INC 693, 481 Dynamics System and Modelling: The Language of Bound Graphs

Dr.-Ing. Sudchai Boonto
Assistant Professor

Department of Control System and Instrumentation Engineering
King Mongkut's Unniversity of Technology Thonburi Thailand

The Language of Bound Graphs

- Lagrangian and Hamiltonian methods can derive the differential equations of any given system.
- The set of linear equations, starting from any initial condition the time evolution of the system can be obtained in a closed form by solving the equations.
- Most the differential equations are nonlinear, which can only be solved numerically with help of computers.
- Human have to derive the differential equations and solving them is performed by a computer.
- Prof. H. M. Paynter of MIT invented a language of system representation through exchange of power and information — by which the job of deriving differential equations can be performed by a computer.

The Language of Bound Graphs

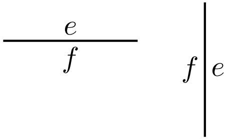
- The concept of bond graphs was originated by Prof. Paynter in 1960.
- The idea was further developed by Prof. Karnopp and Rosenberg.
- There are a lot of softwares supported like Enport, CAMP, 20-SIM.

Power Variables

- In bond graphs two power conjugated variables are assigned to each edge. The are called *effort* and *flow* and are denoted by the letter e and f.

$$\text{Power} = \text{Effort} \times \text{Flow}$$

Annotating a bond with power variables effort and flow:



Bond graph variables used in the various energy domains

Domain	Effort e	Flow f	momentum p	displacement q
Translational Mechanics	Force F	Velocity v	Momentum p	Displacement x x
Rotational Mechanics	Angular Moment M	Angular Velocity ω	Angular Momentum p_ω	Angle θ
Electro	Voltage v	Current i	Linkage Flux λ	Charge q
Hydraulic	Total Pressure P	Volume Flow Q	Pressure Momentum P_p	Volume V_c

One-port Elements

- capacitor, resistor, voltage source and current source are *one-port elements*.
- it is same for mechanical system and hydraulic system.
- the inductor has the property:

$$v = L\frac{di}{dt} \text{ or } i = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$$

$$\text{flow} = \text{ a constant or a function} \times \int_{-\infty}^{t} \text{effort}(\tau)d\tau$$

- the element with this relationship (also the mass in translational mechanics) will be called an *inertial element* and be denoted by

——— I

One-port Elements

- the capacitor has the property:

$$\text{effor} \;=\; \text{a constant or a function} \times \int_{-\infty}^{t} \text{flow}(\tau)d\tau$$

- any element with this relationship (also the spring in translational mechanics) will be called a *compliance element* and be denoted as

$$\rule{2cm}{0.4pt}\ \mathsf{C}$$

- the resistance has the relationship

$$\begin{aligned} \text{effort} &= \text{a constant or a function} \times \text{flow}, \\ \text{flow} &= \text{a constant or a function} \times \text{effort} \end{aligned}$$

One-port Elements

- any element with this property will be called a *resistive element* and will be denoted by the symbol

—— R

- A voltage source in electrical domain and an externally impressed force in mechanical domain are examples of *source of effort*, denoted by the symbol

—— SE

- A current source in an electrical circuit or a cam in a mechanical system, the flow variable is externally determined and the effort variable is decided by the rest of the system. Such an element is called a *source of flow* and is represented by the symbol

—— SF

The Junctions

Junction Structure (JS)

A bound graph in which bounds connect only nodes that instantaneously transfer or distribute power (without energy storage or conversion into heat), is called *Junction Structure* (JS).

There are two kinds of junction:

- 0-junction is the junction that equalized the efforts in all bonds.
- 1-junction is the junction that equalized the flows in all bounds.

The 0-Junction

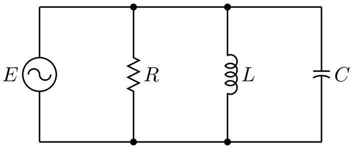
0-junction

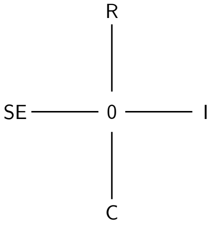
A 0-junction is a multiport element defined by the following equations

$$e_1 = e_2 = \ldots = e_n$$

$$f_1 = f_2 + \ldots + f_n$$

According to both equation above the element is also called *common effort junction*. The lower equation has given rise to the notion of a *flow junction*.





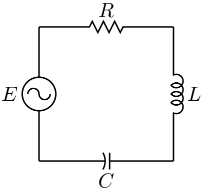
The 1-Junction

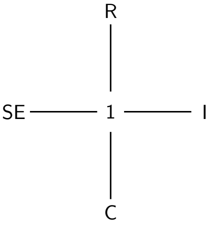
1-junction

A 1-junction is a multiport element defined by the following equations

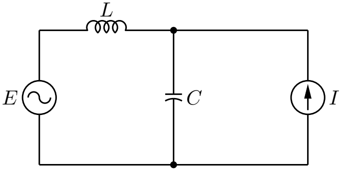
$$f_1 = f_2 = \ldots = f_n$$
$$e_1 - e_2 - \ldots + e_n = 0$$

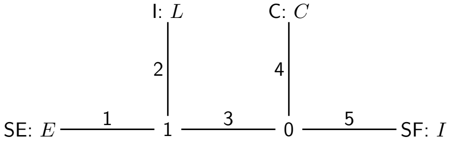
According to both equation above the element is also called *common flow junction*. The lower equation has given rise to the notion of a *effort junction*.

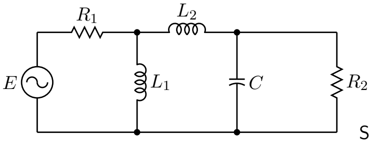


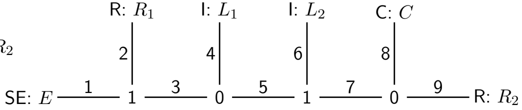


Examples
RLC



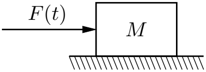


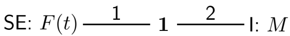


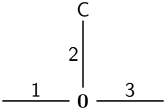


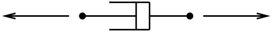
Examples

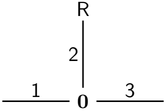
Mechanical Systems





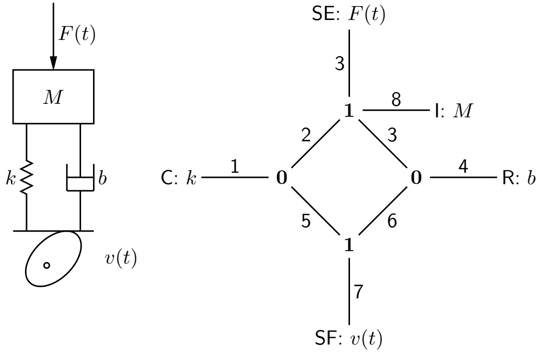






Examples

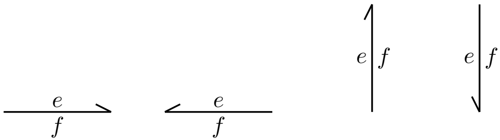
Mechanical Systems



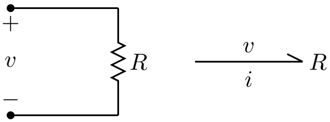
- the mass M shares the same flow with the agent applying the force.
- the top and the bottom ends of the spring and the damper share the same effort.

Reference Power Directions

- Bound graph also assigns a *reference power direction* of each bound.
- Conventions for adding the half arrow to power bonds

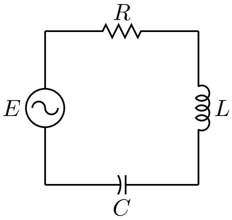


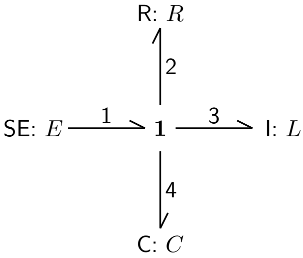
- Resistor circuit example



Reference Power Directions

- with the power directions, we can state the power balance at the 1 junction as





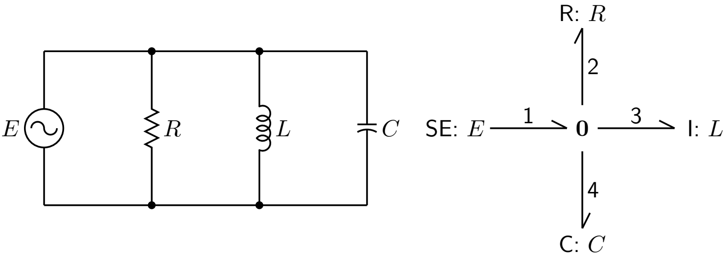
$$e_1 f_1 - e_2 f_2 - e_3 f_3 - e_4 f_4 = 0$$

Since $f_1 = f_2 = f_3 = f_4,$ then

$$e_1 - e_2 - e_3 - e_4 = 0$$

Reference Power Directions

- with the power directions, we can state the power balance at the 0 junction as



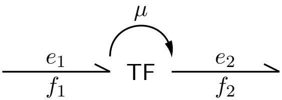
$$e_1 f_1 - e_2 f_2 - e_3 f_3 - e_4 f_4 = 0$$

Since $e_1 = e_2 = e_3 = e_4,$ then

$$f_1 - f_2 - f_3 - f_4 = 0$$

Two-port Elements

Transformer element :



- the power in the two sides must be equal

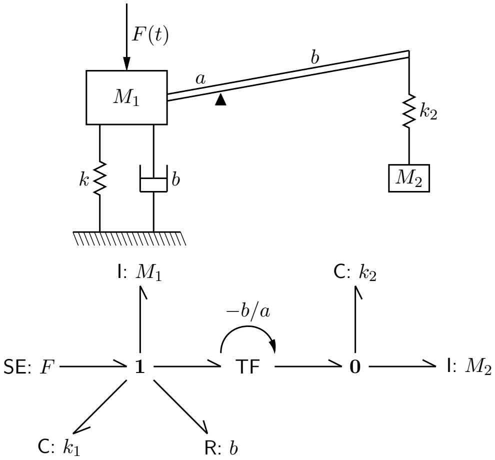
$$e_1 f_1 = e_2 f_2$$

- the notation shown above implies that the flows at two sides are related by

$$f_2 = \mu f_1 \qquad \text{and} \qquad e_2 = \frac{1}{\mu} e_1$$

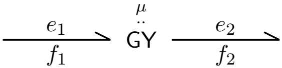
Two-port Elements
transformer system

Transformer element :



Two-port Elements

Gyrator element :



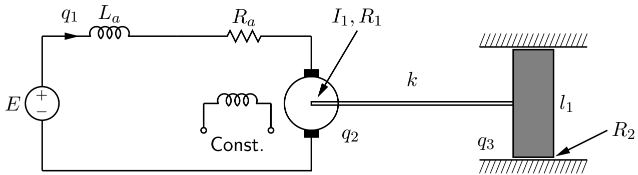
- Here μ is the gyrator modulus and the variables at the two sides are related by

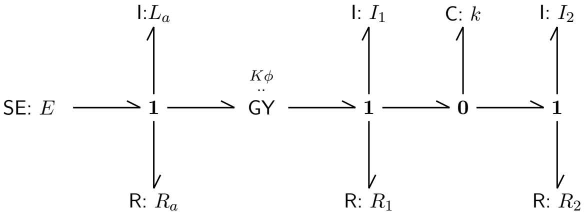
$$e_2 = \mu f_1 \;\; \text{and} \; e_1 = \mu f_2 \;\; \text{and} \; e_1 f_1 = e_2 f_2$$

- the DC machine is a typical example of a gyrator. The armature current and the back emf in the electrical domain are related to the speed and torque in the mechanical domain by

$$\tau = K\phi i_a \;\; \text{and} \;\; e_b = K\phi\omega_r$$

Gyrator element :



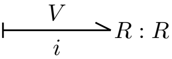


Gyrator element :

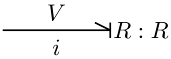
- the voltage source and the armature resistance and inductance share the same flow with the back emf. Therefore, these are connected through a **1** junction.
- The conversion from electrical domain to mechanical domain is represented by the gyrator element.
- The rotor inertia and friction share the same flow (rotor speed) with the coming from the gyrator. So these are connected through a **1** junction.
- The flexible shaft represented by the compliance element, equates the effort at both its ends, and is connected through a **0** junction.
- The load inertia and friction share the same flow with the right-hand end of the shaft. So these are connected through a **1** junction.

Causal Stroke

- The relation between input and output variables can be shown by using a *causal stroke*.
- The causal stroke is a short line that is perpendicular to bond.
- If i is an input and V is an output with $V = Ri$, the causal stroke will be on the outside of the bound.
- If V is an output and i is an input with $i = \frac{1}{R}V$, the causal stroke will be on the same side like R.

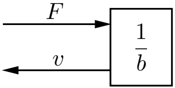


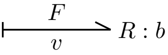
i is an input



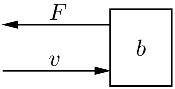
V is an input

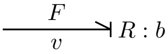
- Consider a damper if velocity is an input and $F = bv$ then bond graph and a block diagram is





- if force is an input and $v = \frac{1}{b}f$ then bond graph and a block diagram is



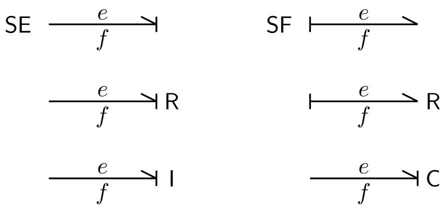


Causal Stroke

- The 1 junction equalizes the flows in the bonds connected to it. Therefore, the information of flow must come from only one bond, and all other bonds must take out the flow information.
- The 0 junction equalizes the efforts in the bounds connected to it. This implies that only one bond must bring in the effort information and all the other bonds must take this information out.
- the above rules of causality for the 1 and 0 junctions are "hard rules", and cannot be violated in a bond graph.
- the bond that brings some information into a junction is called *strong bond* while the other bonds taking out that information are called *weak bonds*.

Causal Stroke

One port element:



Two port element:

Obtaining Differential Equations from Bond Graphs

- An I element the basic differential equation is

$$\dot{p} = e$$

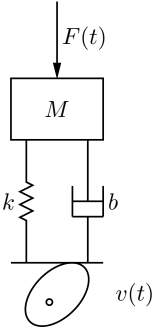
- A C element the basic differential equation is

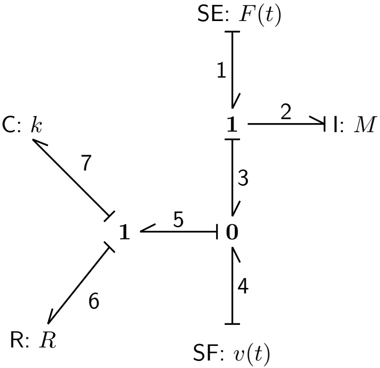
$$\dot{q} = f$$

- Therefor the momentum associated with a mass (or inductor) and the position of a spring (or charge in a capacitor) become the natural choice of generalized variables in the set of first-order differential equations.

Obtaining Differential Equations from Bond Graphs

Mechanical system example





Obtaining Differential Equations from Bond Graphs

What do the elements give to the system?

1. The flow f_2 is given by the mass M. The state variable is $f_2 = p_2/M$.
2. The effort e_7 is given by the spring k. The state variable is $e_7 = kq_7$.
3. The effort e_6 is given by the damper b and $e_b = bf_6$. But f_6 is not a state variable. Since

$$
\begin{aligned}
e_6 &= Rf_6 = Rf_5 \text{ (by junction } \mathbf{1}) \\
e_3 f_3 + e_4 f_4 - e_5 f_5 &= 0 \text{ (by junction } \mathbf{0}) \\
e_6 &= Rf_5 = R(f_3 + f_4) \\
&= R(f_2 + f_4) \text{ (by junction } \mathbf{1}) \\
&= R(\frac{p_2}{M} + v(t)).
\end{aligned}
$$

4. The SE element gives the effort $e_1 = F(t)$ and the SF element gives the flow $f_3 = v(t)$.

What do the integrally causal storage elements receive from the system?

1. I_2 receives the effort e_2 from the system. By the property of the I element, $e_2 = \dot{p}_2$. Then

$$
\begin{aligned}
\dot{p}_2 = e_2 &= e_1 - e_3 \text{ (by junction } \mathbf{1}) \\
&= e_1 - e_5 \text{ (by junction } \mathbf{0}) \\
&= e_1 - e_6 - e_7 \text{ (by junction } \mathbf{1}) \\
&= F(t) - R\left(\frac{p_2}{M} + v(t)\right) - kq_7
\end{aligned}
$$

2. C_7 receive the flow f_7 from the system. We know $f_7 = \dot{q}_7$.

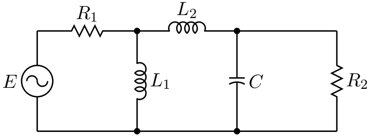
$$\begin{aligned}\dot{q}_7 &= f_7 = f_5 \text{ (by junction } \mathbf{1}) \\ &= f_3 + f_4 \text{ (by junction } \mathbf{0}) \\ &= v(t) + f_2 \text{ (by junction } \mathbf{1}) \\ &= v(t) + \frac{p_2}{M}.\end{aligned}$$

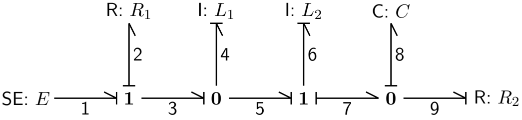
The state-space equation is

$$\begin{bmatrix} \dot{p}_2 \\ \dot{q}_7 \end{bmatrix} = \begin{bmatrix} -\frac{R}{M} & -k \\ \frac{1}{M} & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ q_7 \end{bmatrix} + \begin{bmatrix} 1 & -R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F(t) \\ v(t) \end{bmatrix}$$

Obtaining Differential Equations from Bond Graphs

RLC circuit example





the elements give to the system:

$$\text{SE:} \qquad e_1 = E$$

$$I_4: \qquad f_4 = \frac{p_4}{L_1}$$

$$I_6: \qquad f_6 = \frac{p_6}{L_2}$$

$$C_8: \qquad e_8 = \frac{q_8}{C}$$

R_2: we have

$$\begin{aligned} e_2 &= R_1 f_2 = R_1 f_3 \text{ (by strong bond 3)} \\ &= R_1(f_4 + f_5) \text{ (by junction } \mathbf{0}) \\ &= R_1(f_4 + f_6) \text{ (by junction } \mathbf{1}) \\ &= R_1 \left(\frac{p_4}{L_1} + \frac{p_6}{L_2} \right) \end{aligned}$$

R_9: we have

$$\begin{aligned} f_9 &= \frac{e_9}{R_2} = \frac{e_8}{R_2} \\ &= \frac{q_8}{R_2 C} \text{ (note } q_8 = \frac{e_8}{C}) \end{aligned}$$

Obtaining Differential Equations from Bond Graphs

RLC circuit example

Now the three integrally causal storage elements receive effort and flow from the system.

I_4:

$$\dot{p}_4 = e_4 = e_3 = e_1 - e_2 = E - R_1\left(\frac{p_4}{L_1} + \frac{p_6}{L_2}\right).$$

I_6:

$$\begin{aligned}\dot{p}_6 &= e_6 = e_5 - e_7 = e_3 - e_8 = e_1 - e_2 - e_8 \\ &= E - R_1\left(\frac{p_4}{L_1} + \frac{p_6}{L_2}\right) - \frac{q_8}{C}.\end{aligned}$$

C_8

$$\dot{q}_8 = f_8 = f_7 - f_9 = f_6 - f_9 = \frac{p_6}{L_2} - \frac{q_8}{R_2 C}.$$

Reference

1. Wellstead, P. E. *Introduction to Physical System Modelling*, Electronically published by: www.control-systems-principles.co.uk, 2000
2. Banerjee, S., *Dynamics for Engineers*, John Wiley & Sons, Ltd., 2005
3. Rojas, C. , *Modeling of Dynamical Systems*, Automatic Control, School of Electrical Engineering, KTH Royal Institute of Technology, Sweeden
4. Fabien, B., *Analytical System Dynamics: Modeling and Simulation* Springer, 2009
5. Spong, M. W. and Hutchinson, S. and Vidyasagar, M., *Robot Modeling and Control*, John Wiley & Sons, Inc.