RC2017 Signals and Systems | **Homework 5**
Assign: Thu 21 Feb 2019 | **Due for RC2: Wed 27 Feb 2019**

Instruction: Hand in your work with name and code to my desk by 10.00 am. of the due date. DO NOT copy homework from your classmates or lend it to others. Anyone who violates this regulation will be given -10 for the homework.

1. An LTIC system is specified by the block diagram in Fig. 1 For each input $f(t)$ and

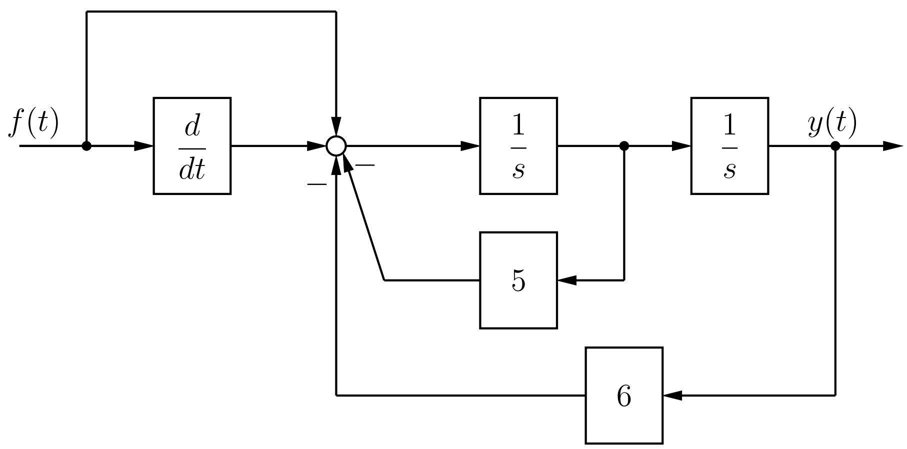

Figure 1: a block diagram of the LTIC system.

the initial values listed below, find zero-input , zero-state and total responses of the system.

a. $f(t) = 6t^2$, $y(0) = 2$ and $\dot{y}(0) = -1$. (5 points)
Solution: From the block diagram the system description is

$$\left(D^2 + 5D + 6\right) = (D + 1)\, f(t)$$

The zero-input response is follow from:

$$\lambda^2 + 5\lambda + 6 = 0$$
$$\lambda = -2, -3$$
$$y_0(t) = C_1 e^{-2t} + C_2 e^{-3t}, t \geq 0$$

By using given initial condition, we obtain

$$C_1 + C_2 = 2 \text{ and } -2C_1 - 3C_2 = -1$$

,then $C_1 = 5$ and $C_2 = -3$. Finally

$$y_0(t) = 5e^{-2t} - 3e^{-3t}, t \geq 0$$

The impulse response $h(t)$ is

$$h(t) = b_m\delta(t) + [P(D)y_n(t)]\mathbb{1}(t).$$

$y_n(t)$ has the same form like $y_0(t)$ but $y_n(0) = 0$ and $\dot{y}_n(0) = 1$, then

$$\begin{aligned} y_n(t) &= 2e^{-2t} - e^{-3t}, t > 0 \\ P(D)y_n(t) = (D+1)y_n(t) &= -e^{-2t} + 2e^{-3t} \end{aligned}$$

Since $b_m = 0$ in this case, hence

$$h(t) = -e^{-2t} + 2e^{-3t}, t \geq 0$$

The zero-state response is

$$y_s(t) = h(t) * f(t) = \left(-e^{-2t} + 2e^{-3t}\right) * 6t^2$$

Using a convolution table in row 7, we have

$$\begin{aligned} y_s(t) &= -6\left[\frac{2!e^{-2t}}{(-2)^3}u(t) - \sum_{j=0}^{2}\frac{2!t^{2-j}}{(-2)^{j+1}(2-j)!}u(t)\right] \\ &\quad + 12\left[\frac{2!e^{-3t}}{(-3)^3}u(t) - \sum_{j=0}^{2}\frac{2!t^{2-j}}{(-3)^{j+1}(2-j)!}\right] \\ &= -6\left[-\frac{1}{4}e^{-2t} - \left(\frac{2}{(-2)(2)}t^2 + \frac{2}{(-2)^2 1!}t + \frac{2}{(-2)^3 0!}\right)\right]u(t) \\ &\quad + 12\left[-\frac{2}{27}e^{-3t} - \left(\frac{2}{(-3)2}t^2 + \frac{2}{(-3)^2 1!}t + \frac{2}{(-3)^3 0!}\right)\right]u(t) \\ &= \left[\frac{3}{2}e^{-2t} - 3t^2 + 3t - \frac{3}{2}\right]u(t) + \left[-\frac{8}{9}e^{-3t} + 4t^2 - \frac{8}{3}t + \frac{8}{9}\right]u(t) \\ &= \frac{3}{2}e^{-2t} - \frac{8}{9}e^{-3t} + t^2 + \frac{1}{3}t - \frac{11}{18}, t \geq 0 \end{aligned}$$

Finally the total response is

$$y(t) = y_0(t) + y_s(t) = \frac{13}{2}e^{-2t} - \frac{35}{9}e^{-3t} + t^2 + \frac{1}{3}t - \frac{11}{18}, t \geq 0$$

■

b. $f(t) = 4e^{-4t}$, $y(0) = 2$ and $\dot{y}(0) = -1$. (5 points)
Solution: The system and all initial conditions are the same like the previous question so the zero-input and impulse response are the same. The only thing that we have to do is re-calculate the zero-state response as follow:

$$\begin{aligned} y_s(t) &= h(t) * f(t) = \left(-e^{-2t} + 2e^{-3t}\right) * 4e^{-4t} \\ &= -4\frac{e^{-2t} - e^{-4t}}{-2-(-4)} + 8\frac{e^{-3t} - e^{-4t}}{-3-(-4)} \\ &= -2e^{-2t} + 2e^{-4t} + 8e^{-3t} - 8e^{-4t} \\ &= -2e^{-2t} + 8e^{-3t} - 6e^{-4t}, t \geq 0 \end{aligned}$$

Finally, the total response is

$$\begin{aligned} y(t) &= 5e^{-2t} - 3e^{-3t} - 2e^{-2t} + 8e^{-3t} - 6e^{-4t} \\ &= 3e^{-2t} + 5e^{-3t} - 6e^{-4t}, t \geq 0 \end{aligned}$$

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2. Determine the convolution $c(t) = f_1(t) * f_2(t)$, where $f_1(t)$ and $f_2(t)$ are shown in Figure 3

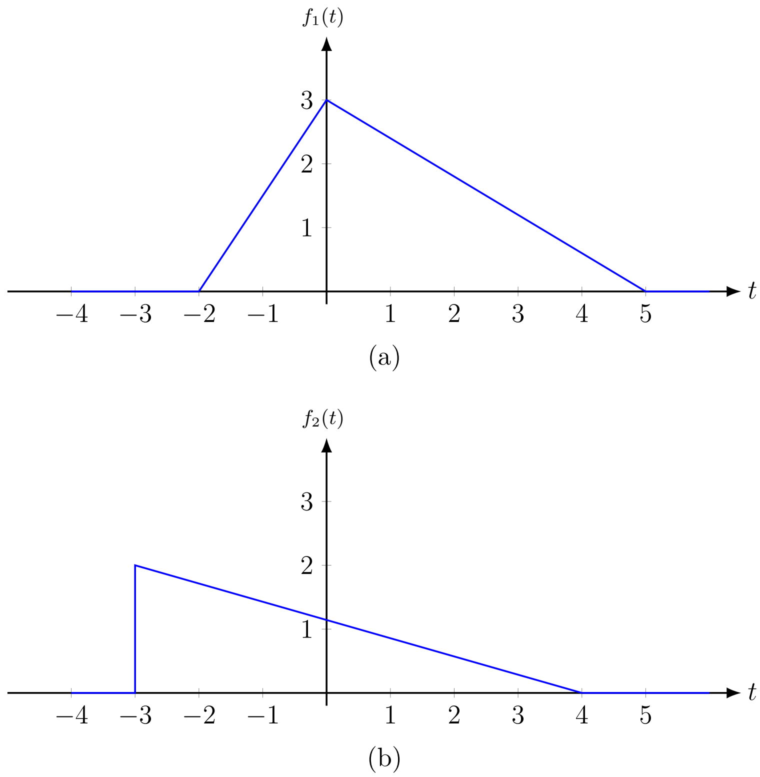


Figure 2: Question 2

Solution: Define the function of each part of the graph as shown in the Figures. In

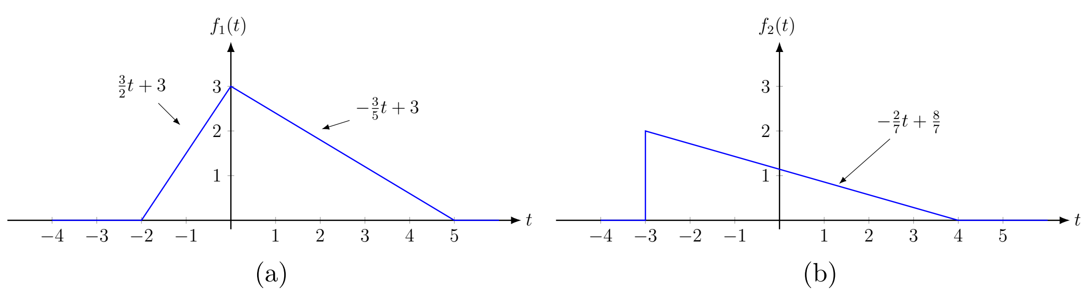
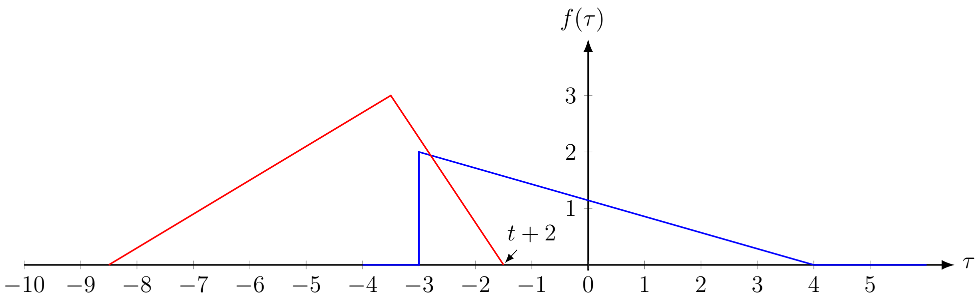


Figure 3: Question 2

our case, the convolution could be separated into six parts: $t < -5$, $-5 \leq t < -3$, $-3 \leq t < 2$, $2 \leq t < 4$, $4 \leq t < 9$, and $t \geq 9$. For simplicity, we use

$$c(t) = f_2(t) * f_1(t) = \int_{-\infty}^{\infty} f_2(\tau) f_1(t - \tau) d\tau$$

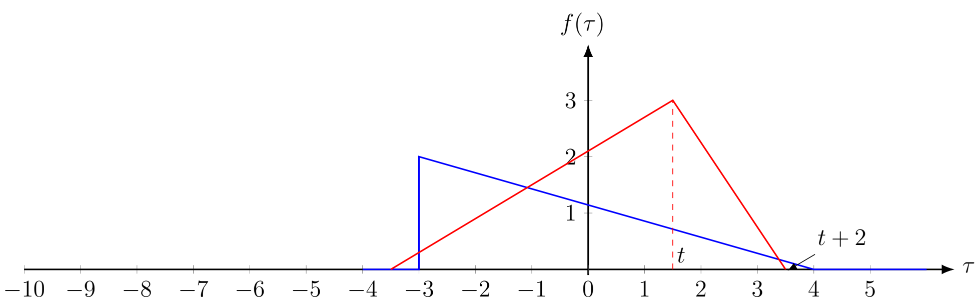


- $-5 \le t < -3$

  The convolution of this period is

$$
\begin{aligned}
c_1(t) &= \int_{-3}^{t+2} \left(-\frac{2}{7}\tau + \frac{8}{7}\right)\left(\frac{3}{2}(t-\tau)+3\right) d\tau \\
&= -\frac{3}{14}t\tau^2 + \frac{12}{7}t\tau + \frac{1}{7}\tau^3 - \frac{9}{7}\tau^2 + \frac{24}{7}\tau \Big|_{-3}^{t+2} \\
&= -\frac{1}{14}t^3 + \frac{3}{7}t^2 + \frac{135}{14}t + \frac{200}{7}
\end{aligned}
$$

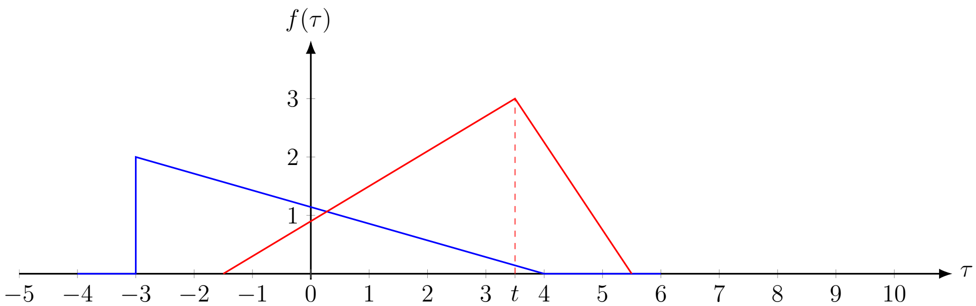
- $-3 \le t < 2$



  The convolution of this period is

$$
\begin{aligned}
c_2(t) &= \int_{-3}^{t} \left(-\frac{2}{7}\tau + \frac{8}{7}\right)\left(\frac{3}{5}(t-\tau)+3\right) d\tau + \int_{t}^{t+2} \left(-\frac{2}{7}\tau + \frac{8}{7}\right)\left(\frac{3}{2}(t-\tau)+3\right) d\tau \\
&= \frac{3}{35}t\tau^2 - \frac{24}{35}t\tau - \frac{2}{35}\tau^3 - \frac{3}{35}\tau^2 + \frac{24}{7}\tau \Big|_{-3}^{t} + \left(-\frac{3}{14}t\tau^2 + \frac{12}{7}t\tau + \frac{1}{7}\tau^3 - \frac{9}{7}\tau^2 + \frac{24}{7}\tau \Big|_{t}^{t+2}\right) \\
&= \frac{1}{35}t^3 - \frac{27}{35}t^2 - \frac{9}{35}t + \frac{433}{35}
\end{aligned}
$$

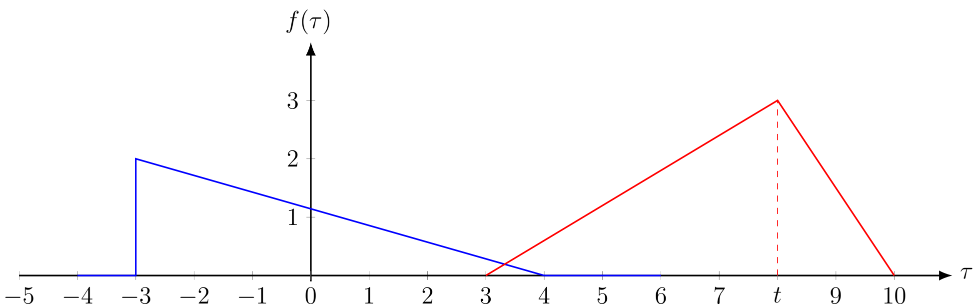
- $2 \le t < 4$



The convolution of this period is

$$
\begin{aligned}
c_3(t) &= \int_{t-5}^{t} \left(-\frac{2}{7}\tau + \frac{8}{7}\right)\left(\frac{3}{5}(t-\tau)+3\right)d\tau + \int_{t}^{4} \left(-\frac{2}{7}\tau + \frac{8}{7}\right)\left(\frac{3}{2}(t-\tau)+3\right)d\tau \\
&= \frac{3}{35}t\tau^2 - \frac{24}{35}t\tau - \frac{2}{35}\tau^3 - \frac{3}{35}\tau^2 + \frac{24}{7}\tau\Big|_{t-5}^{t} + \left(-\frac{3}{14}t\tau^2 + \frac{12}{7}t\tau + \frac{1}{7}\tau^3 - \frac{9}{7}\tau^2 + \frac{24}{7}\tau\Big|_{t}^{4}\right) \\
&= \frac{1}{14}t^3 - \frac{3}{7}t^2 - \frac{15}{7}t + \frac{101}{7}
\end{aligned}
$$

- $4 \le t < 9$



The convolution of this period is

$$
\begin{aligned}
c_4(t) &= \int_{t-5}^{4} \left(-\frac{2}{7}\tau + \frac{8}{7}\right)\left(\frac{3}{5}(t-\tau)+3\right)d\tau \\
&= \frac{3}{35}t\tau^2 - \frac{24}{35}t\tau - \frac{2}{35}\tau^3 - \frac{3}{35}\tau^2 + \frac{24}{7}\tau\Big|_{t-5}^{4} \\
&= -\frac{1}{35}t^3 + \frac{27}{35}t^2 - \frac{243}{35} + \frac{729}{35}
\end{aligned}
$$

$$c(t) = \begin{cases} 0 & , t < -5 \\ -\frac{1}{14}t^3 + \frac{3}{7}t^2 + \frac{135}{14}t + \frac{200}{7} & , -5 \leq t < -3 \\ \frac{1}{35}t^3 - \frac{27}{35}t^2 - \frac{9}{35}t + \frac{433}{35} & , -3 \leq t < 2 \\ \frac{1}{14}t^3 - \frac{3}{7}t^2 - \frac{15}{7}t + \frac{101}{7} & , 2 \leq t < 4 \\ -\frac{1}{35}t^3 + \frac{27}{35}t^2 - \frac{243}{35}t + \frac{729}{35} & , 4 \leq t < 9 \\ 0 & , t \geq 9 \end{cases}$$

Python code

```python
import numpy as np
import matplotlib.pyplot as plt

def step(t, Td):
    x = np.zeros(len(t))
    for k, tt in enumerate(t):
        if tt >= Td:
            x[k] = 1.0

    return x

N = 400
lr = -5
hr = 9
t = np.linspace(lr, hr, N)
T = (hr - lr)/N

f1 = ((3/2)*t + 3)*(step(t, -2) - step(t, 0))
f2 = ((-3/5)*t + 3)*(step(t, 0) - step(t, 5))
f3 = ((-2/7)*t + 8/7) * (step(t, -3) - step(t, 4))

ty = np.linspace(2*lr, 2*hr, 2*N-1)
c = np.convolve(f1 + f2, f3, mode='full') * T
plt.figure(2)
plt.plot(ty, c)
plt.grid(True)
plt.xlabel(r"Time[sec]")
plt.ylabel(r"$c(t)$")
plt.axis([-7, 15, 0, 13])
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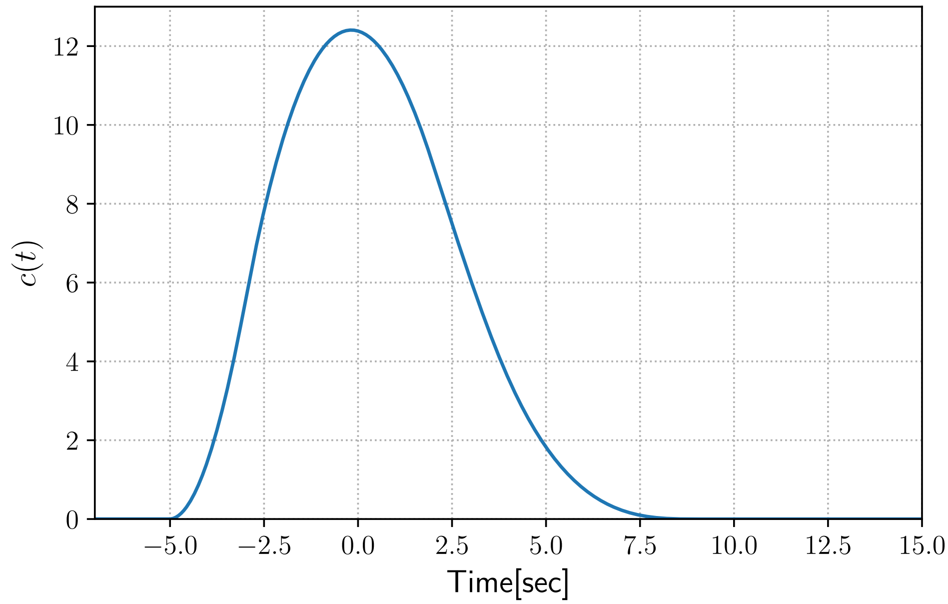

Figure 4: Plot of $c(t)$

3. The circuit shown in Figure 5 has been in the configuration shown (with the switch open) for a long time. At $t = 0$ the switch closes. (a) Find $v_C(0^+)$ and $\dot{v}_C(0^+)$. (b) Find the total response $v_C(t)$ for all t.

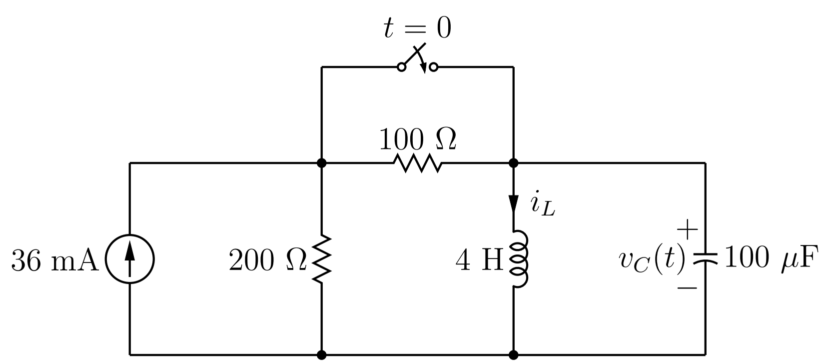

Figure 5: Question 3

a) **Solution**
The circuit at time $t < 0$ is shown in Figure 6.

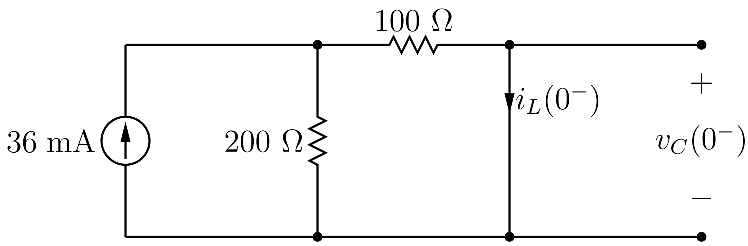

Figure 6: Question 3a

It is clear that $v_C(0^-) = 0$ volt and $i_L(0^-) = 36(200)/(300) = 24$ mA. The circuit at time $t \geq 0$ is shown in Figure 7.

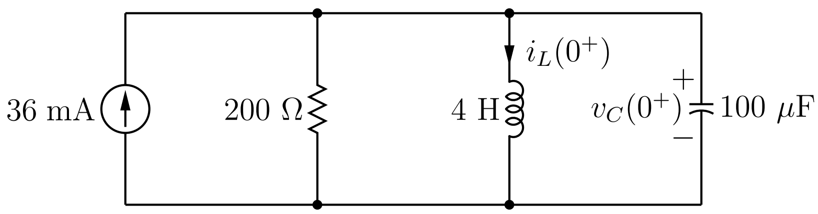

Figure 7: Question 3a2

Since $v_C(t)$ and $i_L(t)$ could not change instantly, then $v_C(0^+) = 0$ volt and $i_L(0^+) = 24$ mA. We have

$$36 \times 10^{-3} = \frac{v_C(0^+)}{200} + i_L(0^+) + 100 \times 10^{-6}\dot{v}_C(0^+)$$
$$\dot{v}_C(0^+) = 120 \text{ volt/sec}$$

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b) From Figure 7, we have

$$i(t) = \frac{1}{200}v_C(t) + \frac{1}{4}\int_{-\infty}^{t} v_C(\tau)d\tau + 100 \times 10^{-6}\frac{d}{dt}v_C(t)$$
$$\frac{d}{dt}i(t) = \frac{1}{200}\frac{d}{dt}v_C(t) + \frac{1}{4}v_C(t) + 100 \times 10^{-6}\frac{d^2}{dt^2}v_C(t)$$

The current source is a constant current source. Hence,

$$(D^2 + 50D + 2500)v_C(t) = 0$$
$$\lambda^2 + 50\lambda + 2500 = 0$$
$$\lambda = -25 \pm j25\sqrt{3}$$

Since the forced response is zero, then the total response is

$$v_C(t) = Ce^{-25t}\cos(25\sqrt{3}t + \theta)$$
$$v_C(0^+) = 0 = C\cos\theta$$
$$\dot{v}_C(0^+) = 120 = -25C\cos\theta - 25\sqrt{3}C\sin\theta \Longrightarrow C\sin\theta = -120/(25\sqrt{3}).$$

We obtain

$$C = \sqrt{0 + (120/(25\sqrt{3}))^2} = 2.771$$
$$\theta = \tan^{-1} -\infty = -90^\circ.$$

Finally, we have

$$v_C(t) = 2.771e^{-25t}\cos(25\sqrt{3}t - 90^\circ) = 2.771e^{-25t}\sin(25\sqrt{3}t)$$

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