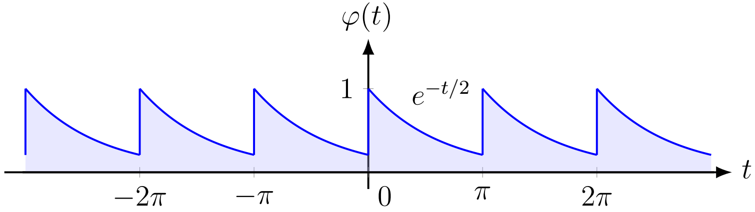


The trigonometric Fourier series is a periodic function of period T_0 (the period of the fundamental). Let us denote the trigonometric Fourier series by $\varphi(t)$. Therefore

$$\varphi(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n), \qquad \forall t$$

and

$$\begin{aligned}
\varphi(t + T_0) &= C_0 + \sum_{n=1}^{\infty} C_n \cos[n\omega_0 (t + T_0) + \theta_n] \\
&= C_0 + \sum_{n=1}^{\infty} C_n \cos[(n\omega_0 t + 2n\pi) + \theta_n] \\
&= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \\
&= \varphi(t) \qquad \forall t
\end{aligned}$$



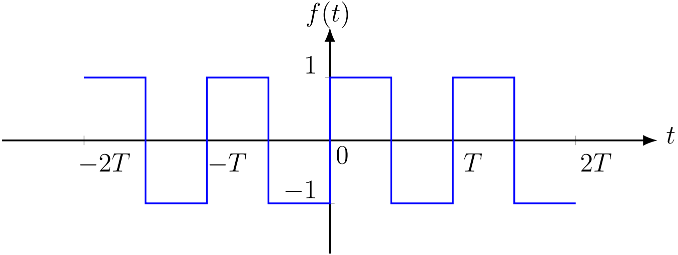
- $\varphi(t)$, the Fourier series of $f(t)$, is a periodic function in which the segment of $f(t)$ over the interval $(0 \leq t \leq \pi)$ repeats periodically every π seconds.
- The function $f(t)$ and its Fourier series $\varphi(t)$ is equal only over that interval of T_0 seconds. Outside this interval, the Fourier series repeats periodically with period T_0.

- If the function $f(t)$ were itself to be periodic with period T_0, then a Fourier series representing $f(t)$ over an interval T_0 will also represent $f(t)$ for all t.
- The periodic signal $f(t)$ can be generated by a periodic repetition of any of its segment of duration T_0.
- The trigonometric Fourier series representing a segment of $f(t)$ of duration T_0 starting at any instant represents $f(t)$ for all t.
- The Fourier coefficients of a series representing a periodic signal $f(t)$ can be expressed as

$$a_0 = \frac{1}{T_0}\int_{T_0} f(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} f(t)\cos n\omega_0 t\,dt, \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{2}{T_0}\int_{T_0} f(t)\sin n\omega_0 t\,dt, \qquad n = 1, 2, 3, \ldots$$



We have $a_0 = 0$ (the average is zero.) Also

$$a_n = \frac{2}{T}\left[\int_0^{T/2} 1 \cdot \cos n\omega_n t dt + \int_{T/2}^{T} (-1)\cos n\omega_0 t dt\right] = 0$$

$$b_n = \frac{2}{T}\left[\int_0^{T/2} 1 \cdot \sin n\omega_0 t dt + \int_{T/2}^{T} (-1)\sin n\omega_0 t dt\right] = \begin{cases} 0, & \text{even } n \\ \frac{4}{n\pi}, & \text{odd } n \end{cases}$$

$$x(t) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin(2k-1)\omega_0 t$$

- The compact trigonometric Fourier series indicates that a periodic signal $f(t)$ can be experessed as a sum of sinusoids of frequencies 0 (dc), ω_0, $2\omega_0$, $\cdots$, $n\omega_0$, $\cdots$, whose amplitudes are C_0, C_1, C_2, $\cdots$, C_n, $\cdots$, and whose phase are 0, θ_1, θ_2, $\cdots$, θ_n, $\cdots$, respectively.
- The plot amplitude C_n vs. ω is called an **amplitude spectrum**.
- The plot phase θ_n vs. ω is called a **phase spectrum**.
- Both plots are called the **frequency spectra** of $f(t)$.
- **Knowing** the frequency spectra, we can reconstruct or synthesize $\varphi(t)$ as

$$
\begin{aligned}
f(t) = 0.504 + 0.244\cos(2t - 75.96^\circ) + 0.125\cos(4t - 82.87^\circ) \\
0.084\cos(6t - 85.24^\circ) + 0.063\cos(8t - 86.42^\circ) + \cdots \qquad 0 \le t \le \pi
\end{aligned}
$$

- From the Example I, we have the **time-domain description** of $\varphi(t)$ and the **frequency-domain description (Fourier spectra)** of $\varphi(t)$.

Fourier Series

Existence of the Fourier Series: Dirichlet Conditions

There are two basic conditions for the existence of the Fourier series

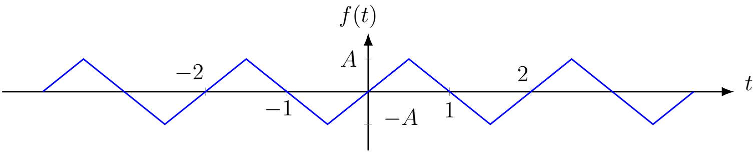
1. For the series to exist, the coefficients a_n, a_n, and b_n must be finite. It follows that the existence of these coefficients is guaranteed if $f(t)$ is absolutely integrable over one period; that is

$$\int_{T_0} |f(t)|\, dt < \infty$$

   This condidtion is known as the **weak Dirichlet condition**. If a function $f(t)$ satisfies this condition the existence of a Fourier series is guaranteed, but the series may not converge at every point.

2. The function $f(t)$ have only a finite number of maxima and minima in pone period, and only a finite number of finite discontinuities in one period. This tow conditions are known as the **strong Dirichlet conditions**. All periodic waveform that can be generated in a laboratory satisfies strong Dirichlet conditions, and hence possesses a convergent Fourier series.

Find the compact trigonometric Fourier series for the periodic square wave $f(t)$ and sketch its amplitude and phase spectra.



Solution In this case the period $T_0 = 2$. Hence

$$\omega_0 = \frac{2\pi}{2} = \pi$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi t + b_n \sin n\pi t,$$

$$f(t) = \begin{cases} 2At, & |t| \leq \frac{1}{2} \\ 2A(1-t) & \frac{1}{2} < t \leq \frac{3}{2} \end{cases}$$

Fourier Series
Example III

The average value (dc) of $f(t)$ is zero, so that $a_0 = 0$.

$$a_n = \frac{2}{2}\int_{-1/2}^{3/2} f(t)\cos n\pi t dt$$

$$= \int_{-1/2}^{1/2} 2At\cos n\pi t dt + \int_{1/2}^{3/2} 2A(1-t)\cos n\pi t dt = 0$$

$$b_n = \int_{-1/2}^{1/2} 2At\sin n\pi t dt + \int_{1/2}^{3/2} 2A(1-t)\sin n\pi t dt$$

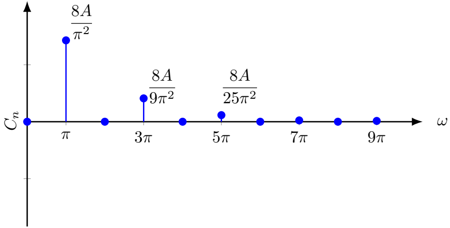
$$= \frac{8A}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \text{ even} \\ \frac{8A}{n^2\pi^2}, & n = 1, 5, 9, 13, \cdots \\ -\frac{8A}{n^2\pi^2}, & n = 3, 7, 11, 15, \cdots \end{cases}$$

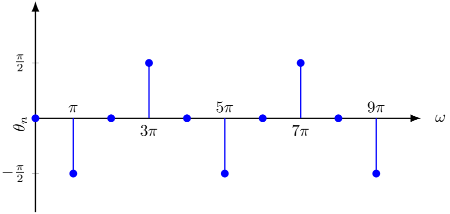
Therefore

$$f(t) = \frac{8A}{\pi^2}\left[\sin \pi t - \frac{1}{9}\sin 3\pi t + \frac{1}{25}\sin 5\pi t - \frac{1}{49}\sin 7\pi t + \cdots\right]$$

In order to plot Fourier spectra, the series must be converted into compact form as:

$$f(t) = \frac{8A}{\pi^2}\left[\cos(\pi t - 90^\circ) + \frac{1}{9}\cos(3\pi t + 90^\circ) + \frac{1}{25}\cos(5\pi t - 90^\circ) + \frac{1}{49}\cos(7\pi t + 90^\circ + \cdots\right]$$





Fourier Series

Symmetry property

For the symmetry (even or odd), the information of one period of $f(t)$ is implicit in only half the period. For this reason, the Fourier coefficients in these cases can be computed by integrating over only half the period rather than a complete period. To prove this, recall that

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\,dt, \quad a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\cos n\omega_0 t\,dt, \quad b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\sin n\omega_0 t\,dt$$

Since $\cos n\omega_0 t$ is an even function and $\sin n\omega_0 t$ is an odd function of t. If $f(t)$ is an even function of t, then $f(t)\cos n\omega_0 t$ is also an even function and $f(t)\sin n\omega_0 t$ is an odd function of t. Therefore

$$a_0 = \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\,dt, \quad a_n = \frac{4}{T_0}\int_{0}^{T_0/2} f(t)\cos n\omega_0 t\,dt, \quad b_n = 0$$

If $f(t)$ is an odd function of t, then $f(t)\cos n\omega_0 t$ is an odd function of t and $f(t)\sin n\omega_0 t$ is an even function of t. Therefore

$$a_0 = a_n = 0, \quad b_n = \frac{4}{T_0}\int_{0}^{T_0/2} f(t)\sin n\omega_0 t\,dt$$

Find the Fourier series of the signal $x(t)$

$$x(t) = \begin{cases} 1 + 4t, & -\frac{1}{2} \le t < 0 \\ 1 - 4t, & 0 \le t < \frac{1}{2} \end{cases}$$

Solution: In this case $T_0 = 1$ and $x(t)$ is an even function. Then

$$\begin{aligned} a_0 &= \frac{2}{T_0} \int_0^{1/2} (1 - 4t)\, dt = 0 \\ a_n &= \frac{4}{T_0} \int_0^{1/2} (1 - 4t) \cos 2n\pi t dt = \frac{4}{n^2\pi^2}(1 - \cos n\pi) \\ &= \begin{cases} 0, & n \text{ even} \\ \dfrac{8}{n^2\pi^2}, & n \text{ odd} \end{cases}, \quad b_n = 0 \\ x(t) &= \frac{8}{\pi^2}\left[\cos 2\pi t + \frac{1}{9}\cos 6\pi t + \frac{1}{25}\cos 10\pi t + \cdots\right] \end{aligned}$$

Fourier Series

Symmetry property: example Maxima

We could code Maxima as follow:

```
(declare(n,integer), assume(n>0),facts());
a0: integrate((1-4*x),x,0,1/2)*2 ;
/* T_0 = 1 */
an: integrate((1-4*x)*cos(2*n*%pi*x),x,0,1/2)*4;
define(a(n),an);
an_list: map('a,[1,2,3,4,5,6]);
```

We get the coefficient of $1, 3, 5$ harmonics as

$$\left[\frac{8}{\pi^2}, \frac{8}{9\pi^2}, \frac{8}{25\pi^2}\right]$$

Reference

1. Naresh, K. Sinha, *Linear Systems*, John Wiley & Sons, Inc., 1991.
2. Lathi, B. P., *Signal Processing & Linear Systems*, Berkeley-Cambridge Press, 1998.
3. Watcharapong Khovidhungij, *Signals, Systems, and Control*, Chulalongkorn University Press, 2016