
Instruction: Hand in your work with name and code directly to my hand by 10.30 am. of the due date. DO NOT copy homework from your classmates or lend it to others. Anyone who violates this regulation will be given -10 for the homework.

1. Find the differential equation relating a current source $i_s(t) = \cos(\omega t)$ with the current $i_L(t)$ in an inductor with inductance $L = 1$ H, connected in parallel with a resistor of $R = 1\ \Omega$. Assume a zero initial current in the inductor.

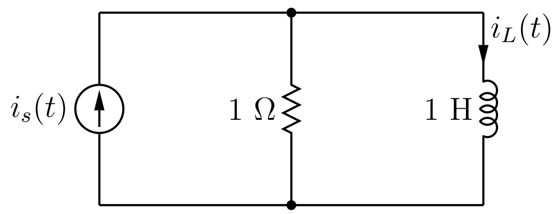

Figure 1: RL circuit: input $i_s(t)$ and output $i_L(t)$

.

(a) Obtain a discrete equation from the differential equation using the trapezoidal approximation of an integral. (5 points)

Solution:

From the circuit we have

$$i_s(t) = \frac{v_L(t)}{R} + i_L(t)$$
$$i_s(t) = \frac{L}{R}\frac{di_L}{dt} + i_L(t) = \frac{di_L}{dt} + i_L(t)$$

By integrating, we have

$$i_L(t_1) - i_L(t_0) = \int_{t_0}^{t_1} i_s(\tau)d\tau - \int_{t_0}^{t_1} i_L(\tau)d\tau$$

Using the formula for the area of a trapezoid we get

$$i_L(t_1) - i_L(t_0) = [i_s(t_1) + i_s(t_0)]\frac{\Delta t}{2} - [i_L(t_1) + i_L(t_0)]\frac{\Delta t}{2}$$
$$i_L(t_1)\left[1 + \frac{\Delta t}{2}\right] = [i_s(t_1) + i_s(t_0)]\frac{\Delta t}{2} + i_L(t_0)\left[1 - \frac{\Delta t}{2}\right]$$

Assuming $\Delta t = T$, we let $t_1 = nT$ and $t_0 = (n-1)T$, we get

$$i_L(nT) = \frac{T}{2+T}[i_s(nT) + i_s((n-1)T)] + \frac{2-T}{2+T}i_L((n-1)T),\ n \geq 1$$

□

(b) Create a Python script to solve the difference equation for $T_s = 0.01$ sec and the frequency of $i_s(t)$ is $\omega = 0.5\pi$. Plot the input current source $i_s(t)$ and the approximate solution $i_L(nT_s)$ in the same figure. (5 points)
Solution:

Python code

```
import numpy as np
import matplotlib
import matplotlib.pyplot as plt

T = 0.01
w = 0.5 * np.pi

tt = np.arange(0,12-T,T)
i_s = 5 * np.cos(w * tt)
i_L = np.zeros(len(tt))

for ii in range(1,len(tt)):
    i_L[ii] = (T/(2+T)) * (i_s[ii] + i_s[ii-1])\
       + ((2-T)/(2+T)) * i_L[ii-1]

plt.plot(tt,i_s,'b--',tt,i_L,'r')
plt.xlabel(r"$t$[sec]")
plt.ylabel(r"$y(t)$")
plt.grid()
plt.axis([0, 12, -1.2, 1.2])
plt.legend([r"$i_s(t)$", r"$i_L(t)$"], loc=2)
```

The plot is shown in Figure 2.

$\square$

2. Define a function $z(t)$ as below,

$$z(t) = \begin{cases} 0, & t < 0 \\ 0.5t^2, & 0 \le t < 0.5 \\ 0.5t - 0.125, & 0.5 \le t < 2.5 \\ -0.5t^2 + 3t - 3.25, & 2.5 \le t < 3.5 \\ -0.5t + 2.875, & 3.5 \le t < 5.5 \\ 0.5t^2 - 6t + 18, & 5.5 \le t < 6 \\ 0, & t > 6 \end{cases}$$

Find $x(t)$ if

$$z(t) = \int_{-\infty}^{t} y(\tau)d\tau \quad \text{and} \quad y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

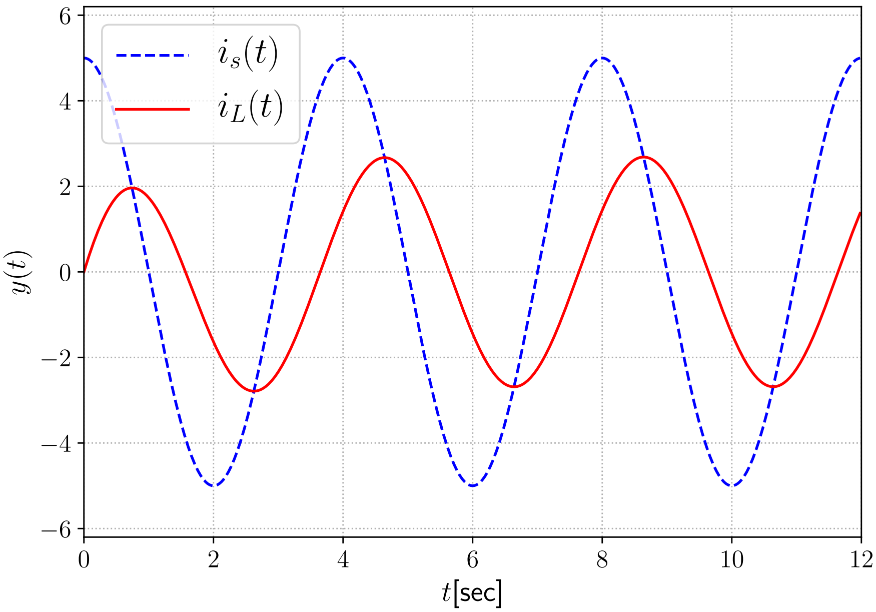

Figure 2: Comparison between i_s and i_L

Test your result with either Matlab or Python. **Hint:** The below listing is an example of python code. If you use Jupyter Notebook, add a magic command `%matplotlib inline` at the beginning of your code to see the plot on the same windows. (10 points)

```
import matplotlib
import matplotlib.pyplot as plt

import numpy as np
def zt(t):
    x = np.zeros(len(t))
    for k, tk in enumerate(t):
        if (tk < 0):
            x[k] = 0
        elif (tk >= 0) and (tk < 0.5):
            x[k] = 0.5*tk**2
        elif (tk >= 0.5) and (tk < 2.5):
            x[k] = 0.5*tk - 0.125
        elif (tk >= 2.5) and (tk < 3.5):
            x[k] = -0.5*tk**2 + 3*tk - 3.25
        elif (tk >= 3.5) and (tk < 5.5):
            x[k] = 2.875 - 0.5*tk
        elif (tk >= 5.5) and (tk < 6):
            x[k] = 18 + 0.5*tk**2 - 6*tk
        elif (tk >= 6):
            x[k] = 0
```

```
    return x

t = np.arange(-2,8,0.01)
z = zt(t)

ax = plt.subplot(3,1,3)
ax.set_yticks([-1.25, 0, 1.25])
ax.set_xticks([0, 0.5, 2.5, 3.5, 5.5, 6])
plt.plot(t,z)
plt.xlim(xmin=-2.0, xmax=8)
plt.ylim(ymin=-1.5, ymax=1.5)
plt.grid()
plt.xlabel("Time(s)")
plt.ylabel("$z(t)$")
```

Solution: Take the derivative on $z(t)$ two time we have

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \leq t < 0.5 \\ 0.5, & 0.5 \leq t < 2.5 \\ -t+3, & 2.5 \leq t < 3.5 \\ -0.5, & 3.5 \leq t < 5.5 \\ t-6, & 5.5 \leq t < 6 \\ 0, & t > 6 \end{cases}$$

$$x(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \leq t < 0.5 \\ 0, & 0.5 \leq t < 2.5 \\ -1, & 2.5 \leq t < 3.5 \\ 0, & 3.5 \leq t < 5.5 \\ 1, & 5.5 \leq t < 6 \\ 0, & t > 6 \end{cases}$$

The Python code is similar to the one given in the hint.

□

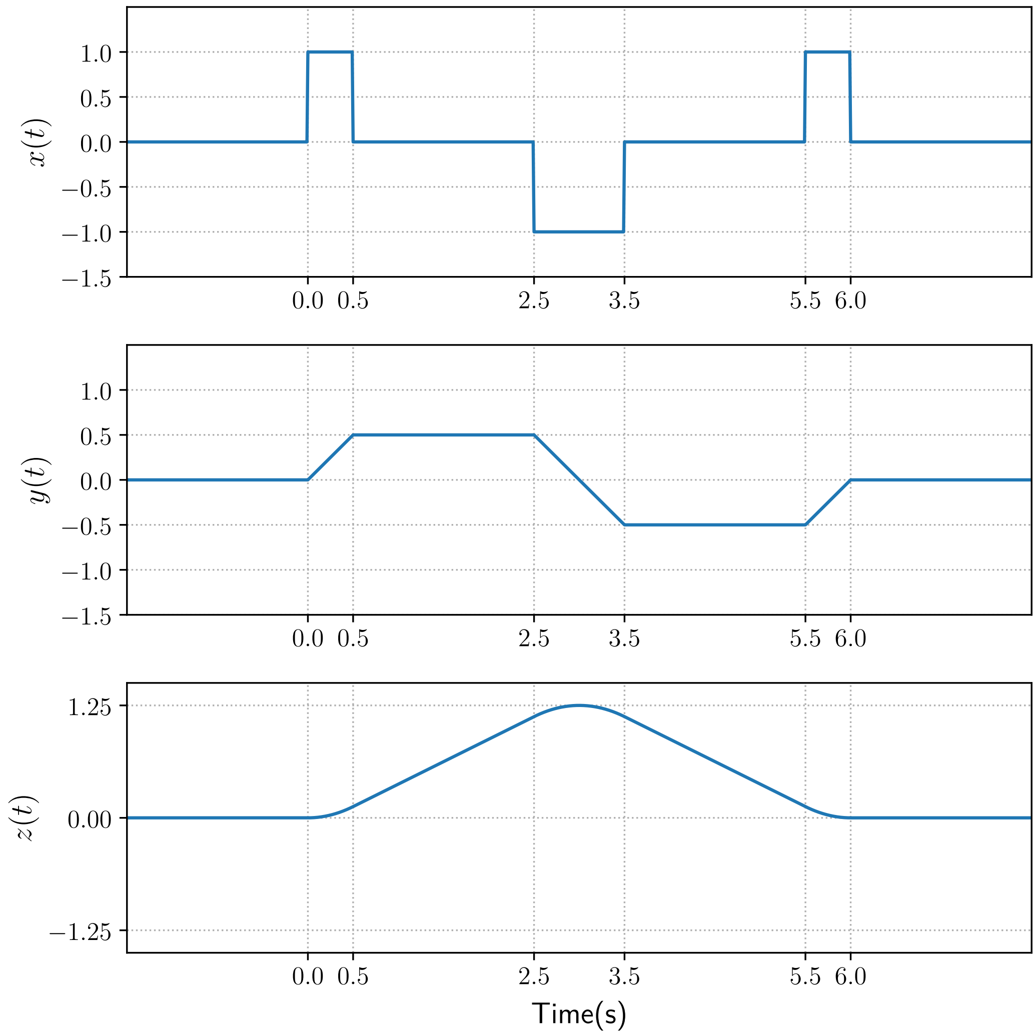

Figure 3: Signal in the question 2

3. Determine the power and the rms value for each of the following signals:

   (a) $10\cos\left(100t+\frac{\pi}{3}\right)+16\sin\left(150t+\frac{\pi}{5}\right)$ (5 points)
   Solution:
   Since the frequency of the first term and the second term are not equal, then

$$
\begin{aligned}
P &= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\left[10\cos(100t+\frac{\pi}{3})+16\sin(150t+\frac{\pi}{5})\right]^2 dt \\
&= \frac{10^2}{2}+\frac{16^2}{2} \\
\text{rms} &= \sqrt{\frac{10^2}{2}+\frac{16^2}{2}}
\end{aligned}
$$

□

(b) $(10 + 2\sin 3t)\cos 10t$ (5 points)
Solution:
Since

$$
\begin{aligned}
(10 + 2\sin 3t)\cos 10t &= 10\cos 10t + 2\sin 3t \cos 10t \\
&= 10\cos 10t + \sin(13t) - \sin(7t),
\end{aligned}
$$

then

$$
\begin{aligned}
P &= \frac{10^2}{2} + 1 = 51 \\
\text{rms} &= \sqrt{51}.
\end{aligned}
$$

Note: $\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$ □

4. An amplifier with a 1Ω output resistance feeds an 8Ω loudspeaker. See Fig. 4a. The voltage source is

$$v_i(t) = 9\,[\sin(100t + 45°) + \cos(10,000t)]$$

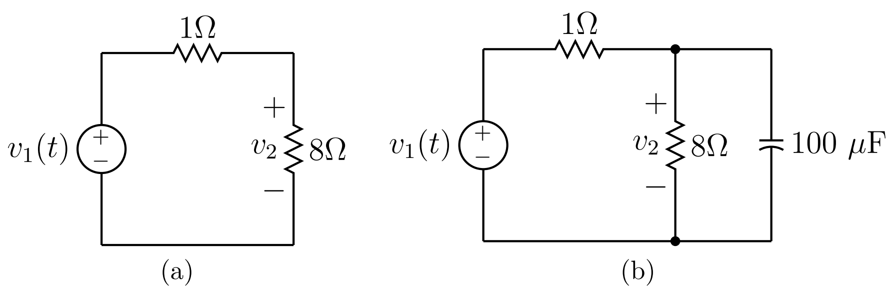

Figure 4: Loud speaker

(a) Find the average power in the loudspeaker. (5 points)
Solution:
From

$$
\begin{aligned}
P &= \frac{1}{2}\frac{|v_2|^2}{R} \\
&= \frac{1}{2}\frac{|\frac{8}{9}v_i|^2}{8} = \frac{8^2}{16} + \frac{8^2}{16} = 8 \text{ W}
\end{aligned}
$$

□

(b) Place a 100 μF capacitor in parallel with the loudspeaker as shown in Fig. 4b. Find the average power in the loudspeaker. (5 points)
Solution: Let R_2 denotes the loudspeaker resistance.

$$Z = \frac{R_2}{j\omega C} = \frac{R_2}{1 + j\omega C R_2}$$

Using a voltage divider technique, we have

$$v_2 = \frac{v_i\left(\frac{R_2}{1+j\omega CR_2}\right)}{R_1 + \frac{R_2}{1+j\omega CR_2}} = \frac{R_2 v_i}{R_1 + R_2 + j\omega CR_1R_2} = \frac{R_2/(R_1+R_2)}{1+j\omega C\frac{R_1R_2}{R_1+R_2}} = \frac{R}{1+j\omega CR},$$

where $R = \frac{8}{9}\Omega$. Then, the average power of the loudspeaker is

$$P = \frac{1}{2}\frac{|v_2|^2}{R_2} = \frac{1}{16}\left(\frac{8}{9}\right)^2\left[\frac{8^2}{1+\omega_1^2C^2R^2} + \frac{8^2}{1+\omega_2^2C^2R^2}\right] = 3.16 + 1.73 = 4.89\ W,$$

where $\omega_1 = 100$ rad/sec and $\omega_2 = 10,000$ rad/sec.

□