Lecture 1: Continuous-Time Signals

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Outline

- Measuring the size of a signal
- Some useful signal operations
- Some useful signal models
- Even and Odd functions
- function plot with MATLAB

Measuring the size of a signal

Size of a signal u is measured in many ways but we consider only three of them:

- **energy (integral-absolute square):**

$$E_f = \int_{-\infty}^{\infty} |u(t)|^2 dt$$

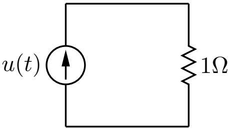
- **power (mean-square):**

$$P_f = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u(t)|^2 dt$$

- **root-mean-square (RMS):**

$$\text{rms} = \left(\lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u(t)|^2 dt \right)^{1/2}$$

Energy and Power signal



- the total energy is $E = \int_{-\infty}^{\infty} |u(t)|^2 dt$
- the average power is $P = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u(t)|^2 dt$
- if $u(t) = a$ A for $t \geq 0$
  - $E = \infty$ (the energy signal is not exist.)
  - $P = a^2$

- $x(t)$ is an energy signal if its energy is finite. A necessary condition for the energy to be finite is that the signal amplitude approach to zero as $|t| \to \infty$.
- $x(t)$ is a power signal if its power is finite and nonzero.
- Since the average power is the averaging over an infinitely large interval, **a signal with finite energy has zero power**, and **a signal with finite power has infinite energy**.
- if $x(t)$ is not satisfied both above conditions, it is not an energy signal or a power signal.

Energy and Power signal
cont.

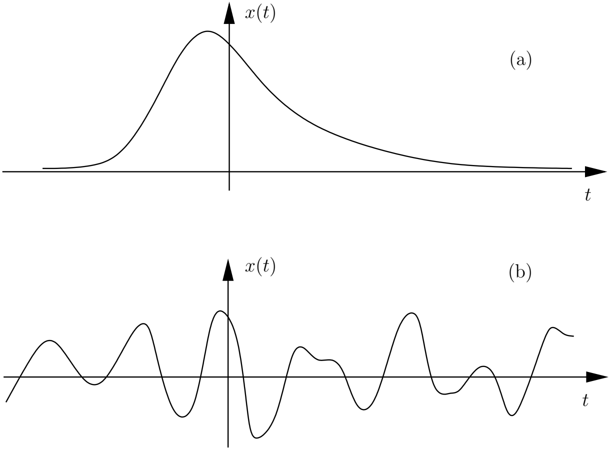
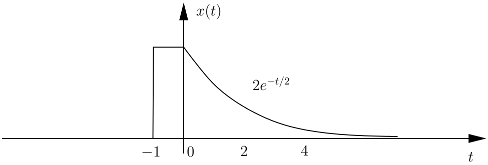


Figure: (a) a signal with finite energy (b) a signal with finite power.



The signal amplitude approaches to 0 as $|t| \to \infty$. Its energy is given by

$$E = \int_{-\infty}^{\infty} x^2(t)dt = \int_{-1}^{0} (2)^2 dt + \int_{0}^{\infty} 4e^{-t} dt = 4 + 4 = 8$$

If we change $2e^{-t/2}$ to a more general $Ae^{-\alpha t}$, then the energy signal is

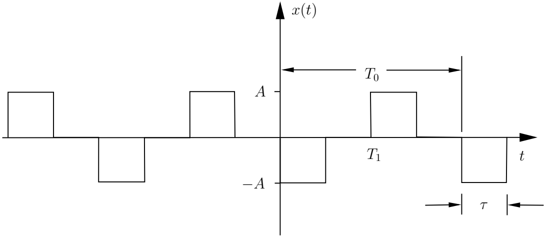
$$E = 4 + \int_{0}^{\infty} A^2 e^{-2\alpha t} dt = 4 - \frac{A^2}{2\alpha} e^{-2\alpha t}\Big|_{0}^{\infty} = 4 + \frac{A^2}{2\alpha}$$

The power signal of the first term is obvious zero and the second term is given by

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{\frac{T}{2}} A^2 e^{-2\alpha t} dt = \lim_{T\to\infty} -\frac{1}{T}\, \frac{A^2}{2\alpha} e^{-2\alpha t}\Big|_{0}^{\frac{T}{2}} = 0$$

Energy and Power signal

examples



It is obvious that the energy signal does not exist (infinite energy since $x(t) \not\to 0$ as $|t| \to \infty$)
Consider the power signal

$$
\begin{aligned}
P = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt &= \frac{1}{T_0}\int_0^{T_0} A^2 dt \\
&= \frac{1}{T_0}\left[A^2 t\big|_0^{\tau} + A^2 t\big|_{T_1}^{T_1+\tau}\right] \\
&= \frac{1}{T_0}\left[A^2\tau + A^2(T_1+\tau) - A^2 T_1\right] \\
&= \frac{2}{T_0}A^2\tau
\end{aligned}
$$

Energy and Power signal

examples

Determine the power and the rms value of $f(t) = C\cos(\omega_0 t + \theta)$

This is a periodic signal with period $T_0 = 2\pi/\omega_0$. It does not converge to 0 when $t \to \infty$. Because it is a periodic signal, we can compute its power by averagin its energy over one period T_0.

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} C^2 \cos^2(\omega_0 t + \theta) dt$$

Since $\cos^2(\omega_0 t + \theta) = 1 - \sin^2(\omega_0 t + \theta) = 1 + \cos(2\omega_0 t + 2\theta) - \cos^2(\omega_0 t + \theta)$ or $\cos^2(\omega_0 t + \theta) = \frac{1}{2}(1 + \cos(2\omega_0 t + 2\theta))$, then

$$\begin{aligned} P &= \lim_{T\to\infty} \frac{C^2}{2T} \int_{-\frac{T}{2}}^{\frac{T}{2}} [1 + \cos(2\omega_0 t + 2\theta)] dt \\ &= \lim_{T\to\infty} \frac{C^2}{2T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt + \lim_{T\to\infty} \frac{C^2}{2T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(2\omega_0 t + 2\theta) dt \\ &= \frac{C^2}{2} \end{aligned}$$

Hence, the rms values is $C/\sqrt{2}$

Energy and Power signal

examples

Determine the power and the rms value of

$$f(t) = C_1 \cos(\omega_1 t + \theta_1) + C_2 \cos(\omega_2 t + \theta_2)(\omega_1 \neq \omega_2)$$

$$\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[C_1 \cos(\omega_1 t + \theta_1) + C_2 \cos(\omega_2 t + \theta_2)\right]^2 dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} C_1^2 \cos^2(\omega_1 t + \theta_1) dt + \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} C_2^2 \cos^2(\omega_2 t + \theta_2) dt \\
&\quad + \lim_{T\to\infty} \frac{2C_1C_2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(\omega_1 t + \theta_1)\cos(\omega_2 t + \theta_2) dt
\end{aligned}$$

The first and second integrals on the right-hand side are the powers of the two sinusoids, which are $C_1^2/2$ and $C_2^2/2$ as found in the previous example. The third term is zero if $\omega_1 \neq \omega_2$, and we have

$$P = \frac{C_1^2}{2} + \frac{C_2^2}{2} \text{ and the rms value is } \sqrt{(C_1^2 + C_2^2)/2}$$

This result can be extended to a sum of any number of sinusoids with distinct frequencies. If

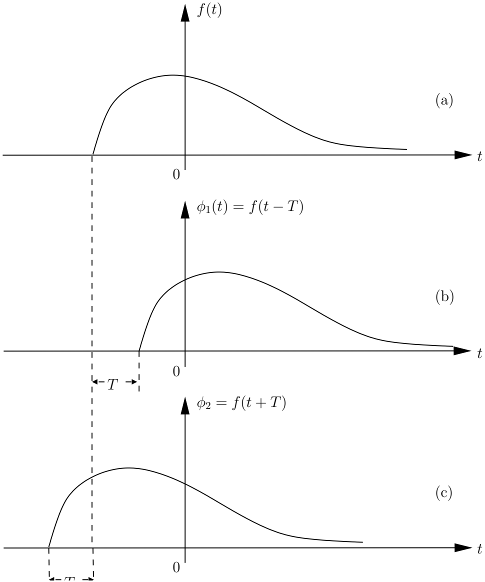
$$f(t) = \sum_{n=1}^{\infty} C_n \cos(\omega_n t + \theta_n)$$

where none of the two sinusoids have identical frequencies, then

$$P = \frac{1}{2} \sum_{n=1}^{\infty} C_n^2$$

Note: This is ture only if $\omega_1 \neq \omega_2$. If $\omega_1 = \omega_2$, the integrand of the third term contains a constant $\cos(\theta_1 - \theta_2)$, and the third term $\to 2C_1C_2\cos(\theta_1 - \theta_2)$ as $T \to \infty$.

Time Shifting



- $f(t)$ in Fig. (b) is the same signal like (a) but delayed by T seconds
- $f(t)$ in Fig. (c) is the same signal like (b) but advanced by T seconds
- Whatever happens in $f(t)$ in at some instant t also happens in $\phi_1(t)$ T seconds later and happens in $\phi_2(t)$ T seconds before.
- we have

  delay:

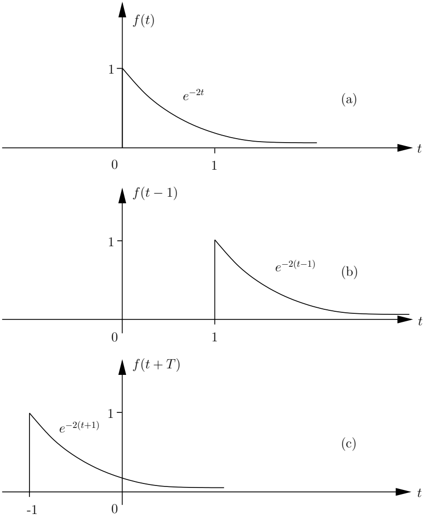
  $$\phi_1(t+T) = f(t) \text{ or } \phi_1(t) = f(t-T)$$

  advance:

  $$\phi_2(t-T) = f(t) \text{ or } \phi_2(t) = f(t+T)$$

Time Shifting

example



(a) signal $f(t)$ (b) $f(t)$ delayed by 1 second (c) $f(t)$ advanced by 1 second

Time Shifting

example

An exponential function $f(t) = e^{-2t}$ shown in the previous Fig. is delayed by 1 second. Sketch and mathematically describe the delayed function. Repeat the problem if $f(t)$ is advanced by 1 second. The function $f(t)$ can be described mathematically as

$$f(t) = \begin{cases} e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

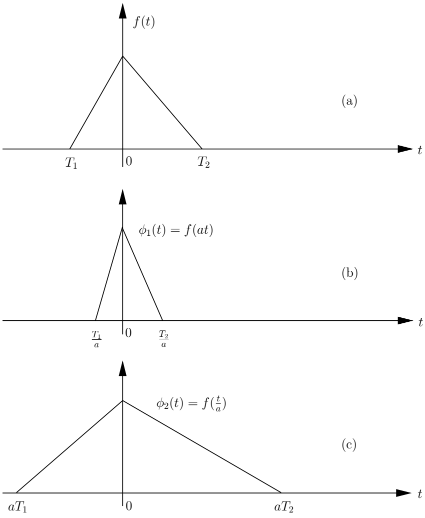
Let $f_d(t)$ represent the function $f(t)$ delayed by 1 second. Replace t with $t-1$, thus

$$f_d(t) = f(t-1) = \begin{cases} e^{-2(t-1)} & t-1 \geq 0 \text{ or } t \geq 1 \\ 0 & t-1 < 0 \text{ or } t < 1 \end{cases}$$

Let $f_a(t)$ represent the function $f(t)$ advanced by 1 second. Replace t with $t+1$, thus

$$f_a(t) = f(t+1) = \begin{cases} e^{-2(t+1)} & t+1 \geq 0 \text{ or } t \geq -1 \\ 0 & t+1 < 0 \text{ or } t < -1 \end{cases}$$

Time Scaling



Time scaling a signal.

- The compression or expansion of a signal in time is known as **time scaling**.
- $f(t)$ in Fig. (b) is $f(t)$ compressed in time by a factor of a.
- $f(t)$ in Fig. (c) is $f(t)$ expanded in time by a factor of a.
- Whatever happens in $f(t)$ at some instant t also happens to $\phi(t)$ at the instant t/a or at
- we have

compress:

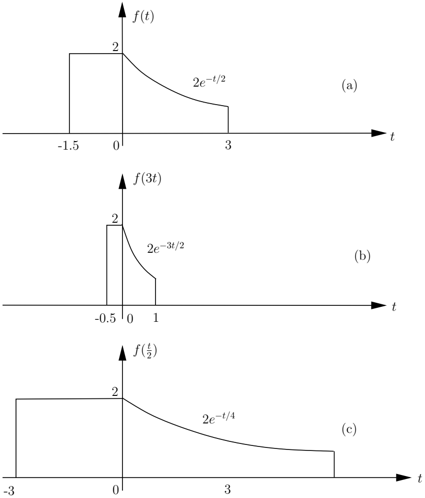
$$\phi(\frac{t}{a}) = f(t) \text{ or } \phi(t) = f(at)$$

expand:

$$\phi(at) = f(t) \text{ or } \phi(t) = f(\frac{t}{a})$$

Time Scaling

example



(a) signal $f(t)$ (b) signal $f(3t)$ (c) signal $f(\frac{t}{2})$

Time Scaling

example

A signal $f(t)$ shown in the Fig. Sketch and describe mathematically this isgnal time-compressed by factor 3. Repeat the problem for the same signal time-expanded by factor 2. The signal $f(t)$ can be described as

$$f(t) = \begin{cases} 2 & -1.5 \leq t < 0 \\ 2e^{-t/2} & 0 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}$$

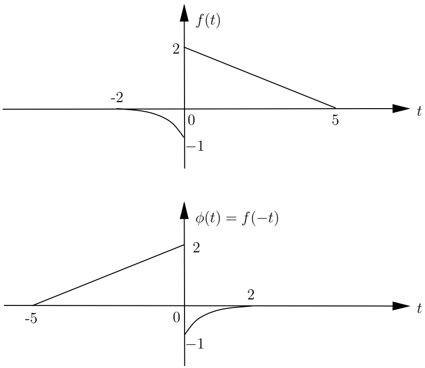
Let $f_c(t)$ is a time-compressed of $f(t)$ by factor 3. Replace t with $3t$, we have

$$f_c(t) = f(3t) = \begin{cases} 2 & -1.5 \leq 3t < 0 \text{ or } -0.5 \leq t < 0 \\ 2e^{-3t/2} & 0 \leq 3t < 3 \text{ or } 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $f_e(t)$ is a time-expanded by factor 2. Replace t with $t/2$, we have

$$f_e(t) = f(\frac{t}{2}) = \begin{cases} 2 & -1.5 \le \frac{t}{2} < 0 \text{ or } -3 \le t < 0 \\ 2e^{-t/4} & 0 \le \frac{t}{2} < 3 \text{ or } 0 \le t < 6 \\ 0 & \text{otherwise} \end{cases}$$

Time inversion



Time inversion (reflection) of a signal.

The inversion or folding [the reflection of $f(t)$ about the vertical axis] given us the signal $\phi(t)$. Observe that whatever happens in $f(t)$ at the instant t also happens in $\phi(t)$ at the time instant $-t$. Therefore

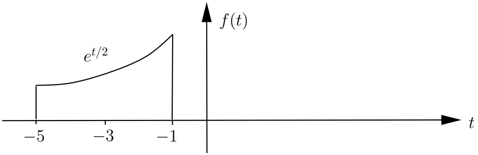
$$\phi(-t) = f(t)$$

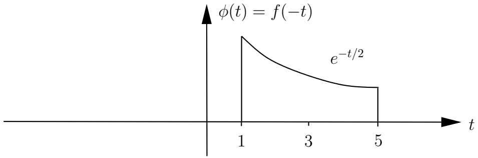
and

$$\phi(t) = f(-t)$$

note: the mirror image of $f(t)$ about the horizontal axis is $-f(t)$.

Time inversion
example





The instants -1 and -5 in $f(t)$ are mapped into instants 1 and 5 in $f(-t)$. Because $f(t) = e^{t/2}$, we have $f(-t) = e^{-t/2}$. The signal $f(-t)$ is depicted above.

$$f(t) = \begin{cases} e^{t/2} & -1 \geq t > -5 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f(-t) = \begin{cases} e^{-t/2} & -1 \geq -t > -5 \text{ or } 1 \leq t < 5 \\ 0 & \text{otherwise} \end{cases}$$

Combined operations

The most general operation involving all the three operations is $f(at - b)$, which is realized in two possible sequences of operation:

- Time-shift $f(t)$ by b to obtain $f(t - b)$. Then time-scale the shifted signal $f(t - b)$ by a (that is, replace t with at) to obtain $f(at - b)$.
- Time-scale $f(t)$ by a to obtain $f(at)$. Then time-shift $f(at)$ by $\frac{b}{a}$ (that is replace t with $(t - \frac{b}{a})$ to obtain $f[a(t - \frac{b}{a})] = f(at - b)$). In either case, if a is negative, time scaling involves time inversion.

Combined operations
examples

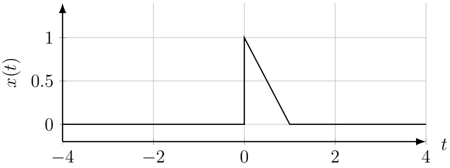
Consider a signal

$$x(t) = \begin{cases} 1 - t, & 0 < t \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

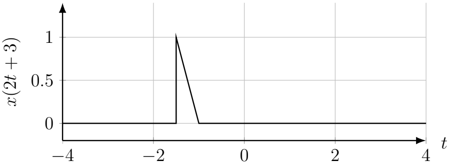
If we substitute t with $2t + 3$, we have

$$x(2t+3) = \begin{cases} 1 - (2t+3), & 0 < 2t+3 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} -2t - 2, , & -\frac{3}{2} < t \le -1 \\ 0, & \text{otherwise.} \end{cases}$$



(a)



(b)

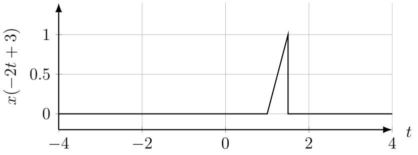
Combined operations

examples

If we substitute t with $-2t+3$, we have

$$x(-2t+3) = \begin{cases} 1-(-2t+3), & 0 < -2t+3 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

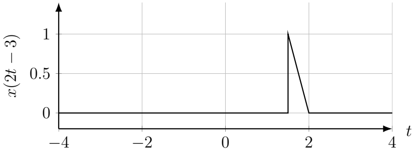
$$= \begin{cases} 2t-2,, & \frac{3}{2} < t \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$



(c)

Similarly,

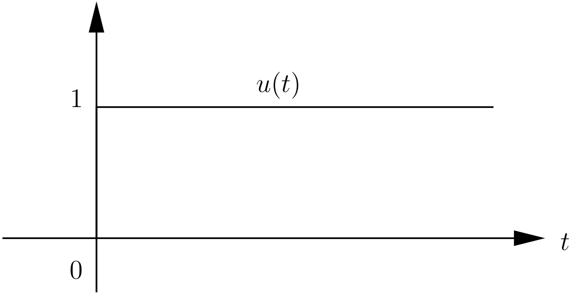
$$x(2t-3) = \begin{cases} -2t+4, & \frac{3}{2} < t \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$



(d)

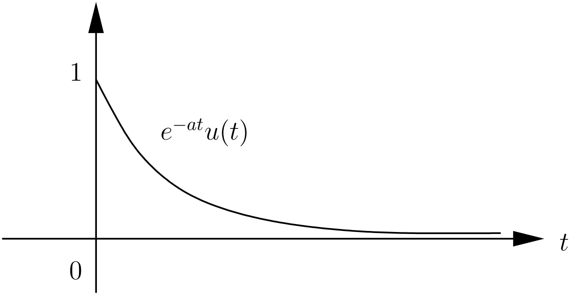
A unit step function $u(t)$ is defined by

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



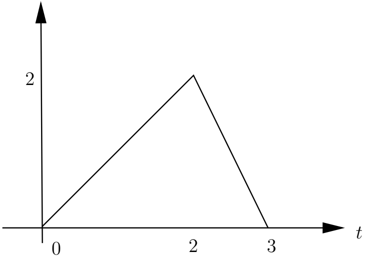
Unit step function $u(t)$

- We can force a signal to start at $t = 0$ by multiplying the signal with $u(t)$ (the signal has a value of zero for $t < 0$).
- For example, the signal e^{-at} represents an everlasting exponential that starts at $t = -\infty$. the causal form of this exponential shown in Fig. can be describe as $e^{-at}u(t)$.



Unit step function $u(t)$

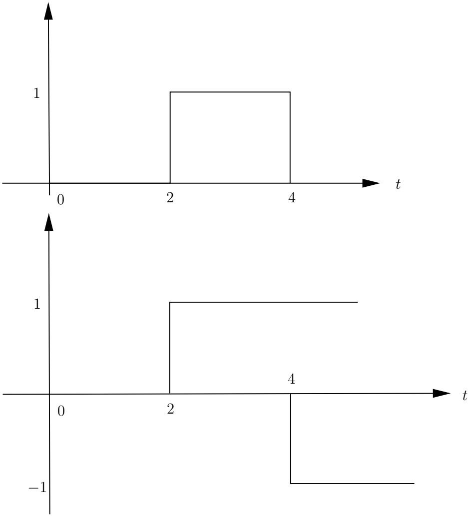
- The unit step function can be used to specify a function with different mathematical descriptions over different intervals.



- a mathematic description of above signal is inconvenient.
- we can describe a signal by using unit step signals.

Unit step function $u(t)$

example

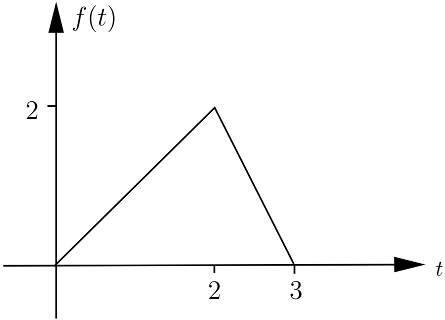


- the unit step function $u(t)$ delayed by T seconds is $u(t - T)$
- from the lower Fig., it is clear that

$$f(t) = u(t - 2) - u(t - 4)$$

Unit step function $u(t)$

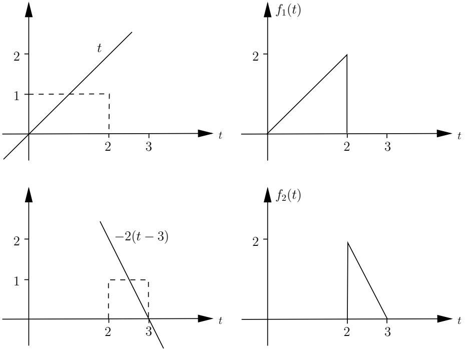
example



Describe the signal in Fig. by using unit step signals.

Unit step function $u(t)$

example

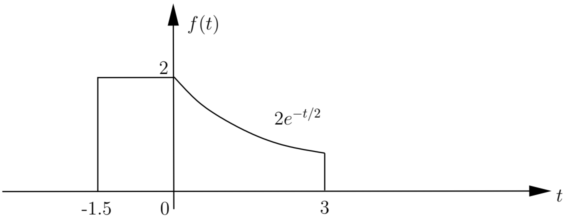


- $f_1(t)$ can be obtained by multiplying the ramp t by the gate pulse $u(t) - u(t-2)$, then $f_1(t) = t[u(t) - u(t-2)]$
- $f_2(t)$ can be obtained by multiplying another ramp by the gate pulse $u(t-2) - u(t-3)$, then $f_2(t) = -2(t-3)[u(t-2) - u(t-3)]$

$$
\begin{aligned}
f(t) = f_1(t) + f_2(t) &= t[u(t) - u(t-2)] - 2(t-3)[u(t-2) - u(t-3)] \\
&= tu(t) - 3(t-2)u(t-2) + 2(t-3)u(t-3)
\end{aligned}
$$

Unit step function $u(t)$

example



Describe the signal in Fig. by a single expression valid for all t. (Recall in previous example, we use 3 equations to describe the signal.)

- Over the interval from -1.5 to 0, the signal can be described by a constant 2, and over the interval from 0 to 3, it can be described by $2e^{-t/2}$. Therefore

$$
\begin{aligned}
f(t) &= \underbrace{2[u(t+1.5) - u(t)]}_{f_1(t)} + \underbrace{2e^{-t/2}[u(t) - u(t-3)]}_{f_2(t)} \\
&= 2u(t+1.5) - 2(1 - e^{-t/2})u(t) - 2e^{-t/2}u(t-3)
\end{aligned}
$$

The Unit Impulse Function $\delta(t)$

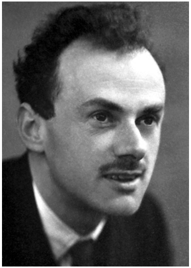
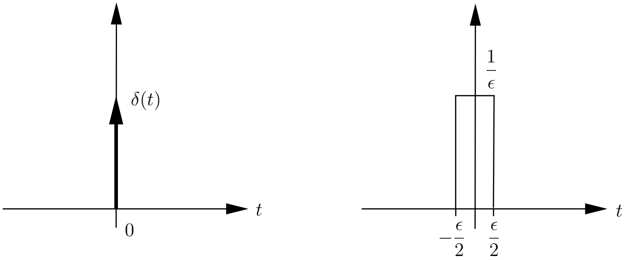


Figure: P. A. M. Dirac

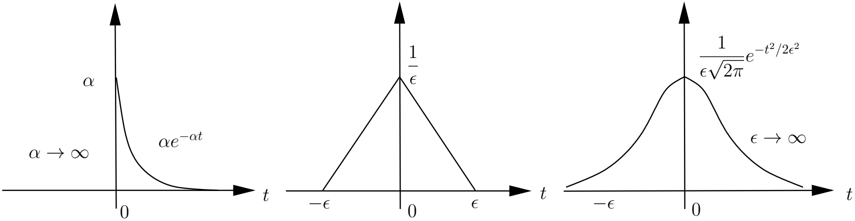
(Dirac's) **delta function** or **impulse** δ is an *idealization* of a signal that

- it is very large near $t = 0$ and very small away from $t = 0$, hence $\delta(t) = 0, t \neq 0$
- it has integral 1 or $\int_{-\infty}^{\infty} \delta(t)dt = 1$

The Unit Impulse Function $\delta(t)$



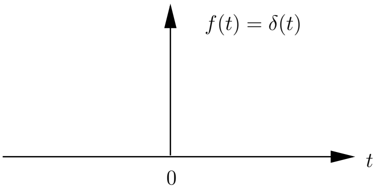
A unit impulse and its approximation.

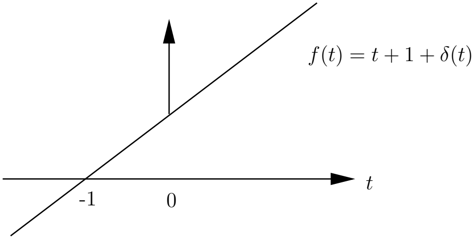


Other possible approximations to a unit impulse.

The Unit Impulse Function $\delta(t)$

on plots δ is shown as a solid arrow:





Impulse functions are used to model physical signals

- functions that act over short time intervals
- functions whose effect depends on integral of signal

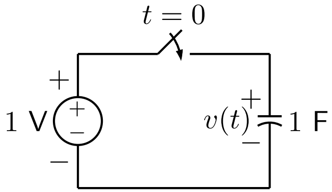
example:

- hammer blow, or bat hitting ball, at $t = 2$
- force f acts on mass m between $t = 1.999$ sec and $t = 2.001$ sec
- $\displaystyle\int_{1.999}^{2.001} f(t)dt = I$ (mechanical impulse, N·sec)
- blow induces change in velocity of

$$v(2.001) - v(1.999) = \frac{1}{m}\int_{1.999}^{2.001} f(\tau)d\tau = I/m$$

for applications we can model force as an impulse at $t = 2$, with magnitude I.

Rapid charging of capacitor

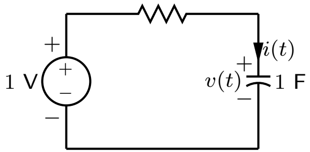


assuming $v(0) = 0$, what is $v(t), i(t)$ for $t > 0$

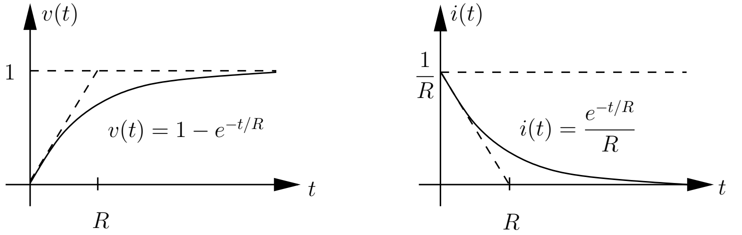
- $i(t)$ is very large, for a very short time
- a unit charge is transferred to the capacitor 'almost instantaneously'
- $v(t)$ increases to $v(t) = 1$ 'almost instantaneously'

to calculate i, v, we need a more detailed model.

Include small resistance



$$i(t) = \frac{dv(t)}{dt} = \frac{1 - v(t)}{R},\ v(0) = 0$$



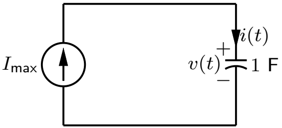
as $R \to 0$, i approaches an impulse, v approaches a unit step

The Unit Impulse Function $\delta(t)$

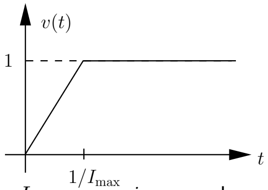
Physical interpretation example

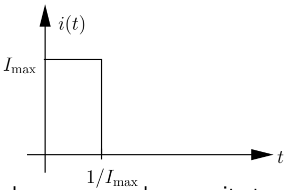
Assume the current delivered by the source is limited:

if $v(t) \leq 1$, the source acts as a current source $i(t) = I_{\text{max}}$



$$i(t) = \frac{dv(t)}{dt} = I_{\text{max}}, \qquad v(0) = 0 \qquad \text{(open circuit when } v(t) = 1)$$





as $I_{\text{max}} \to \infty$, i approaches an impulse, v approaches a unit step.

The Unit Impulse Function $\delta(t)$

Physical interpretation example

in conclusion,

- large current i acts over very short time between $t = 0$ and ϵ
- total charge transfer is $\displaystyle\int_0^{\epsilon} i(t)dt = 1$
- resulting change in $v(t)$ is $v(\epsilon) - v(0) = 1$
- can approximate i as impulse at $t = 0$ with magnitude 1

modeling current as impulse

- obscures details of current signal
- obscures details of voltage change during the rapid charging
- preserves total change in charge, voltage
- is reasonable model for time scales $\gg \epsilon$

Let us consider what happens when we multiply the unit impulse $\delta(t)$ by a function $\phi(t)$ that is known to be continuous at $t = 0$.

$$\phi(t)\delta(t) = \phi(0)\delta(t),$$

where $\phi(t)$ at $t = 0$ is $\phi(0)$.

Similarly, if $\phi(t)$ is multiplied by an impulse $\delta(t - T)$ (impulse located at $t = T$) then

$$\phi(t)\delta(t - T) = \phi(T)\delta(t - T)$$

By the multiplication property, it follows that

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0)\int_{-\infty}^{\infty} \delta(t)dt = \phi(0)$$

provided $\phi(t)$ is continuous at $t = 0$.

- the area under the product of a function with an impulse $\delta(t)$ to the value of the function at the instant where the unit impulse is located.
- This property is known as the **sampling** or **sifting property** of the unit impulse.
- If $\phi(t)$ continuous at $t = T$ we have $\displaystyle\int_{-\infty}^{\infty} \phi(t)\delta(t - T)dt = \phi(T)$

The Unit Impulse Function $\delta(t)$

Sampling Property of the Unit Impulse Function example.

example:

$$
\begin{aligned}
&\int_{-2}^{3} f(t)\left(2+\delta(t+1)-3\delta(t-1)+2\delta(t+3)\right)dt \\
&\quad = 2\int_{-2}^{3} f(t)dt + \int_{-2}^{3} f(t)\delta(t+1)dt - 3\int_{-2}^{3} f(t)\delta(t-1)dt \\
&\qquad + 2\int_{-2}^{3} f(t)\delta(t+3))dt \\
&\quad = 2\int_{-2}^{3} f(t)dt + f(-1) - 3f(1) + 0
\end{aligned}
$$

The Unit Impulse Function $\delta(t)$

Unit impulse as a generalized function

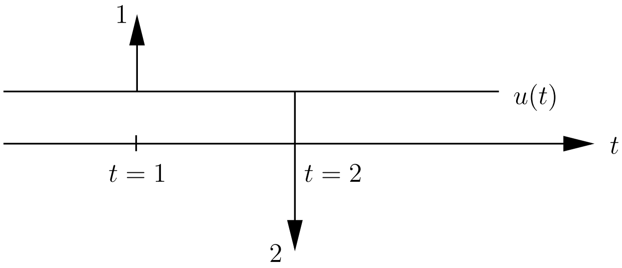
The definition of the unit impulse function given before is not mathematically rigorous:

- it is not a real function (Dirac's δ is what is called a *distribution*)
- it does not define a unique function: for example, it can be shown that $\delta(t) + \dot{\delta}(t)$ also satisfies the definition.
- some innocent looking expressions don't make any sense at all (e.g., $\delta^2(t)$ or $\delta(t^2)$)

A **genrealized function** is defined by its effect on other functions instead of by its value at every instant of time. This approach the impulse function is defined by the smpling property.

Integral of a function with impulses has jump at each impulse, equal to the magnitude of impulse.

example: $x(t) = 1 + \delta(t-1) - 2\delta(t-2)$; define $f(t) = \int_0^t x(\tau)d\tau$



$f(t) = t$ for $t < 1$ ($f(1)$ and $f(2)$ are undefined),

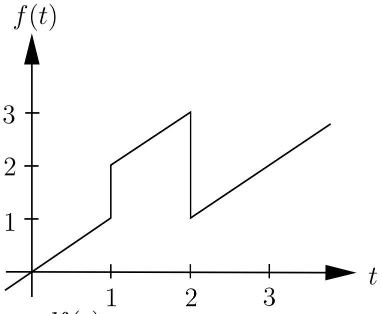
$f(t) = t + 1$ for $1 < t < 2$, $\quad f(t) = t - 1$ for $t > 2$

The Unit Impulse Function $\delta(t)$

Derivatives of discontinuous functions

We now present an interesting application of the generalized function definition of an impulse:

- derivative of unit step function is $\delta(t)$
- derivative of discontinuous functions $f(t)$ of the previous page



we have $\dfrac{df(t)}{dt} = 1 + \delta(t-1) - 2\delta(t-2)$

Because the unit step function $u(t)$ is discontinuous at $t = 0$, its derivative du/dt does not exist at $t = 0$ in the ordinary sense. We will shown that this derivative does exist in the generalized sense, and it is $\delta(t)$.

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{du}{dt}\phi(t)dt &= u(t)\phi(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\dot{\phi}(t)dt \\
&= \phi(\infty) - 0 - \int_{0}^{\infty} \dot{\phi}(t)dt \\
&= \phi(\infty) - \phi(t)|_{0}^{\infty} \\
&= \phi(0)
\end{aligned}
$$

This result shows that du/dt satisfies the sampling property of $\delta(t)$. Therefore it is an impulse $\delta(t)$ in the generalized sense.

That is

$$\frac{du}{dt} = \delta(t)$$

Consequently

$$\int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$$

The area from $-\infty$ to t under the limiting form of $\delta(t)$ is zero if $t < 0$ and unity if $t \geq 0$. Consequently

$$\int_{-\infty}^{t} \delta(\tau) d\tau = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} = u(t)$$

Integration by parts suggests we define

$$\int_{-\infty}^{\infty} \dot{\delta}(t) f(t) dt = \delta(t) f(t) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t) \dot{f}(t) dt = -\dot{f}(0)$$

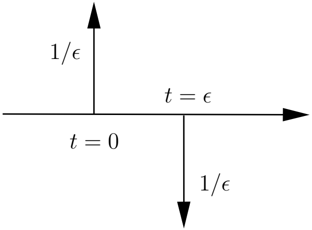
provided $a < 0, b > 0$, and $\dot{f}(t)$ continuous at $t = 0$

- $\dot{\delta}(t)$ is called **doublet**
- $\dot{\delta}(t), \ddot{\delta}(t)$, etc. are called **higher-order impulses**
- similar rules for higher-order impulses:

$$\int_{-\infty}^{\infty} \delta^{(k)}(t) f(t) dt = (-1)^k f^{(k)}(0)$$

  if $f^{(k)}$ continuous at $t = 0$.

Interpretation of doublet $\dot{\delta}(t)$: take two impulses with magnitude $\pm 1/\epsilon$, a distance ϵ apart, and let $\epsilon \to 0$



$$\int_{-\infty}^{\infty} f(t) \left(\frac{\delta(t)}{\epsilon} - \frac{\delta(t-\epsilon)}{\epsilon} \right) dt = \frac{f(0) - f(\epsilon)}{\epsilon}$$

converges to $-\dot{f}(0)$ if $\epsilon \to 0$.

Derivatives of impulse functions
Examples

Determine the value of

$$\int_{-\infty}^{\infty} \sin t \ \delta'(2t-\pi)dt$$

Let $\tau = 2t - \pi$ we have $t = (\tau + \pi)/2$ and $dt = d\tau/2$. Hence,

$$\begin{aligned}\int_{-\infty}^{\infty} \sin t\delta'(2t-\pi)dt &= \int_{-\infty}^{\infty} \sin\left(\frac{\tau+\pi}{2}\right)\delta'(\tau)\frac{d\tau}{2} \\ &= \frac{1}{2}(-1)\frac{d}{d\tau}\sin\left(\frac{\tau+\pi}{2}\right)\Bigg|_{\tau=0} \\ &= -\frac{1}{4}\cos\frac{\pi}{2} = 0\end{aligned}$$

The scaling property of the delta function is

$$\delta(at+b) = \frac{1}{|a|}\delta\left(t+\frac{b}{a}\right)$$

We need to show that

$$\int_{-\infty}^{\infty} f(t)\delta(at+b)dt = \int_{-\infty}^{\infty} f(t)\frac{1}{|a|}\delta\left(1+\frac{b}{a}\right)dt$$

Consider the left hand side of the above equation: Let $\tau = at+b$, then $t = (\tau - b)/a$ and $dt = d\tau/a$
(If $a > 0$) we have $dt = d\tau/|a|$ and

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\delta(at+b)dt &= \int_{-\infty}^{\infty} f\left(\frac{\tau-b}{a}\right)\delta(\tau)\frac{1}{|a|}d\tau \\ &= \frac{1}{|a|}f\left(\frac{\tau-b}{a}\right)\bigg|_{\tau=0} = \frac{1}{|a|}f\left(-\frac{b}{a}\right)\end{aligned}$$

(If $a < 0$) we have $dt = -d\tau/|a|$ and (note that $\tau = \mp\infty$ when $t = \pm\infty$) then

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\delta(at+b)dt &= \int_{\infty}^{-\infty} f\left(\frac{\tau-b}{a}\right)\delta(\tau)\left(-\frac{1}{|a|}\right)d\tau \\ &= \int_{-\infty}^{\infty} f\left(\frac{\tau-b}{a}\right)\delta(\tau)\frac{1}{|a|}d\tau \\ &= \frac{1}{|a|}f\left(-\frac{b}{a}\right)\end{aligned}$$

From sifting property we obtain,

$$\int_{-\infty}^{\infty} f(t)\frac{1}{|a|}\delta\left(t+\frac{b}{a}\right)dt = \frac{1}{|a|}f\left(-\frac{b}{a}\right)$$

Since $f(t)$ is a arbitrary function, then

$$\delta(at+b) = \frac{1}{|a|}\delta\left(t-\frac{b}{a}\right)$$

- $$\int_{-\infty}^{\infty} e^{-t}\delta(2t+3)dt = \int_{-\infty}^{\infty} e^{-t}\frac{1}{|2|}\delta\left(t+\frac{3}{2}\right)dt$$
  $$= \frac{1}{2}e^{-t}\Big|_{t=-\frac{3}{2}} = \frac{1}{2}e^{3/2}$$

- $$\int_{-\infty}^{\infty} e^{-t}\delta(-2t+3)dt = \int_{-\infty}^{\infty} e^{-t}\frac{1}{|-2|}\delta\left(t-\frac{3}{2}\right)dt$$
  $$= \frac{1}{|-2|}e^{-t}\Big|_{t=\frac{3}{2}} = \frac{1}{2}e^{-3/2}$$

The Exponential Function e^{st}

Let s be a complex number $s = \sigma + j\omega$.

$$\begin{aligned} e^{st} &= e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t} \\ &= e^{\sigma t}(\cos \omega t + j \sin \omega t) \end{aligned}$$

If $s^* = \sigma - j\omega$ (the conjugate of s), then

$$\begin{aligned} e^{s^* t} &= e^{(\sigma - j\omega)t} = e^{\sigma t} e^{-j\omega t} \\ &= e^{\sigma t}(\cos \omega t - j \sin \omega t) \end{aligned}$$

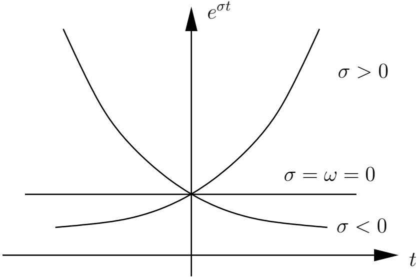
and

$$e^{\sigma t} \cos \omega t = \frac{1}{2}(e^{st} + e^{s^* t})$$

The Exponential Function e^{st}

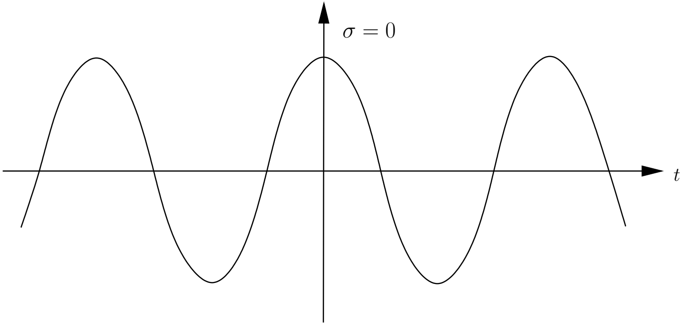
The function e^{st} encompasses a large class of functions:

- A constant $k = ke^{0t}$ $(s = 0)$
- A monotonic exponential $e^{\sigma t}$ $(\omega = 0, s = \sigma)$
- For signals whose complex frequencies lie on the real axis (σ-axis, where $\omega = 0$). these signals are monotonically increasing or decreasing exponentials.The case $s = 0$ $(\sigma = \omega = 0)$ corresponds to a constant (dc) signal because $e^{0t} = 1$.



The Exponential Function e^{st}

- For signals whose frequencies lie on the imaginary axis ($j\omega$ axis where $\sigma = 0$), $e^{\sigma t} = 1$.
- These signals are conventional sinusoids with constant amplitude.



- a constant amplitude sinusoid $\cos(\omega t + \theta)$ can be expressed as a sum of exponentials $e^{j\omega t}$ and $e^{-j\omega t}$

The Exponential Function e^{st}

- An exponentially varying sinusoid $e^{\sigma t}\cos(\omega t)$ $(s = \sigma t \pm j\omega)$
- Let $\sigma > 0$ and $\omega > 0$, then an exponentially growing sinusoid $e^{at}\cos(\omega t)$ can be expressed as

$$f(t) = \frac{1}{2}(e^{(\sigma+j\omega)t} + e^{(\sigma-j\omega)t})$$

  with complex frequencies $\sigma + j\omega$ and $\sigma - j\omega$.
- Let $\sigma > 0$ and $\omega > 0$, then an exponentially decaying sinusoid $e^{-at}\cos(\omega t + \theta)$ can be expressed as

$$f(t) = \frac{1}{2}(e^{(-\sigma+j\omega)t} + e^{(-\sigma-j\omega)t})$$

  with complex frequencies $-\sigma + j\omega$ and $-\sigma - j\omega$.

The Exponential Function e^{st}

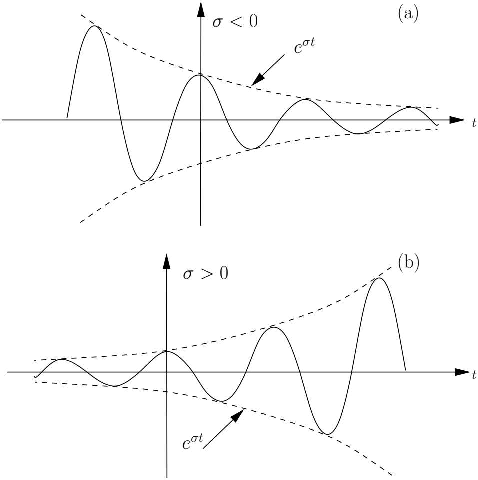


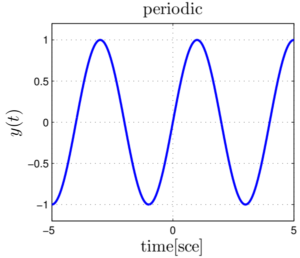
Figure: (a) decaying sinusoid (b) growing sinusoid

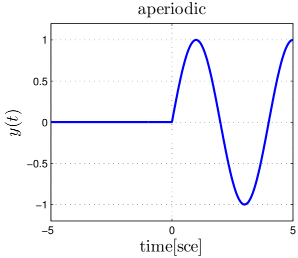
Periodic and aperiodic signals

A continuous-time signal $x(t)$ is said to be *periodic* if it satisfies the following property:

$$x(t) = x(t + T_0) \qquad \exists T_0 > 0 \text{ and } \forall t.$$

The smallest positive value of T_0 that satisfies the periodicity condition is referred to as the *fundamental period* of $x(t)$. A signal that is not periodic is called an *aperiodic* or *non-periodic* signal.





Periodic and aperiodic signals

- The reciprocal of the fundamental period of a signal is called the *fundamental frequency*. The fundamental frequency is expressed as follows

$$f_0 = \frac{1}{T_0}, \text{ for CT signals },$$

  where T_0 is the fundamental periods of the continuous-time signal.
- The frequency of a signal provides useful information regarding how fast the signal changes its amplitude.
- the unit of frequency is *cycles per second* (c/s) or *hertz* (Hz).
- we also use *radians per second* as a unit of frequency. Since there are 2π radians (or 360°) is one cycle, a frequency of f_0 hertz is equivalent to $2\pi f_0$ radians per second.

Periodic and aperiodic signals

- If radians per second is used as a unit of frequency, the frequency is referred to as the *angular frequency* and is given by

$$\omega_0 = \frac{2\pi}{T_0}, \text{ for CT signals.}$$

- A familiar example of a periodic signal is a sinusoidal function represented mathematically by the following expression:

$$x(t) = A\sin(\omega_0 t + \theta).$$

- the sinusoidal signal $x(t)$ has a fundamental period $T_0 = 2\pi/\omega_0$.
- Substituting t by $t + T_0$ in the sinusoidal function, yields

$$x(t + T_0) = A\sin(\omega_0 t + \omega_0 T_0 + \theta).$$

Periodic and aperiodic signals

- Since

$$x(t) = A\sin(\omega_0 t + \theta) = A\sin(\omega_0 t + 2m\pi + \theta),$$

  for $m = 0, \pm 1, \pm 2, \ldots$.
- the above two expressions are equal iff $\omega_0 T_0 = 2m\pi$. Selecting $m = 1$, the fundamental period is given by $T_0 = 2\pi/\omega_0$.

Periodic and aperiodic signals

Example

(i) CT sine wave: $x_1(t) = \sin(4\pi t)$ is a periodic signal with period $T_1 = 2\pi/4\pi = 1/2$;

(ii) CT cosine wave: $x_2(t) = \cos(3\pi t)$ is a periodic signal with period $T_2 = 2\pi/3\pi = 2/3$;

(iii) CT tangent wave: $x_3(t) = \tan(10t)$ is a periodic signal with period $T_3 = \pi/5$;

(iv) CT complex exponential: $x_4(t) = e^{j(2t+7)}$ is a periodic signal with period $T_4 = 2\pi/2 = \pi$;

(v) CT sin wave of limited duration: $x_6(t) = \begin{cases} \sin 4\pi t & -2 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$ is an aperiodic signal;

(vi) CT linear relationship: $x_7(t) = 2t + 5$ is an aperiodic signal;

Periodic and aperiodic signals

Linear Combination of Two Signals

- A signal $g(t)$ is a linear combination of two periodic signals, $x_1(t)$ with fundamental period T_1 and $x_2(t)$ with fundamental period T_2 as follows:

$$g(t) = ax_1(t) + bx_2(t)$$

is periodic iff

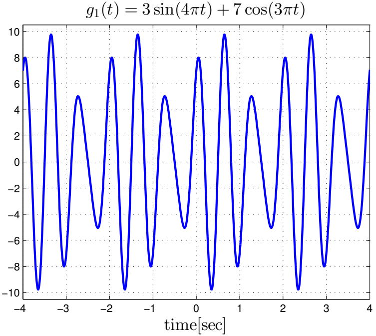
$$\frac{T_1}{T_2} = \frac{m}{n} = \text{ rational number }.$$

The funcdamental period of $g(t)$ is given by $nT_1 = mT_2$ provided that the values of m and n are chosen such that the greatest common divisor (gcd) between m and n is 1.

Determine if the following signals are periodic. If yes, determine the fundamental period

$$g_1(t) = 3\sin(4\pi t) + 7\cos(3\pi t)$$

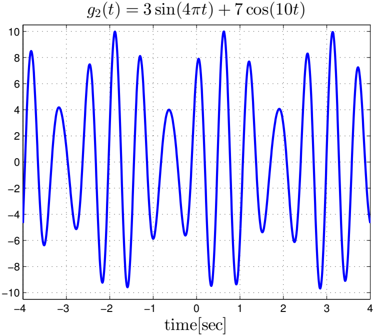
Signals $\sin(4\pi t)$ and $\cos(3\pi t)$ are both periodic signals with fundamental periods $1/2$ and $2/3$ second, respectively. The ratio of the two fundamental periods yields $\frac{T_1}{T_2} = \frac{1/2}{2/3} = \frac{3}{4}$ which is a rational number. Hence, the linear combination $g_1(t)$ is a periodic signal.



Determine if the following signals are periodic. If yes, determine the fundamental period

$$g_2(t) = 3\sin(4\pi t) + 7\cos(10t)$$

Signals $\sin(4\pi t)$ and $\cos(10t)$ are both periodic signals with fundamental periods $1/2$ and $\pi/5$ second, respectively. The ratio of the two fundamental periods yields $\frac{T_1}{T_2} = \frac{1/2}{\pi/5} = \frac{5}{2\pi}$ which is not a rational number. Hence, the linear combination $g_2(t)$ is a periodic signal.



Even and Odd Functions

A function $f_e(t)$ is said to be an **even function** of t if

$$f_e(t) = f_e(-t)$$

and a function $f_o(t)$ is said to be and **odd function** of t if

$$f_o(t) = -f_o(-t)$$

- an even function $f_e(t)$ has the same value at the instants t and $-t$ for all values of t. Clearly, it is symmetrical about the vertical axis.
- for an odd function $f_o(t)$, the value at the instant t is the negative of its values at the instant $-t$. Therefore, $f_o(t)$ is anti-symmetrical about the vertical axis.

Even and Odd Functions

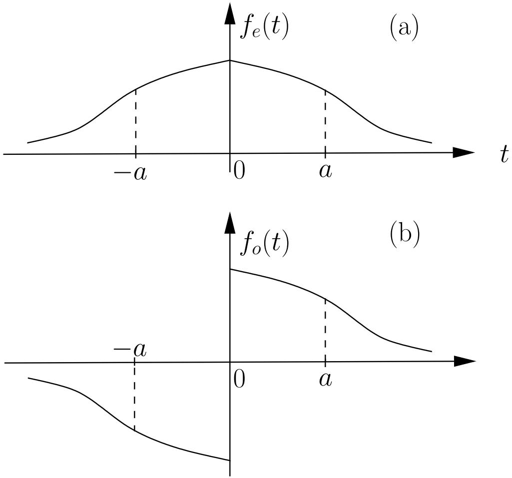


Figure: (a) even function (b) odd function

Even and odd functions have the following property:

- even function $\times$ odd function $=$ odd function
- odd function $\times$ odd function $=$ even function
- even function $\times$ even function $=$ even function

Area

- $f_e(t)$ is symmetrical about the vertical axis, it follows that

$$\int_{-a}^{a} f_e(t)dt = 2\int_{0}^{a} f_e(t)dt$$

and it is clear that

$$\int_{-a}^{a} f_o(t)dt = 0$$

Every signal $f(t)$ can be expressed as a sum of even and odd components because

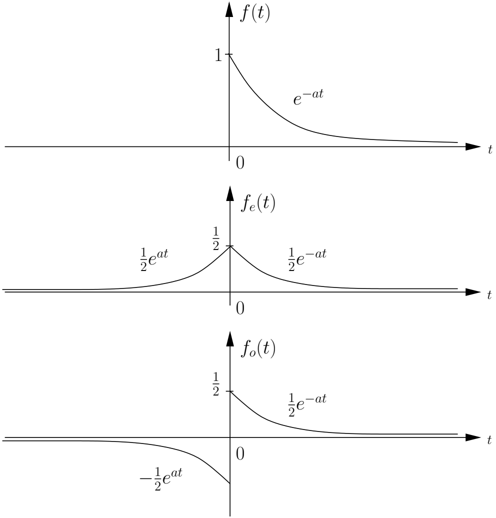
$$f(t) = \underbrace{\frac{1}{2}[f(t) + f(-t)]}_{\text{even function}} + \underbrace{\frac{1}{2}[f(t) - f(-t)]}_{\text{odd function}}$$

Consider the function $f(t) = e^{-at}u(t)$

$$\begin{aligned} f(t) &= f_e(t) + f_o(t) \\ f_e(t) &= \frac{1}{2}\left[e^{-at}u(t) + e^{at}u(-t)\right] \\ f_o(t) &= \frac{1}{2}\left[e^{-at}u(t) - e^{at}u(-t)\right] \end{aligned}$$

Even and Odd Functions

Even and Odd Components of a Signal



Find the even and odd components of e^{jt}

$$e^{jt} = f_e(t) + f_o(t)$$

where

$$f_e(t) = \frac{1}{2}\left[e^{jt} + e^{-jt}\right] = \cos t$$

and

$$f_o(t) = \frac{1}{2}\left[e^{jt} - e^{-jt}\right] = j\sin t$$

Consider the signal

$$x(t) = \begin{cases} 2\cos(4t) & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find its even and odd decomposition. What would happen if $x(0) = 2$ instead of 0–that is, when we define the sinusoid at $t = 0$? Explain.

Solution:

The signal $x(t)$ is neither even nor odd given that its values for $t \leq 0$ are zero. For its even-odd decomposition, the even component is given by

$$x_e(t) = 0.5[x(t) + x(-t)] = \begin{cases} \cos(4t) & t > 0 \\ \cos(4t) & t < 0 \\ 0 & t = 0 \end{cases}$$

and the odd component is given by

$$x_0(t)0.5[x(t) - x(-t)] = \begin{cases} \cos(4t) & t > 0 \\ -\cos(4t) & t < 0 \\ 0 & t = 0 \end{cases}$$

which when added together become the given signal. If $x(0) = 2$, we have

$$x_e(t) = 0.5[x(t) + x(-t)] = \begin{cases} \cos(4t) & t > 0 \\ \cos(4t) & t < 0 \\ 2 & t = 0 \end{cases}$$

while the odd component is the same. The even component has a discontinuity at $t = 0$.

Unit Step Function with MATLAB

```
function y = ustep(t,ad)
% generation of unit step
% t : time
% ad : advance (positive),
% delay (negative)
% USE y = ustep(t,ad)
N = length(t); y = zeros(1,N);
for i = 1:N,
    if t(i) >= -ad, y(i) = 1;
    end
end
```

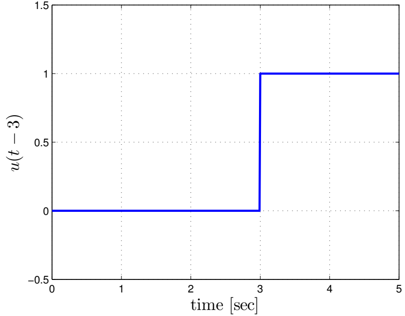


Figure: Using commands `t=0:5`, `y = ustep(t,-3)` and `stairs(t,y)`

Unit Step Function with Scilab

To use Scilab function, you need to load it first with `exec("ustep.sce")`.

```
function y = ustep(t,ad)
// generation of unit step
// t : time
// ad : advance (positive)
// delay (negative)
// USE y = ustep(t,ad)
N = length(t); y = zeros(1,N);
for i = 1:N,
    if t(i) >= -ad, y(i) = 1;
    end
end
endfunction
```

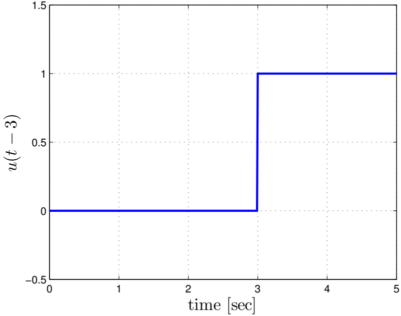


Figure: Using commands `t=0:5`, `y = ustep(t,-3)` and `plot2d2(t,y)`

Ramp Function with MATLAB

```
function y = ramp(t,m,ad)
% generation of ramp
% t : time
% ad : advance (positive), delay (negative
% USE y = ramp(t,m,ad)
N = length(t);
y = zeros(1,N);
for i = 1:N,
   if t(i) >= -ad,
   y(i) = m*(t(i)+ad);
   end
end
```

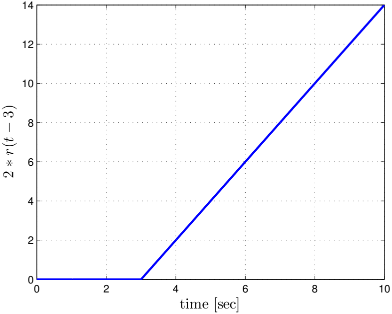


Figure: Using commands `t=0:10`, `y = ramp(t,2,-3)` and `plot(t,y)`

Ramp Function with Scilab

```
function y = ramp(t,m,ad)
// generation of ramp
// t : time
// ad : advance (positive), delay (negativ
// USE y = ramp(t,m,ad)
N = length(t);
y = zeros(1,N);
for i = 1:N,
   if t(i) >= -ad,
   y(i) = m*(t(i)+ad);
   end
end
endfunction
```

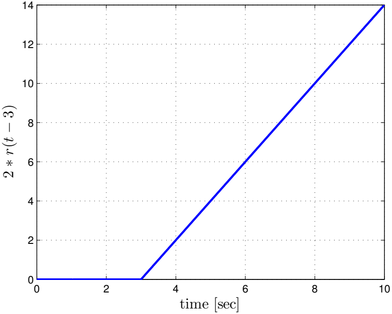


Figure: Using commands `t=0:10`, `y = ramp(t,2,-3)` and `plot(t,y)`

Even/Odd Function with MATLAB

```
function [ye,yo] = evenodd(t,y)
% even/odd decomposition
% t : time
% y : analog signal
% ye, yo: even and odd components
% USE [ye,yo] = evenodd(t,y)
yr = fliplr(y);
ye = 0.5*(y+yr);
yo = 0.5*(y-yr);
```

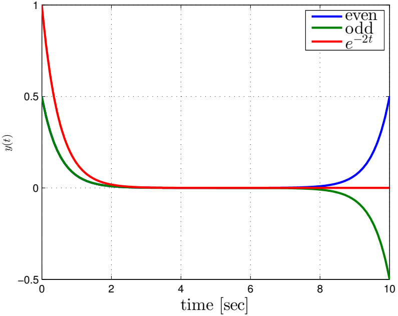


Figure: Using commands `t = 0:0.1:10`, `y= exp(-2*t)`, `[ye,yo]=evenodd(t,y)`

Even/Odd Function with Scilab

```
function [ye,yo] = evenodd(t,y)
// even/odd decomposition
// t : time
// y : analog signal
// ye, yo: even and odd components
// USE [ye,yo] = evenodd(t,y)
yr = mtlb_fliplr(y);
ye = 0.5*(y+yr);
yo = 0.5*(y-yr);
```

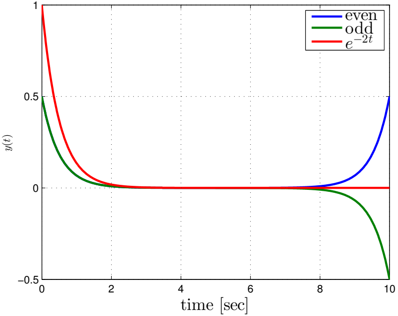
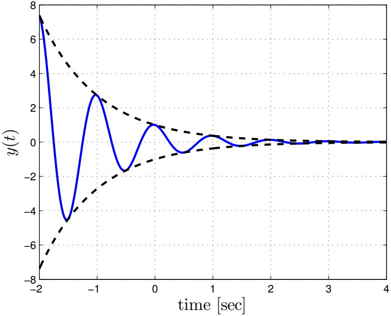


Figure: Using commands `t = 0:0.1:10`, **y= exp(-2*t)**, `[ye,yo]=evenodd(t,y)`

Use MATLAB to generate the following analog signals.

(a) For the damped sinusoid signal $x(t) = e^{-t}\cos(2\pi t)$ obtain a script to generate $x(t)$ and its envelope.

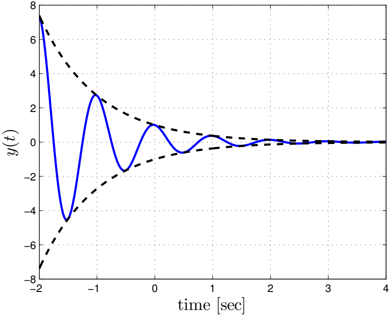
```
% damped sinusoid
  t = -2:0.01:4;
  x = exp(-t).*cos(2*pi*t);
  y = exp(-t);
  plot(t,x,'b-','linewidth',2);
  grid
  hold on
  plot(t,y,'--k','linewidth',2);
  hold on
  plot(t,-y,'--k','linewidth',2);
  axis([-2 4 -8 8]);
  hold off
```



Use Scilab to generate the following analog signals.

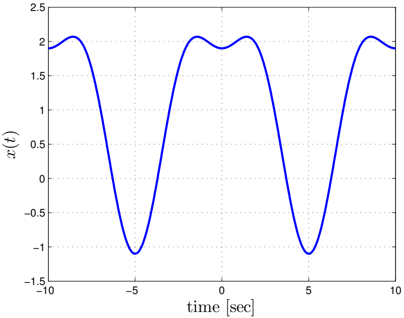
(a) For the damped sinusoid signal $x(t) = e^{-t}\cos(2\pi t)$ obtain a script to generate $x(t)$ and its envelope.

```
// damped sinusoid
  t = -2:0.01:4;
// using %pi instead of pi
  x = exp(-t).*cos(2*%pi*t);
  y = exp(-t);
  plot(t,x,'b-','linewidth',2);
  xgrid
  mtlb_hold on
  plot(t,y,'--k','linewidth',2);
  hold on
  plot(t,-y,'--k','linewidth',2);
  mtlb_axis([-2 4 -8 8]);
  mtlb_hold off
```



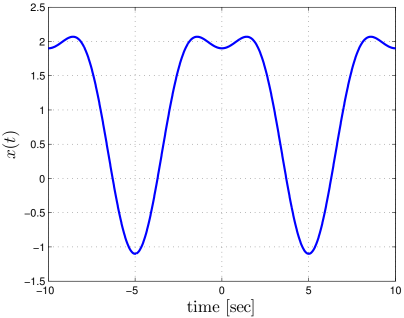
(b) For a rough approximation of a periodic pulse generated by adding three cosines of frequencies multiples of $\Omega_0 = pi/10$–that is $x(t) = 1 + 1.5\cos(2\Omega_0 t) - 0.6\cos(4\Omega_0 t)$ write a script to generate $x_1(t)$.

```
% weigthed cosines approximating a pulse
t = -10:0.01:10;
x = 1 + 1.5*cos(2*pi*t/10)...
    -.6*cos(4*pi*t/10);
plot(t,x,'linewidth',2);
grid;
```



(b) For a rough approximation of a periodic pulse generated by adding three cosines of frequencies multiples of $\Omega_0 = pi/10$–that is $x(t) = 1 + 1.5\cos(2\Omega_0 t) - 0.6\cos(4\Omega_0 t)$ write a script to generate $x_1(t)$.

```
% weigthed cosines approximating a pulse
t = -10:0.01:10;
x = 1 + 1.5*cos(2*%pi*t/10)...
    -.6*cos(4*%pi*t/10);
plot(t,x,'linewidth',2);
xgrid;
```



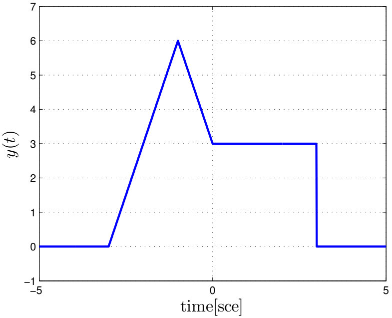
Functions with MATLAB

Example

Write a script and the necessary function to generate a signal,

$$y(t) = 3(t+3) - 6(t+1) + 3t - 3u(t-3).$$

```
% ex 1.16
clear all; clf;
Ts = 0.01; t = -5:Ts:5;
% ramp with support [-5,5]
% slope of 3 and advanced
% with respect to the origin by 3
y1 = ramp(t,3,3);
y2 = ramp(t,-6,1);
y3 = ramp(t,3,0)
% unit-step function with support
% [-5,5] delayed by 3
y4 = -3*ustep(t,-3);
y = y1 + y2 + y3 + y4;
plot(t,y,'linewidth',2); axis([-5,5,-1,7])
grid;
```



Reference

1. Luis F. Chaparro, *Signals and Systems using MATLAB*, Academic Press, 2011
2. Lecture note on *Signals and Systems* Boyd, S., Stanford, USA.
3. Lathi, B. P., *Signal Processing & Linear Systems*, Berkeley-Cambridge Press, 1998
4. Chaisawadi, A.*Signals and Systems*, The Engineering Institute of Thailand, 2543 (in thai)