INC 693, 481 Dynamics System and Modelling: Lagrangian Method III

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First-order Equations for the Lagrangian Method

The second-order differential equation can be expressed in the from of two first-order equations by defining an additional variable. Define the additional variables as the *generalized momenta*, given by

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad \text{Since} \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \dot{p}_i$$

The Lagrangian equation

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathcal{R}}{\partial \dot{q}_i} &= 0 \\ \dot{p}_i - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathcal{R}}{\partial \dot{q}_i} &= 0 \end{aligned}$$

These are first-order equations. The desirable form of the first-order equations is such that the derivative quantities ($\dot{q}_i$ and $\dot{p}_i$) may be eqpressed as function of the fundamental variables.

First-order Equations for the Lagrangian Method

Spring Pendulum

Recall the Lagrangian of the spring pendulum system is given by

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(a+r)^2\dot{\theta}^2 - \frac{1}{2}k\left(r + \frac{mg}{k}\right)^2 + mg(a+r)\cos\theta - mga.$$

Here the generalized coordinates are $q_1 = r$ and $q_2 = \theta$. Then

$$\mathcal{L} = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}m(a+q_1)^2\dot{q}_2^2 - \frac{1}{2}k\left(q_1 + \frac{mg}{k}\right)^2 + mg(a+q_1)\cos q_2 - mga$$

Hence, the generalized momenta are

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = m\dot{q}_1, \qquad p_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = m(a+q_1)^2\dot{q}_2.$$

Putting the derivative quantities, we have

$$\begin{aligned}\dot{q}_1 &= \frac{p_1}{m},\\ \dot{q}_2 &= \frac{p_2}{m(a+q_1)^2}\end{aligned}$$

First-order Equations for the Lagrangian Method

Spring Pendulum

The first equation is obtained from

$$\dot{p}_1 - \frac{\partial \mathcal{L}}{\partial q_1} = 0$$

$$\dot{p}_1 - m\dot{q}_2^2(a + q_1) + k\left(q_1 + \frac{mg}{k}\right) - mg\cos q_2 = 0$$

Substituting $\dot{q}_2$, we get

$$\dot{p}_1 = \frac{p_2^2}{m(a+q_1)^3} - k\left(q_1 + \frac{mg}{k}\right) + mg\cos q_2$$

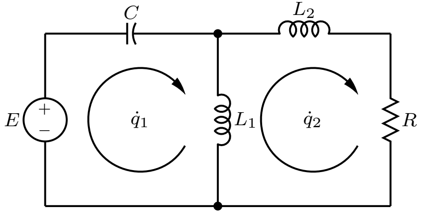
The equation in the second coordinate is obtained from

$$\dot{p}_2 - \frac{\partial \mathcal{L}}{\partial q_2} = 0$$

$$\dot{p}_2 = -mg(a + q_1)\sin q_2$$

Thus we get four first-order equations of $\dot{q}_1$, $\dot{q}_2$, $\dot{p}_1$ and $\dot{p}_2$. To get the linear state-space form, we need to linearize the equations.

Consider a RLC circuit



We have

$$T = \frac{1}{2}L_1(\dot{q}_1 - \dot{q}_2)^2 + \frac{1}{2}L_2\dot{q}_2^2$$

$$V = \frac{1}{2C}q_1^2 - q_1 E, \qquad \mathcal{R} = \frac{1}{2}R\dot{q}_2^2$$

$$\mathcal{L} = \frac{1}{2}L_1(\dot{q}_1 - \dot{q}_2)^2 + \frac{1}{2}L_2\dot{q}_2^2 - \frac{1}{2C}q_1^2 + q_1 E$$

First-order Equations for the Lagrangian Method

RLC Circuit

The Lagrangian equations become

$$L_1(\ddot{q}_1 - \ddot{q}_2) + \frac{1}{C}q_1 - E = 0$$
$$-L_1(\ddot{q}_1 - \ddot{q}_2) + L_2\ddot{q}_2 + R\dot{q}_2 = 0$$

The conjugate momenta are

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = L_1(\dot{q}_1 - \dot{q}_2), \quad p_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = -L_1(\dot{q}_1 - \dot{q}_2) + L_2\dot{q}_2$$

These provide the equations for $\dot{q}_1$ and $\dot{q}_2$ as

$$\dot{q}_1 = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)p_1 + \frac{1}{L_2}p_2, \qquad \dot{q}_2 = \frac{1}{L_2}p_1 + \frac{1}{L_2}p_2.$$

In terms of p_1 and p_2, the Lagrangian equations become

$$\dot{p}_1 = -\frac{1}{C}q_1 + E, \qquad \dot{p}_2 = -R\dot{q}_2 = -\frac{R}{L_2}(p_1 + p_2).$$

The Hamiltonian Formalism

- To derive the system equations directly in the first order, can be done by using Hamiltonian method. Instead of the Lagrangian function $\mathcal{L} = T - V$, we shall use the total energy function and denote it by $\mathcal{H}$. Hence

$$\mathcal{H} = T + V$$

  Note that the potential V is dependent only on the generalized coordinates and not on the generalized velocities. Hence

$$\frac{\partial V}{\partial \dot{q}_i} = 0, \quad \text{and therefore }, \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial (T - V)}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

- the kinetic energy is a homogeneous function of degree 2 in the generalized velocities.
- To see this consider

$$T(k\dot{q}_1, k\dot{q}_2) = k^2 T(\dot{q}_1, \dot{q}_2)$$

The Hamiltonian Formalism

- Differentiate both sides of the equation, to get

$$\dot{q}_1 \frac{\partial T}{\partial k\dot{q}_1} + \dot{q}_2 \frac{\partial T}{\partial k\dot{q}_2} = 2kT(\dot{q}_1, \dot{q}_2)$$

Since k is arbitrary, this equation would be valid for $k = 1$. Hence

$$\dot{q}_1 \frac{\partial T}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial T}{\partial \dot{q}_2} = 2T(\dot{q}_1, \dot{q}_2)$$

For higher dimensional system, we have

$$\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

$$\sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 2T$$

The total energy function $\mathcal{H}$ can be written as

$$\mathcal{H} = T + V = 2T - (T - V) = 2T - \mathcal{L}$$

The Hamiltonian Formalism

- or

$$\mathcal{H} = \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} = \sum_i \dot{q}_i p_i - \mathcal{L}.$$

  The function in the right-hand side is called the *Hamiltonian function* (denote by H) in classical mechanics.

- The Hamiltonian function is the total energy (i.e. $H = \mathcal{H}$) for systems which the above functional forms of T and V hold. H is a function of p_i, $\dot{q}_i$, and t. Moreover, from the previous discussion, we have expressed T as a function of $\dot{q}_i$, it will be necessary to substitute $\dot{q}_i$ in terms of p_i or $\dot{q}_i$. This can be written explicitly as $\dot{q}(p, q)$.
- After $H(p_i, q_i, t)$ is obtained, the equations of motion in terms of H can be obtained by taking the taking the differential of H as

$$dH = \sum_i \frac{\partial H}{\partial p_i} dp_i + \sum_i \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt$$

The Hamiltonian Formalism

- From

$$
\begin{aligned}
\frac{\partial H(q,p)}{\partial p} &= \frac{\partial (p\dot{q}(p,q))}{\partial p} - \frac{\partial \mathcal{L}(q,\dot{q}(p,q))}{\partial p} \\
&= \left(\dot{q} + p\frac{\partial \dot{q}}{\partial p}\right) - \frac{\mathcal{L}(q,\dot{q})}{\partial \dot{q}}\frac{\partial \dot{q}(p,q)}{\partial p} = \left(\dot{q} + p\frac{\partial \dot{q}}{\partial p}\right) - p\frac{\partial \dot{q}}{\partial p} = \dot{q} \\
\frac{\partial H(q,p)}{\partial q} &= \frac{\partial (p\dot{q}(p,q))}{\partial q} - \frac{\partial \mathcal{L}(q,\dot{p}(p,q))}{\partial q} \\
&= p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q} - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}}\frac{\partial \dot{q}(p,q)}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q} - p\frac{\partial \dot{q}}{\partial q} \\
&= -\frac{\partial \mathcal{L}(q,\dot{q})}{\partial q}
\end{aligned}
$$

- Then, we can also write dH as

$$
\begin{aligned}
dH &= \sum_i \dot{q}_i dp_i - \sum_i \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial t} dt \\
&= \sum_i \dot{q}_i dp_i + \sum_i \left(-\dot{p}_i - \frac{\partial \mathcal{R}}{\partial \dot{q}_i}\right) dq_i - \frac{\partial \mathcal{L}}{\partial t} dt
\end{aligned}
$$

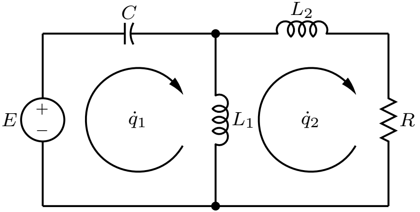
The Hamiltonian Formalism

- Finally we have the relations (Comparing dH in page 9 and in page 10)

$$\frac{\partial H}{\partial p_i} = \dot{q}_i$$
$$\frac{\partial H}{\partial q_i} = -\dot{p}_i - \frac{\partial \mathcal{R}}{\partial \dot{q}_i},$$
$$\frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

- the first two equations, expressed in the first-order form, are called the Hamiltonian equations of motion.
- the last equation is not a differential equation and hence is not needed to represent the dynamics.
- the dynamical systems where these equations hold are called *Hamiltonian systems*.

Consider a RLC circuit



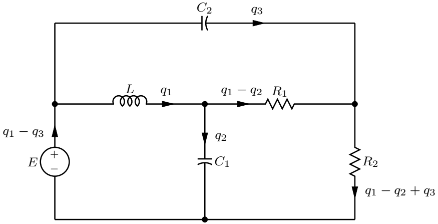
$$
\begin{aligned}
H = T + V &= \frac{1}{2}L_1(\dot{q}_1 - \dot{q}_2)^2 + \frac{1}{2}L_2\dot{q}_2^2 + \frac{1}{2C}q_1^2 - q_1 E \\
&= \frac{1}{2}L_1\left(\frac{1}{L_1}p_1\right)^2 + \frac{1}{2}L_2\left(\frac{1}{2}(p_1 + p_2)\right)^2 + \frac{1}{2C}q_1^2 - q_1 E \\
&= \frac{1}{2L_1}p_1^2 + \frac{1}{2L_2}(p_1 + p_2)^2 + \frac{1}{2C}q_1^2 - q_1 E
\end{aligned}
$$

Hamiltonian method

RLC Circuit

Hence, the Hamiltonian equations are

$$
\begin{aligned}
\dot{q}_1 &= \frac{\partial H}{\partial p_1} = \frac{1}{L_1}p_1 + \frac{1}{L_2}(p_1 + p_2), \\
\dot{q}_2 &= \frac{\partial H}{\partial p_2} = \frac{1}{L_2}(p_1 + p_2), \\
\dot{p}_1 &= -\frac{\partial H}{\partial q_1} - \frac{\partial \mathcal{R}}{\partial \dot{q}_1} = -\frac{1}{C}q_1 + E, \\
\dot{p}_2 &= -\frac{\partial H}{\partial q_1} - \frac{\partial \mathcal{R}}{\partial \dot{q}_2} = -R\dot{q}_2 = -\frac{R}{L_2}(p_1 + p_2)
\end{aligned}
$$



$$\mathcal{L} = \frac{1}{2} L\dot{q}_1^2 - \frac{1}{2C_1} q_2^2 - \frac{1}{2C_2} q_3^2 + E(q_1 + q_3),$$

$$\mathcal{R} = \frac{1}{2} R_1 \left(\dot{q}_1 - \dot{q}_2\right)^2 + \frac{1}{2} R_2 \left(\dot{q}_1 - \dot{q}_2 + \dot{q}_3\right)^2$$

This gives

$$p_1 = L_1 \dot{q}_1, \qquad p_2 = 0, \qquad p_3 = 0$$

Thus, H can be written as

$$H = \frac{1}{2L_1}p_1^2 + \frac{1}{2C_1}q_2^2 + \frac{1}{2C_2}q_3^2 - E(q_1 + q_3).$$

The first set of Hamiltonian equations give $\dot{q}_1 = \partial H/\partial p_1 = p_1/L_1$.

Since $p_2 = p_3 = 0$ $\partial H/\partial p_2$ and $\partial H/\partial p_3$ are undefined, and $\dot{p}_2 = \dot{p}_3 = 0$.
The second set of Hamiltonian equations give

$$\begin{aligned}
\dot{p}_1 &= E - R_1\left(\dot{q}_1 - \dot{q}_2\right) - R_2(\dot{q}_1 - \dot{q}_2 + \dot{q}_3)\,,\\
\dot{p}_2 &= -\frac{q_2}{C_1} + R_1\left(\dot{q}_1 - \dot{q}_2\right) + R_2\left(\dot{q}_1 - \dot{q}_2 + \dot{q}_3\right) = 0,\\
\dot{p}_3 &= -\frac{q_3}{C_2} + E - R_2\left(\dot{q}_1 - \dot{q}_2 + \dot{q}_3\right) = 0.
\end{aligned}$$

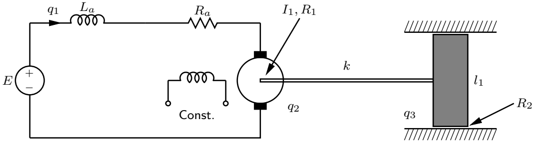
Algebraic manipulation of these three equations yield

$$\dot{p}_1 = E - \frac{q_2}{C_1},$$
$$\dot{q}_2 = -\frac{q_2}{R_1 C_1} - \frac{q_3}{R_1 C_2} + \frac{p_1}{L_1} + \frac{E}{R_1},$$
$$\dot{q}_3 = \frac{q_2}{R_2 C_1} - \frac{R_1 + R_2}{R_1 R_2}\left(\frac{q_2}{C_1} + \frac{q_3}{C_2} - E\right)$$

These three are the first-order differential equations of the system.

Hamiltonian method

Separately excited DC motor and mechanical load system with a flexible shaft.



- q_1: the charge flowing in the armature circuit
- q_2: the angle of the rotor
- q_3: the angle of the load wheel

There are two sub-systems: the electrical circuit and the mechanical part.

- These two sub-systems interact through the torque F exerted by the electrical side one the mechanical side.

$$E_b = K\phi\dot{q}_2,$$

- The back emf E_b exerted by the mechanical side on the electrical side.

$$F = K\phi\dot{q}_1$$

Hamiltonian method

Separately excited DC motor and mechanical load system with a flexible shaft.

The electrical sub-system:

$$T_e = \frac{1}{2} L_a \dot{q}_1^2, \qquad V_e = -(E - E_b) q_1$$
$$\mathcal{R}_e = \frac{1}{2} R_a \dot{q}_1^2$$
$$p_1 = \frac{\partial \mathcal{L}_e}{\partial \dot{q}_1} = \frac{\partial T_e}{\partial \dot{q}_1} = L_a \dot{q}_1$$

The Hamiltonian H_e of the electrical sub-system in terms of q_1 and p_1 as

$$H_e = \frac{1}{2L_a} p_1^2 - E q_1 + E_b q_1$$

This gives the first-order equations as

$$\dot{q}_1 = \frac{\partial H_e}{\partial p_1} = \frac{p_1}{L_a}$$
$$\dot{p}_1 = \frac{\partial H_e}{\partial q_1} - \frac{\partial \mathcal{R}_e}{\partial \dot{q}_1} = E - E_b - R_a \dot{q}_1 = E - E_b - \frac{R_a}{L_a} p_1$$

Hamiltonian method

Separately excited DC motor and mechanical load system with a flexible shaft.

The mechanical sub-system:

$$T_m = \frac{1}{2} I_1 \dot{q}_2^2 + \frac{1}{2} I_2 \dot{q}_3^2, \qquad V_m = \frac{1}{2} k (q_2 - q_3)^2 - F q_2$$

$$\mathcal{R}_m = \frac{1}{2} R_1 \dot{q}_2^2 + \frac{1}{2} R_2 \dot{q}_3^2$$

The generalized momenta, p_2 and p_3 are given by

$$p_2 = \frac{\partial T_m}{\partial \dot{q}_2} = I_1 \dot{q}_2, \qquad p_3 = \frac{\partial T_m}{\partial \dot{q}_3} = I_2 \dot{q}_3$$

The Hamiltonian function of the mechanical system is

$$H_m = \frac{1}{2I_1} p_2^2 + \frac{1}{2I_2} p_3^2 + \frac{1}{2} k (q_2 - q_3)^2 - F q_2.$$

Hamiltonian method

Separately excited DC motor and mechanical load system with a flexible shaft.

The first-order equations for the mechanical sub-system:

$$\dot{q}_2 = \frac{\partial H_m}{\partial p_2} = \frac{p_2}{I_1}$$

$$\dot{q}_3 = \frac{\partial H_m}{\partial p_3} = \frac{p_3}{I_2}$$

$$\dot{p}_2 = -\frac{\partial H_m}{\partial q_2} - \frac{\partial \mathcal{R}_m}{\partial \dot{q}_2} = -k(q_2 - q_3) + F - R_1\dot{q}_2 = -k(q_2 - q_3) + F - \frac{R_1}{I_1}p_2$$

$$\dot{p}_3 = -\frac{\partial H_m}{\partial q_3} - \frac{\partial \mathcal{R}_m}{\partial \dot{q}_3} = k(q_2 - q_3) - R_2\dot{q}_3 = k(q_2 - q_3) - \frac{R_2}{I_2}p_3.$$

The interaction between the mechanical and electrical sub-system:

$$\dot{p}_1 = E - K\phi\frac{p_2}{I_1} - \frac{R_a}{L_a}p_1$$

$$\dot{p}_2 = -k(q_2 - q_3) + K\phi\frac{p_1}{L_a} - \frac{R_1}{I_1}p_2.$$

Lagrangian method

Separately excited DC motor and mechanical load system with a flexible shaft.

The total kinetic energy without separating the sub-system is

$$T = \frac{1}{2}L_a\dot{q}_1^2 + \frac{1}{2}I_1\dot{q}_2^2 + \frac{1}{2}I_2\dot{q}_3^2$$

The total potential energy is

$$\begin{aligned} V &= -(E - E_b)q_1 + \frac{1}{2}k(q_2 - q_3)^2 - Fq_2 \\ &= -(E - K\phi\dot{q}_2)q_1 + \frac{1}{2}k(q_2 - q_3)^2 - K\phi\dot{q}_1 q_2, \end{aligned}$$

The total Rayleigh function is

$$\mathcal{R} = \frac{1}{2}R_a\dot{q}_1^2 + \frac{1}{2}R_1\dot{q}_2^2 + \frac{1}{2}R_2\dot{q}_3^2$$

Lagrangian method

Separately excited DC motor and mechanical load system with a flexible shaft.

Since $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \neq \frac{\partial T}{\partial \dot{q}_1}$, we have to use

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathcal{R}}{\partial \dot{q}_i} = 0$$

We get the second-order equations as

$$\begin{aligned} L_a\ddot{q}_1 - (E - K\phi\dot{q}_2) + R_a\dot{q}_1 0 \\ I_1\ddot{q}_2 - K\phi\dot{q}_1 + k(q_2 - q_3) + R_1\dot{q}_2 = 0 \\ I_2\ddot{q}_3 - k(q_2 - q_3) + R_2\dot{q}_3 = 0 \end{aligned}$$

The expression forthe generalized momenta will be

$$p_i = \frac{\partial T}{\partial \dot{q}_i}$$

which gives

$$p_1 = L_a\dot{q}_1, \qquad p_2 = I_1\dot{q}_2, \qquad p_3 = I_2\dot{q}_3.$$

The first-order equations are obtained from

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} - \frac{\partial \mathcal{R}}{\partial \dot{q}_i}$$

as

$$\begin{aligned}
\dot{p}_1 &= E - K\phi\dot{q}_2 - R_a\dot{q}_1 \\
&= E - K\phi\frac{p_2}{I_1} - R_a\frac{p_1}{L_a} \\
\dot{p}_2 &= K\phi\dot{q}_1 - k(q_2 - q_3) - R_1\dot{q}_2 \\
&= K\phi\frac{p_1}{L_a} - k(q_2 - q_3) - R_1\frac{p_2}{I_1} \\
\dot{p}_3 &= k(q_2 - q_3) - R_2\dot{q}_3 \\
&= k(q_2 - q_3) - R_2\frac{p_3}{I_2}
\end{aligned}$$

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