RC2017 Signals and Systems | **Homework 8**
Assign: Thu 25 April 2019 | **Due for RC2: 1 May 2019**

Instruction: Hand in your work with name and code to my desk by 10.00 am. of the due date. DO NOT copy homework from your classmates or lend it to others. Anyone who violates this regulation will be given -10 for the homework.

1. Compute the Fourier transform of each of the following siganals:

   (a) $[e^{-\alpha t}\cos\omega_0 t]u(t),\ \alpha > 0$
   Solution:
   The given signal is

$$e^{-\alpha t}\cos(\omega_0 t)u(t) = \frac{1}{2}e^{-\alpha t}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-\alpha t}e^{-j\omega_0 t}u(t)$$

   Therefore,

$$X(j\omega) = \frac{1}{2}\frac{1}{j\omega + \alpha - j\omega_0} - \frac{1}{2}\frac{1}{j\omega + \alpha - j\omega_0}$$

□

   (b) $e^{-3|t|}\sin 2t$
   Solution: The given signal is

$$x(t) = e^{-3t}\sin(2t)u(t) + e^{3t}\sin(2t)u(-t)$$

   We have

$$x_1(t) = e^{-3t}\sin(2t)u(t) \quad \overset{FT}{\longleftrightarrow} \quad X_1(j\omega) = \frac{1}{2j}\frac{1}{j\omega + 3 - j2} - \frac{1}{2j}\frac{1}{j\omega + 3 + j2}$$

   Also,

$$x_2(t) = e^{3t}\sin(2t)u(-t) = -x_1(-t) \quad \overset{FT}{\longleftrightarrow}$$
$$X_2(j\omega) = -X_1(-j\omega) = \frac{1}{2j}\frac{1}{-j\omega + 3 - j2} - \frac{1}{2j}\frac{1}{-j\omega + 3 + j2}$$

   Therefore,

$$X(j\omega) = X_1(j\omega) + X_2(j\omega) = \frac{3j}{9 + (\omega + 2)^2} - \frac{3j}{9 + (\omega - 2)^2}$$

□

(c) $x(t) = \begin{cases} 1 + \cos \pi t, & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$
Solution: Using the equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

we have

$$X(j\omega) = \frac{2\sin\omega}{\omega} + \frac{\sin\omega}{\pi - \omega} - \frac{\sin\omega}{\pi + \omega}$$

□

(d) $\sum_{k=0}^{\infty} \alpha^k \delta(t - kT), \ |\alpha| < 1$
Solution: Using the equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

we have

$$X(j\omega) = \frac{1}{1 - \alpha e^{-j\omega T}}$$

□

(e) $[te^{-2t} \sin 4t]u(t)$
Solution: We have

$$x(t) = \frac{1}{2j} te^{-2t} e^{j4t} u(t) - \frac{1}{2j} te^{-2t} e^{-j4t} u(t)$$

Therefore,

$$X(j\omega) = \frac{1}{2j} \frac{1}{(j\omega + 2 - j4)^2} - \frac{1}{2j} \frac{1}{(-j\omega + 2 + j4)^2}$$

□

(f) $\left[\frac{\sin \pi t}{\pi t}\right] \left[\frac{\sin 2\pi(t-1)}{\pi(t-1)}\right]$
Solution: We have

$$x_1(t) = \frac{\sin \pi t}{\pi t} \qquad \overset{FT}{\longleftrightarrow} \qquad X_1(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}$$

Also

$$x_2(t) = \frac{\sin 2\pi(t-1)}{\pi(t-1)} \quad \stackrel{FT}{\longleftrightarrow} \quad X_2(j\omega) = \begin{cases} e^{-2\omega}, & |\omega| < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

$$x(t)x_1(t)x_2(t) \quad \stackrel{FT}{\longleftrightarrow} \quad X(j\omega) = \frac{1}{2\pi}\left\{X_1(j\omega) * X_2(j\omega)\right\}.$$

Therefore,

$$X(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < \pi \\ (1/2\pi)(2\pi+\omega)e^{-j\omega}, & -3\pi < \omega < -\pi \\ (1/2\pi)(3\pi-\omega)e^{-j\omega}, & \pi < \omega < 3\pi \\ 0, & \text{otherwise} \end{cases}$$

□

(g) $x(t)$ as shown in Figure 1 .

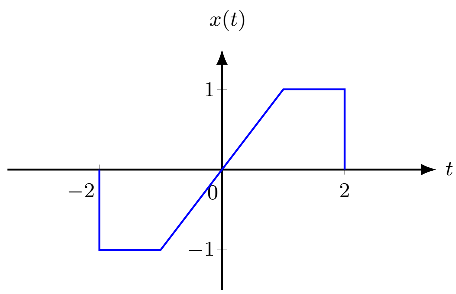


Figure 1: Question 1g

Solution: Using Fourier Transform formula we obtain

$$X(j\omega) = \frac{2j}{\omega}\left[\cos 2\omega - \frac{\sin\omega}{\omega}\right]$$

□

(h) $x(t)$ as shown in Figure 2 .
Solution: If

$$x_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-2k),$$

then

$$x(t) = 2x_1(t) + x_1(t-1).$$

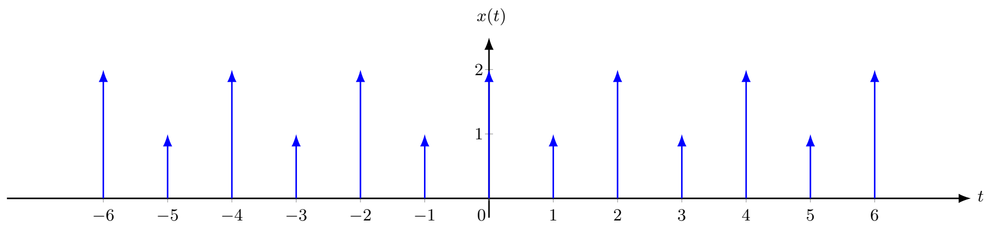


Figure 2: Question 1h

Therefore,

$$X(j\omega) = X_1(j\omega)[2 + e^{-\omega}] = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\pi)[2 + (-1)^k]$$

□

(i) $x(t) = \begin{cases} 1 - t^2, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

Solution: Using the Fourier transform formula, we obtain

$$X(j\omega) = \frac{1}{j\omega} + \frac{2e^{-j\omega}}{-\omega^2} - \frac{2e^{-j\omega} - 2}{j\omega^2}$$

□

(j) $\sum_{n=-\infty}^{\infty} e^{-|t-2n|}$

Solution: Since, using convolution with $\delta(t)$,

$$x(t) = \left[\sum_{n=-\infty}^{\infty} \delta(t - 2n)\right] * e^{-|t|} = \sum_{n=-\infty}^{\infty} e^{-|t-2n|}$$

$$X(\omega) = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - 2n)\right\} \mathcal{F}\left\{e^{-|t|}\right\}$$

From Fourier Transform Table, we have

$$\mathcal{F}\left\{e^{-|t|}\right\} = \frac{2}{1 + \omega^2}$$

$$\mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - 2n)\right\} = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) = \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)$$

Therefore,

$$X(\omega) = \pi \sum_{n=-\infty}^{\infty} \frac{2}{1 + \omega^2} \delta(\omega - n\pi)$$

□

2. Determine the continuous-time signal corresponding to each of the following transforms

   (a) $X(\omega) = \dfrac{2\sin[3(\omega - 2\pi)]}{(\omega - 2\pi)}$
   Solution: From table, we have

$$\frac{W}{\pi}\sin(Wt) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \text{rect}\left(\frac{\omega}{2W}\right)$$

   Using delay and duality properties, then

$$\frac{3}{\pi}\frac{\sin[3(t+2\pi)]}{3(t+2\pi)} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \text{rect}\left(\frac{\omega}{6}\right)e^{j2\pi\omega}$$
$$\text{rect}\left(\frac{t}{6}\right)e^{j2\pi t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad 2\pi\frac{3}{\pi}\frac{\sin[3(-\omega+2\pi)}{3(-\omega+2\pi)} = 2\frac{\sin(3(\omega-2\pi)}{(\omega-2\pi)}$$

   Note: $\sin(-A) = -\sin(A)$. Therefore,

$$x(t) = \text{rect}\left(\frac{t}{6}\right)e^{j2\pi t}$$

$\square$.

   (b) $X(\omega) = \cos(4\omega + \pi/3)$
   Solution: From

$$\cos(4t + \frac{\pi}{3}) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \pi[\delta(\omega+4)e^{j\frac{\pi}{3}} + \delta(\omega-4)e^{-j\frac{\pi}{3}}$$

   By duality, we obtain

$$\pi[\delta(t+4)e^{j\frac{\pi}{3}} + \delta(t-4)e^{-j\frac{\pi}{3}}] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad 2\pi\cos(-4\omega + \frac{\pi}{3}) = 2\pi\cos(4\omega + \frac{\pi}{3})$$

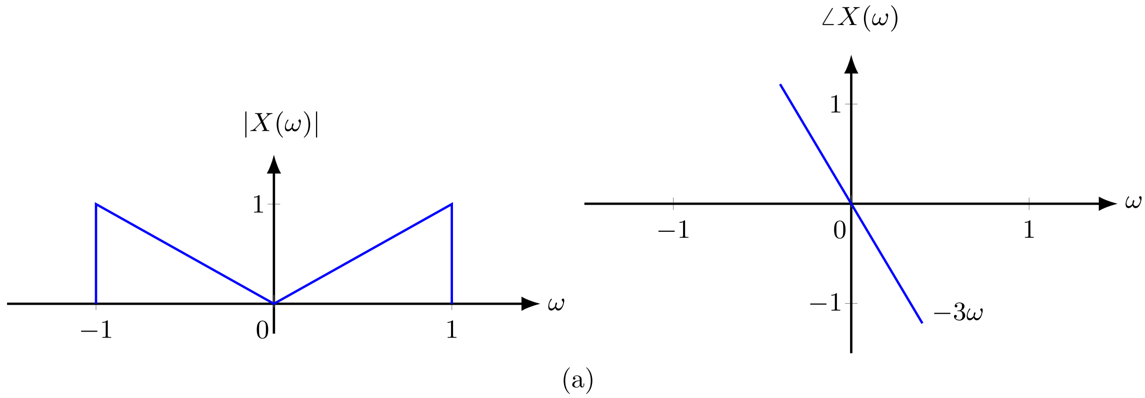
   Therefore

$$\cos(4\omega + \frac{\pi}{3}) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{2}[\delta(t+4)e^{j\frac{\pi}{3}} + \delta(t-4)e^{-j\frac{\pi}{3}}]$$

$\square$

   (c) $X(\omega)$ as given by the magnitude and phase plots of Figure 3.
   Solution: From

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|e^{j\angle X(\omega)}e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\left[\int_{-1}^{0} -\omega e^{j\omega(t-3)}d\omega + \int_{0}^{1} \omega e^{j\omega(t-3)}d\omega\right]
\end{aligned}$$



(a)

Figure 3: Question 2c

Using by-part integral, $u = \omega,\ du = d\omega$ and $dv = e^{j\omega(t-3)}d\omega,\ v = \frac{1}{j(t-3)}e^{j\omega(t-3)}$, we have

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\Bigg[-\left(\frac{\omega}{j(t-3)}e^{j\omega(t-3)}\Big|_{-1}^{0} - \int_{-1}^{0}\frac{1}{j(t-3)}e^{j\omega(t-3)}d\omega\right) \\
&\quad + \left(\frac{\omega}{j(t-3)}e^{j\omega(t-3)}\Big|_{0}^{1}\right) - \int_{0}^{1}\frac{1}{j(t-3)}e^{j\omega(t-3)}d\omega\Bigg] \\
&= \frac{1}{2\pi}\Bigg[\left(-\frac{1}{j(t-3)}e^{-j(t-3)} - \frac{1}{(t-3)^2} + \frac{1}{(t-3)^2}e^{-j(t-3)}\right) \\
&\quad + \left(\frac{1}{j(t-3)}e^{j(t-3)} + \frac{1}{(t-3)^2}e^{j(t-3)} - \frac{1}{(t-3)^2}\right)\Bigg] \\
&= \frac{1}{2\pi}\left[\frac{\sin(t-3)}{t-3} + \frac{\cos(t-3)-1}{(t-3)^2}\right]
\end{aligned}
$$

□

(d) $X(\omega) = 2[\delta(\omega-1) - \delta(\omega+1)] + 3[\delta(\omega-2\pi) + \delta(\omega+2\pi)]$
Solution: From table,

$$x(t) = \frac{2j}{\pi}\sin(t) + \frac{3}{\pi}\cos(2\pi t)$$

□

(e) $X(\omega)$ as in Figure 4.

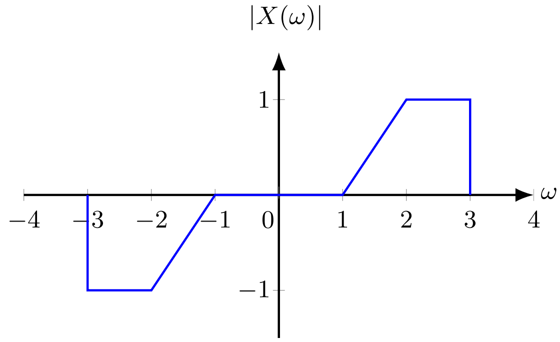


Figure 4: Question 2e

Solution: We have

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\left[\int_{-3}^{-2} -1e^{j\omega t}d\omega + \int_{-2}^{-1}(\omega+1)e^{j\omega t}d\omega \right.\\
&\quad \left. + \int_{1}^{2}(\omega-1)e^{j\omega t}d\omega + \int_{2}^{3} 1e^{j\omega t}d\omega\right] \\
&= \frac{1}{2\pi}\left[\frac{-1}{jt}e^{j\omega t}\Big|_{-3}^{-2} + \frac{\omega}{jt}e^{j\omega t}\Big|_{-2}^{-1} - \int_{-2}^{-1}\frac{1}{jt}e^{j\omega t}d\omega + \frac{1}{jt}e^{j\omega t}\Big|_{-2}^{-1} \right.\\
&\quad \left. + \frac{\omega}{jt}e^{j\omega t}\Big|_{1}^{2} - \int_{1}^{2}\frac{1}{jt}e^{j\omega t}d\omega - \frac{1}{jt}e^{j\omega t}\Big|_{1}^{2} + \frac{1}{jt}e^{j\omega t}\Big|_{2}^{3}\right] \\
&= \frac{1}{2\pi}\left[\frac{-1}{jt}e^{-j2t} + \frac{1}{jt}e^{-j3t} - \frac{1}{jt}e^{-jt} + \frac{2}{jt}e^{-j2t} + \frac{1}{t^2}e^{-jt} - \frac{1}{t^2}e^{-j2t} + \frac{1}{jt}e^{-jt} - \frac{1}{jt}e^{-j2t} \right.\\
&\quad \left. + \frac{2}{jt}e^{j2t} - \frac{1}{jt}e^{jt} + \frac{1}{t^2}e^{j2t} - \frac{1}{t^2}e^{jt} - \frac{1}{jt}e^{j2t} + \frac{1}{jt}e^{jt} + \frac{1}{jt}e^{j3t} - \frac{1}{jt}e^{j2t}\right] \\
&= \frac{1}{2\pi}\left[\frac{-2\cos(2t)}{jt} + \frac{2\cos(3t)}{jt} + \frac{4\cos(2t)}{jt} - \frac{2j\sin(t)}{t^2} + \frac{2j\sin(2t)}{t^2} - \frac{2\cos(2t)}{jt}\right] \\
&= \frac{1}{2\pi}\left[\frac{2\cos(3t)}{jt} - \frac{2j\sin(t)}{t^2} + \frac{2j\sin(2t)}{t^2}\right] \\
&= \frac{\cos(3t)}{j\pi t} + \frac{\sin(t)}{j\pi t^2} - \frac{\sin(2t)}{j\pi t^2}
\end{aligned}
$$

□