Lecture 10: Discrete-Time System Analysis Using the Z-Transform

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Outline

- Discrete-Time System Equations
- E operator
- Response of Linear Discrete-Time Systems
- Useful Signal Operations
- Examples

z-Transform

The z-transform is defined by

$$F[z] = \sum_{n=-\infty}^{\infty} f[n] z^{-n}$$

$$f[n] = \frac{1}{2\pi j} \oint F[z] z^{n-1} \, dz$$

We are restricted only to the analysis of causal systems with causal input. In **the unilateral z-transform**, the signals are restricted to be causal; that is, they start at $n = 0$. The unilateral z-transform is defined by

$$F[z] = \sum_{n=0}^{\infty} f[n] z^{-n},$$

where z is complex in general.

z-Transform
Examples

Find the z-transform of a signal $\gamma^n \mathbb{1}[n]$.

$$F[z] = \sum_{n=0}^{\infty} \gamma^n \mathbb{1}[n] z^{-n} = \sum_{n=0}^{\infty} \gamma^n z^{-n}$$
$$= 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots$$

From the well-known geometric progression and its sum:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \text{ if } |x| < 1$$

we have

$$F[z] = \frac{1}{1 - \frac{\gamma}{z}}, \quad \left|\frac{\gamma}{z}\right| < 1$$
$$= \frac{z}{z-\gamma}, \quad |z| > |\gamma|$$

Find the z-transforms of (a) $\delta[n]$, (b) $\mathbb{1}[n]$, (c) $\cos \beta k \mathbb{1}[n]$
By definition

$$
\begin{aligned}
F[z] &= \sum_{n=0}^{\infty} f[n] z^{-n} \\
&= f[0] + \frac{f[1]}{z} + \frac{f[2]}{z^2} + \frac{f[3]}{z^3} + \cdots
\end{aligned}
$$

(a) For $f[n] = \delta[n]$, $f[0] = 1$, and $f[2] = f[3] = f[4] = \cdots = 0$. Therefore

$$\delta[n] \overset{\mathcal{Z}}{\longleftrightarrow} 1 \quad \text{for all } z$$

(b) For $f[n] = \mathbb{1}[n], f[0] = f[1] = f[3] = \cdots = 1$. Therefore

$$
\begin{aligned}
F[z] &= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots = \frac{1}{1 - \frac{1}{z}} \qquad \left|\frac{1}{z}\right| < 1 \\
&= \frac{z}{z-1} \qquad |z| > 1
\end{aligned}
$$

Therefore

$$\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-1} \qquad\qquad |z| > 1.$$

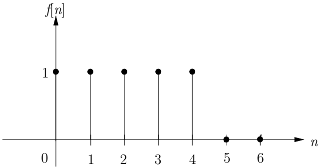
(c) Recall that $\cos \beta k = \left(e^{j\beta n} + e^{-j\beta n}\right)/2$. Moreover,

$$e^{\pm j\beta k}\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z - e^{\pm j\beta}} \qquad\qquad |z| > |e^{\pm j\beta}| = 1$$

Therefore

$$\begin{aligned} F[z] &= \frac{1}{2}\left[\frac{z}{z - e^{j\beta}} + \frac{z}{z - e^{-j\beta}}\right] \\ &= \frac{z(z - \cos\beta)}{z^2 - 2z\cos\beta + 1} \qquad\qquad |z| > 1 \end{aligned}$$

Find the z-transforms of a signal shown in Figure below



Here $f[0] = f[1] = f[2] = f[3] = f[4] = 1$ and $f[5] = f[6] = \cdots = 0$. Therefore,

$$F[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}$$
$$= \frac{z^4 + z^3 + z^2 + z + 1}{z^4}$$

or

$$F[z] = \frac{\left(\frac{1}{z}\right)^5 - \left(\frac{1}{z}\right)^0}{\frac{1}{z} - 1} = \frac{z}{z-1}(1 - z^{-5})$$

Inverse z-Transform

Many of the transforms $F[z]$ of practical interest are rational functions. Such functions can be expressed as a sum of simpler functions using partial fraction expansion.

Find the inverse z-transform of

$$F[z] = \frac{8z - 19}{(z-2)(z-3)}$$

Expanding $F[z]$ into partial fractions yields

$$F[z] = \frac{8z - 19}{(z-2)(z-3)} = \frac{3}{z-2} + \frac{5}{z-3}$$

From z-transform Table Pair 6, we have

$$f[n] = \left[3(2)^{n-1} + 5(3)^{n-1}\right] \mathbb{1}[n-1]$$

This result is not convenient. We prefer the form that in multiplied by $\mathbb{1}[n]$ rather than $\mathbb{1}[n-1]$. We will expand $F[z]/z$ instead of $F[z]$. For this case

$$\frac{F[z]}{z} = \frac{8z - 19}{z(z-2)(z-3)} = \frac{(-19/6)}{z} + \frac{(3/2)}{z-2} + \frac{(5/3)}{z-3}$$

Inverse z-Transform

Examples

Multiplying both sides by z yields

$$F[z] = -\frac{19}{6} + \frac{3}{2}\left(\frac{z}{z-2}\right) + \frac{5}{3}\left(\frac{z}{z-3}\right)$$

From Pairs 1 and 7 in the z-transform table, it follows that

$$f[n] = -\frac{19}{6}\delta[n] + \left[\frac{3}{2}(2)^n + \frac{5}{3}(3)^n\right]\mathbb{1}[n]$$

Find the inverse z-transform of

$$F[z] = \frac{z(2z^2 - 11z + 12)}{(z-1)(z-2)^3}$$

and

$$\begin{aligned} \frac{F[z]}{z} &= \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} \\ &= \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)} \end{aligned}$$

Using a cover up method yields

$$\begin{aligned} k &= \left.\frac{2z^2 - 11z + 12}{(z-2)^3}\right|_{z=1} = -3 \\ a_0 &= \left.\frac{2z^2 - 11z + 12}{(z-1)}\right|_{z=2} = -2 \end{aligned}$$

Therefore

$$\frac{F[z]}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

By using short cuts method, we multiply both sides of the equation by z and let $z \to \infty$. This yields

$$0 = -3 - 0 + 0 + a_2 \Longrightarrow a_2 = 3$$

Another unknown a_1 is readily determined by letting z take any convenient value, $z = 0$, on both sides. This step yields

$$\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{3}{2} \Longrightarrow a_1 = -1$$

Therefore

$$\frac{F[z]}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{z-2}$$

Inverse z-Transform

Examples

$$F[z] = -3\frac{z}{z-1} - 2\frac{z}{(z-2)^3} - \frac{z}{(z-2)^2} + 3\frac{z}{z-2}$$

Now the use of Table, Pairs 7 and 10 yields

$$\begin{aligned} f[n] &= \left[-3 - 2\frac{n(n-1)}{8}(2)^n - \frac{n}{2}(2)^n + 3(2)^n\right]\mathbb{1}[n] \\ &= -\left[3 + \frac{1}{4}(n^2 + n - 12)2^n\right]\mathbb{1}[n] \end{aligned}$$

Complex Poles

$$F[z] = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$$

Method of First-Order Factors

$$\frac{F[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25} = \frac{2(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

We find the partial fraction of $F[z]/z$ using the "cover up" method:

$$\frac{F[z]}{z} = \frac{2}{z-1} + \frac{1.6e^{-j2.246}}{z-3-j4} + \frac{1.6e^{j2.246}}{z-3+j4}$$

and

$$F[z] = 2\frac{z}{z-1} + (1.6e^{-j2.246})\frac{z}{z-3-j4} + (1.6e^{j2.246})\frac{z}{z-3+j4}$$

The inverse transform of the first time on the right-hand side is $2\mathbb{1}[n]$. The inverse transform of the remeaining two terms can be obtained from z-Transform Table Pair 12b by identifying $\frac{r}{2} = 1.6$, $\theta = -2.246$ rad, $\gamma = 3 + j4 = 5e^{j0.927}$, so that $|\gamma| = 5$, $\beta = 0.927$. Therefore

$$f[n] = [2 + 3.2(5)^n \cos(0.927k - 2.246)]\mathbb{1}[n]$$

Method of Quadratic Factors

$$\frac{F[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

Multiplying both sides by z and letting $z \to \infty$, we find

$$0 = 2 + A \implies A = -2$$

and

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

To find B we let z take any convenient value, say $z = 0$. This step yields

$$\frac{-34}{25} = -2 + \frac{B}{25}$$

Multiplying both sides by 25 yields

$$-34 = -50 + B \Longrightarrow B = 16$$

Therefore

$$\frac{F[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25}$$

and

$$F[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

Inverse z-Transform

Examples

We now usr z-Transform table Pair 12c where we identify $A = -2$, $B = 16$, $|\gamma| = 5$, $a = -3$. Therefore

$$r = \sqrt{\frac{100 + 256 - 192}{25 - 9}} = 3.2, \; \beta = \cos^{-1}\left(\frac{3}{5}\right) = 0.927 \text{ rad}$$

and

$$\theta = \tan^{-1}\left(\frac{-10}{-8}\right) = -2.246 \text{ rad.}$$

so that

$$f[n] = [2 + 3.2(5)^n \cos(0.927n - 2.246)]\, \mathbb{1}[n]$$

Right Shift (Delay)
If

$$f[n]\mathbb{1}[n] \Longleftrightarrow F[z]$$

then

$$f[n-1]\mathbb{1}[n-1] \Longleftrightarrow \frac{1}{z}F[z]$$

and

$$f[n-m]\mathbb{1}[n-m] \Longleftrightarrow \frac{1}{z^m}F[z]$$

and

$$f[n-1]\mathbb{1}[n] \Longleftrightarrow \frac{1}{z}F[z] + f[-1]$$

Repeated application of this property yields

$$f[n-2]\mathbb{1}[n] \Longleftrightarrow \frac{1}{z}\left[\frac{1}{z}F[z] + f[-1]\right] + f[-2]$$

$$f[n-2]\mathbb{1}[n] = \frac{1}{z^2}F[z] + \frac{1}{z}f[-1] + f[-2]$$

and

$$f[n-m]\mathbb{1}[n] \Longleftrightarrow z^{-m}F[z] + z^{-m}\sum_{n=1}^{m} f[-n]z^{n}$$

Proof:

$$\mathcal{Z}\left\{f[n-m]\mathbb{1}[n-m]\right\} = \sum_{n=0}^{\infty} f[n-m]\mathbb{1}[n-m]z^{-n}$$

Recall that $f[n-m]\mathbb{1}[n-m] = 0$ for $k < m$, so that the limits on the summation on the right-hand side can be taken form $n = m$ to ∞.

Some properties of the z-Transform

Right-Shifty (Delay) cont.

$$\mathcal{Z}\left\{f[n-m]\mathbb{1}[n-m]\right\} = \sum_{n=m}^{\infty} f[n-m]z^{-n}$$
$$= \sum_{r=0}^{\infty} f[r]z^{-(r+m)} = \frac{1}{z^m}\sum_{r=0}^{\infty} f[r]z^{-r} = \frac{1}{z^m}F[z]$$

$$\mathcal{Z}\left\{f[n-m]\mathbb{1}[n]\right\} = \sum_{n=0}^{\infty} f[n-m]z^{-n} = \sum_{r=-m}^{\infty} f[r]z^{-(r+m)}$$
$$= z^{-m}\left[\sum_{r=-m}^{-1} f[r]z^{-r} + \sum_{r=0}^{\infty} f[r]z^{-r}\right]$$
$$= z^{-m}\sum_{n=1}^{m} f[-n]z^{n} + z^{-m}F[z]$$

If

$$f[n]\mathbb{1}[n] \Longleftrightarrow F[z]$$

then

$$f[n+1]\mathbb{1}[n] \Longleftrightarrow zF[z] - zf[0]$$

Repeated application of this property yields

$$\begin{aligned} f[n+2]\mathbb{1}[n] \Longleftrightarrow & z\{z(F[z] - zf[0]) - f[1]\} \\ &= z^2F[z] - z^2f[0] - zf[1] \end{aligned}$$

and

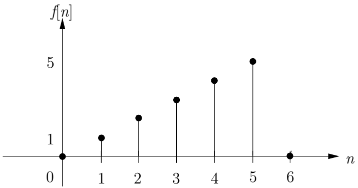
$$f[n+m]\mathbb{1}[n] \Longleftrightarrow z^mF[z] - z^m\sum_{n=0}^{m-1} f[n]z^{-n}$$

Some properties of the z-Transform
Left-Shifty (Advance) cont.

Proof: By definition

$$\begin{aligned}
\mathcal{Z}\left\{f[n+m]\mathbb{1}[n]\right\} &= \sum_{n=0}^{\infty} f[n+m]z^{-n} \\
&= \sum_{r=m}^{\infty} f[r]z^{-(r-m)} \\
&= z^m \sum_{r=m}^{\infty} f[r]z^{-r} \\
&= z^m \left[\sum_{r=0}^{\infty} f[r]z^{-r} - \sum^{m-1} r = 0 f[r]z^{-r}\right] \\
&= z^m F[z] - z^m \sum_{r=0}^{m-1} f[r]z^{-r}
\end{aligned}$$

Find the z-transform of the signal $f[n]$ depicted in a Figure below.



The signal can be expressed as a product of k and a gate pulse $\mathbb{1}[n] - \mathbb{1}[n-6]$. Therefore

$$f[n] = n\mathbb{1}[n] - n\mathbb{1}[n-6] = n\mathbb{1}[n] - (n-6+6)\mathbb{1}[n-6]$$
$$= n\mathbb{1}[n] - (n-6)\mathbb{1}[n-6] + 6\mathbb{1}[n-6]$$

Because $\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-1}$ and $k\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{(z-1)^2}$,

$$\mathbb{1}[n-6] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z^6}\frac{z}{z-1} = \frac{1}{z^5(z-1)}, \text{ and } (n-6)\mathbb{1}[n-6] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z^6}\frac{z}{(z-1)^2} = \frac{1}{z^5(z-1)^2}$$

Some properties of the z-Transform

Left-Shifty (Advance) cont.

Therefore

$$F[z] = \frac{z}{(z-1)^2} - \frac{1}{z^5(z-1)^2} - \frac{6}{z^5(z-1)}$$
$$= \frac{z^6 - 6z + 5}{z^5(z-1)^2}$$

Some properties of the z-Transform
Convolution

The time convolution property state that if

$$f_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} F_1[n] \text{ and } f_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} F_2[z],$$

then (**time convolution**)

$$f_1[n] * f_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} F_1[z]F_2[z]$$

Proof:

$$\begin{aligned} \mathcal{Z}\left\{f_1[n] * f_2[n]\right\} &= \mathcal{Z}\left[\sum_{m=-\infty}^{\infty} f_1[m]f_2[n-m]\right] \\ &= \sum_{n=-\infty}^{\infty} z^{-n} \sum_{m=-\infty}^{\infty} f_1[m]f_2[n-m] \end{aligned}$$

Some properties of the z-Transform

Convolution cont.

Interchanging the order of summation,

$$\begin{aligned}
\mathcal{Z}\left[f_1[n] * f_2[n]\right] &= \sum_{m=-\infty}^{\infty} f_1[m] \sum_{n=-\infty}^{\infty} f_2[n-m] z^{-n} \\
&= \sum_{m=-\infty}^{\infty} f_1[m] \sum_{r=-\infty}^{\infty} f_2[r] z^{-(r+m)} \\
&= \sum_{m=-\infty}^{\infty} f_1[m] z^{-m} \sum_{r=-\infty}^{\infty} f_2[r] z^{-r} \\
&= F_1[z] F_2[z]
\end{aligned}$$

Some properties of the z-Transform

Multiplication by γ^n

If

$$f[n]\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} F[z]$$

then

$$\gamma^n f[n]\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} F\left[\frac{z}{\gamma}\right]$$

Proof:

$$\mathcal{Z}\left\{\gamma^n f[n]\mathbb{1}[n]\right\} = \sum_{n=0}^{\infty} \gamma^n f[n] z^{-n} = \sum_{n=0}^{\infty} f[n]\left(\frac{z}{\gamma}\right)^{-n} = F\left[\frac{z}{\gamma}\right]$$

Some properties of the z-Transform

Multiplication by n

If

$$f[n]\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} F[z]$$

then

$$kf[n]\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}F[z]$$

Proof:

$$-z\frac{d}{dz}F[z] = -z\frac{d}{dz}\sum_{n=0}^{\infty} f[n]z^{-n} = -z\sum_{n=0}^{\infty} -nf[n]z^{-n-1}$$

$$= \sum_{n=0}^{\infty} kf[n]z^{-n} = \mathcal{Z}\left\{kf[n]\mathbb{1}[n]\right\}$$

z-Transform Solution of Linear Difference Equations

- The time-shift (left- or right-shift) property has set the stage for solving linear difference equations with constant coefficients.
- As in the case of the Laplace transform with differential equations, the z-transform converts difference equations into algebraic equations which are readily solved to find the solution in the z-domain.
- Taking the inverse z-transform of the z-domain solution yields the desired time-domain solution.

Solve

$$y[n+2] - 5y[n+1] + 6y[n] = 3f[n+1] + 5f[n]$$

if the initial conditions are $y[-1] = \frac{11}{6}$, $y[-2] = \frac{37}{36}$, and the input $f[n] = (2)^{-n}\mathbb{1}[n]$. Since the given initial conditions are not suitable for the forward form, we transform the equation to the delay form:

$$y[n] - 5y[n-1] + 6y[n-2] = 3f[n-1] + 5f[n-2].$$

Clearly that we consider the solution when $k \geq 0$ then $y[n-j]$ means $y[n-j]\mathbb{1}[n]$. Now

$$\begin{aligned}
y[n]\mathbb{1}[n] &\overset{\mathcal{Z}}{\longleftrightarrow} Y[z] \\
y[n-1]\mathbb{1}[n] &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z}Y[z] + y[-1] = \frac{1}{z}Y[z] + \frac{11}{6} \\
y[n-2]\mathbb{1}[n] &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z^2}Y[z] + \frac{1}{z}y[-1] + y[-2] = \frac{1}{z^2}Y[z] + \frac{11}{6z} + \frac{37}{36}
\end{aligned}$$

z-Transform Solution of Linear Difference Equations

Examples cont.

Also

$$f[n] = (2)^{-n}\mathbb{1}[n] = (2^{-1})\mathbb{1}[n] = (0.5)^n\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-0.5}$$

$$f[n-1]\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z}F[z] + f[-1] = \frac{1}{z}\frac{z}{z-0.5} + 0 = \frac{1}{z-0.5}$$

$$f[n-2]\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z^2}F[z] + \frac{1}{z}f[-1] + f[-2] = \frac{1}{z^2}F[z] + 0 + 0 = \frac{1}{z(z-0.5)}$$

Note that for causal input $f[n]$,

$$f[-1] = f[-2] = \cdots = f[-n] = 0$$

Hence

$$f[n-r]\mathbb{1}[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z^r}F[z]$$

Taking the z-transform of the difference equation and substituting the above results, we obtain

$$Y[z] - 5\left[\frac{1}{z}Y[z] + \frac{11}{6}\right] + 6\left[\frac{1}{z^2}Y[z] + \frac{11}{6z} + \frac{37}{36}\right] = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

or

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)Y[z] - \left(3 - \frac{11}{z}\right) = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

and

$$\begin{aligned}\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)Y[z] &= \left(3 - \frac{11}{z}\right) + \frac{3z+5}{z(z-0.5)} \\ &= \frac{3z^2 - 9.5z + 10.5}{z(z-0.5)}\end{aligned}$$

Multiplication of both sides by z^2 yields

$$(z^2 - 5z + 6)\,Y[z] = \frac{z(3z^2 - 9.5z + 10.5)}{(z - 0.5)}$$

so that

$$Y[z] = \frac{z(3z^2 - 9.5z + 10.5)}{(z - 0.5)(z^2 - 5z + 6)}$$

and

$$\begin{aligned}\frac{Y[z]}{z} &= \frac{3z^2 - 9.5z + 10.5}{(z - 0.5)(z - 2)(z - 3)} \\ &= \frac{(26/15)}{z - 0.5} - \frac{(7/3)}{z - 2} + \frac{(18/5)}{z - 3}\end{aligned}$$

Therefore

$$Y[z] = \frac{26}{15}\left(\frac{z}{z - 0.5}\right) - \frac{7}{3}\left(\frac{z}{z - 2}\right) + \frac{18}{5}\left(\frac{z}{z - 3}\right)$$

and

$$y[n] = \left[\frac{26}{15}(0.5)^n - \frac{7}{3}(2)^n + \frac{18}{5}(3)^n\right] \mathbb{1}[n]$$

- From the previous example, we found the total solution of the difference equation.
- It is easy to separate the solution into zero-input and zero-state components.
- We have to separate the response into terms arising from the input and terms arising from initial conditions.

From the previous example:

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right) Y[z] - \underbrace{\left(3 - \frac{11}{z}\right)}_{\text{initial condition terms}} = \underbrace{\frac{3}{z-0.5} + \frac{5}{z(z-0.5)}}_{\text{terms arising from input}}$$

z-Transform Solution of Linear Difference Equations

Zero-Input and Zero-State Components cont.

Therefore

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right) Y[z] = \underbrace{\left(3 - \frac{11}{z}\right)}_{\text{initial condition terms}} + \underbrace{\frac{(3z+5)}{z(z-0.5)}}_{\text{input terms}}$$

Multiplying both sides by z^2 yields

$$(z^2 - 5z + 6)\, Y[z] = \underbrace{z(3z-11)}_{\text{initial condition terms}} + \underbrace{\frac{z(3z+5)}{z-0.5}}_{\text{input terms}}$$

and

$$Y[z] = \underbrace{\frac{z(3z-11)}{z^2 - 5z + 6}}_{\text{zero-input response}} + \underbrace{\frac{z(3z+5)}{(z-0.5)(z^2 - 5z + 6)}}_{\text{zero-state response}}$$

z-Transform Solution of Linear Difference Equations

Zero-Input and Zero-State Components cont.

We expand both terms on the right-hand side into modified partial fractions to yield

$$Y[z] = \underbrace{\left[5\left(\frac{z}{z-2}\right)\right]}_{\text{zero-input}} + \underbrace{\left[\frac{26}{15}\left(\frac{z}{z-0.5}\right) - \frac{22}{3}\left(\frac{z}{z-2}\right) + \frac{28}{5}\left(\frac{z}{z-3}\right)\right]}_{\text{zero-state}}$$

and

$$\begin{aligned} y[n] &= \left[\underbrace{5(2)^n - 2(3)^n}_{\text{zero-input}} \underbrace{- \frac{22}{3}(2)^n + \frac{28}{5}(3)^n + \frac{26}{15}(0.5)^n}_{\text{zero-state}}\right] \mathbb{1}[n] \\ &= \left[-\frac{7}{3}(2)^n + \frac{18}{5}(3)^n + \frac{26}{15}(0.5)^n\right] \mathbb{1}[n] \end{aligned}$$

Reference

1. Lathi, B. P., *Signal Processing & Linear Systems*, Berkeley-Cambridge Press, 1998.