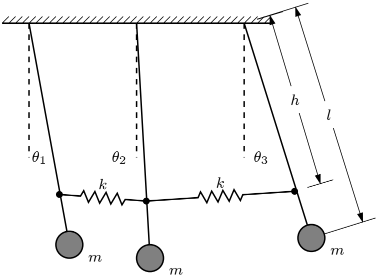
INC 693, 481 Dynamics System and Modelling: Lagrangian Method II

Dr.-Ing. Sudchai Boonto
Assistant Professor

Department of Control System and Instrumentation Engineering
King Mongkut's Unniversity of Technology Thonburi Thailand

Spring-connected triple pendulum system



The minimum set of coordinates are the three angles θ_1, θ_2 and θ_3

$$T = \frac{1}{2} m l^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 \right)$$

The potential energy consists of two parts: the energy due to the gravitational force and the strain energy of the spring

$$V_g = mlg(1 - \cos\theta_1) + mgl(1 - \cos\theta_2) + mgl(1 - \cos\theta_3) \approx \frac{1}{2} mgl \left(\theta_1^2 + \theta_2^2 + \theta_3^2 \right)$$

, where $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$

The elongations of the springs are given by

$$h(\sin\theta_2 - \sin\theta_1) \approx h(\theta_2 - \theta_2)$$
$$h(\sin\theta_3 - \sin\theta_2) \approx h(\theta_3 - \theta_2)$$

Spring-connected triple pendulum system

Hence, the energy stored in the springs is

$$V_s = \frac{1}{2}kx^2 = \frac{1}{2}kh^2\left[(\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2\right]$$

This gives the total potential energy as

$$V = \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2 + \theta_3^3\right) + \frac{1}{2}kh^2\left[(\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2\right]$$

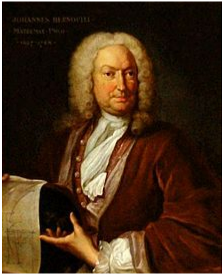
There are three generalized coordinates and hence the system will be described by three Lagrangian equations. For θ_1, θ_2, and θ_3, the Lagrangian equation gives respectively

$$l^2m\ddot{\theta}_1 + mgl\theta_1 + kh^2(\theta_1 - \theta_2) = 0,$$
$$l^2m\ddot{\theta}_2 + mgl\theta_2 + kh^2(\theta_2 - \theta_1) + kh^2(\theta_2 - \theta_3) = 0,$$
$$l^2m\ddot{\theta}_3 + mgl\theta_3 + kh^2(\theta_3 - \theta_2) = 0$$

Principle of Least Action

The principle of least action:

- **mechanical system:** if the system moved from $q = x_1$ at time t_1 to $q = x_2$ at time t_2, the path in between would be the one form which the integral of the Lagrangian function is a minimum.
- **electrical system:** the system always change in a fashion that minimizes the integral of the difference between the energy stored in the inductors and the energy stored in the capacitors.



"What shape a frictionless wire should have in order that ta bead can slide sown it in minimum time." One have to minimize an integral of a function.

Principle of Least Action

The rule can be stated that a term , called S, would be minimized over the path where

$$S = \int_{t_1}^{t_2} f \; dt$$

For dynamical systems, f is the Lagrangian function $\mathcal{L}$. We will find the solution for one-dimensional systems for which $\mathcal{L}$ is a function of q, $\dot{q}$ and t. And it is the integral

$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t) dt$$

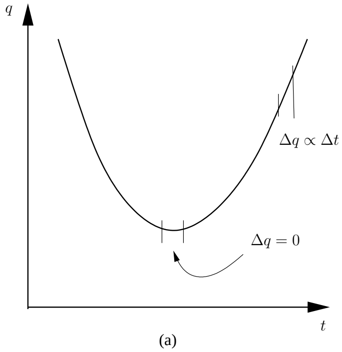
that we have to minimize.

- if a function q of an independent variable t, the minima has the particular property that if t is varied slightly, the variation in q is negligible. In other word $\frac{dq}{dt} = 0$
- In general $\Delta q \propto \Delta t$
- To vary the path, we arbitrarily draw a function $\eta(t)$ and obtain a varied path as

$$\tilde{q}(t) = q(t) + \alpha\eta(t),$$

where α is a variable quantity. A small α will make $\tilde{q}(t)$ deviate slightly from $q(t)$ and a large α will cause a large variation.

Principle of Least Action



(a) the minimal point in a family of points is obtained by varying t and observing the resulting change in q. (b) The minimal function in a family of functions obtained by varying α and observing the resulting variation in S.

Principle of Least Action

To obtain the varied path $\eta(t_1) = \eta(t_2) = 0.$ and

$$\dot{\tilde{q}}(t) = \dot{q}(t) + \alpha\dot{\eta}(t)$$

The new S is given by

$$S = \int_{t_1}^{t_2} \mathcal{L}(\tilde{q}, \dot{\tilde{q}}, t)dt = \int_{t_1}^{t_2} \mathcal{L}(q + \alpha\eta, \dot{q} + \alpha\dot{\eta}, t)dt$$

This function is minimized if $\frac{\partial S}{\partial \alpha} = 0$ at $\alpha = 0$

$$\frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} \frac{\partial}{\partial \alpha}\mathcal{L}(\tilde{q}, \dot{\tilde{q}}, t)dt = \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial \tilde{q}}\frac{\partial \tilde{q}}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}}\frac{\partial \dot{\tilde{q}}}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial t}\frac{\partial t}{\partial \alpha}\right] dt$$

Since t is independent of α, $\frac{\partial t}{\partial \alpha} = 0$, then

$$\frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q}\eta(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\eta}(t)\right] dt = 0$$

Principle of Least Action

Integrating the second term by parts, we get

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial q}\eta(t)dt + \left.\frac{\partial \mathcal{L}}{\partial \dot{q}}\eta(t)\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\eta(t)dt = 0$$

The second term would always vanish. Thus the condition for minimum S becomes

$$\int_{t_1}^{t_2} \eta(t)\left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right]dt = 0$$

Since $\eta(t)$ can be any arbitrary function and the integral must vanish for all ηs, the term in bracket must be zero. Hence

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

This is the Lagrangian equation for one generalized coordinate. Thus, the principle of least action yields the same result as Newtonian or Lagrangian mechanics.

Systems with two degrees of freedom

The system has two degrees of freedom q_1 and q_2 then

$$S = \int_{t_2}^{t_1} \mathcal{L}(q_1, \dot{q}_1, q_2, \dot{q}_2, t) dt$$

We define two arbitrary functions $\eta_1(t)$ and $\eta_2(t)$ with the boundary conditions

$$\eta_1(t_1) = \eta_1(t_2) = \eta_2(t_1) = \eta_2(t_2) = 0$$

Then we obtain varied functions $\tilde{q}_1$ and $\tilde{q}_2$ with the help of a variable α as

$$\begin{aligned} \tilde{q}_1(t) &= q_1(t) + \alpha\eta_1(t) \\ \tilde{q}_2(t) &= q_2(t) + \alpha\eta_2(t) \end{aligned}$$

At the minimum,

$$\frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q_1} \eta_1(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}_1} \dot{\eta}_1(t) + \frac{\partial \mathcal{L}}{\partial q_2} \eta_2(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}_2} \dot{\eta}_2(t) \right] dt = 0$$

Systems with two degrees of freedom

The terms involving η_1 and η_2 are then integrated by parts that yield

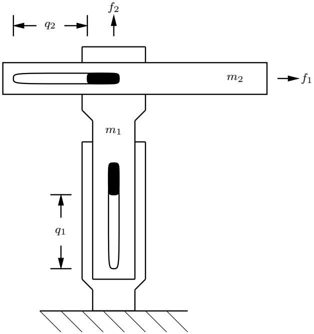
$$\int_{t_1}^{t_2} \left\{ \eta_1(t) \left[\frac{\partial \mathcal{L}}{\partial q_1} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \right) \right] + \eta_2(t) \left[\frac{\partial \mathcal{L}}{\partial q_2} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2} \right) \right] \right\} dt = 0$$

Since this must hold for all arbitrary choices of the functions $\eta_1(t)$ and $\eta_2(t)$ we get two Euler's equations:

$$\left[\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \right) - \frac{\partial \mathcal{L}}{\partial q_1} \right] = 0$$
$$\left[\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2} \right) - \frac{\partial \mathcal{L}}{\partial q_2} \right] = 0$$

The law of least action yields two Lagrangian equations for q_1 and q_2 . Generalizing the line of argument, one can see that for a system with n configuration coordinates, the same least action principle is equivalent to a set of n Lagrangian equations, one for each of the independent degrees of freedom.

Two-link cartesian robot



Consider the manipulator, consisting of two links and two prismatic joints. The masses of the two links by m_1 and m_2 and denote the displacement of the two prismatic joints by q_1 and q_2, respectively. There are 2 generalized coordinates for the manipulator.

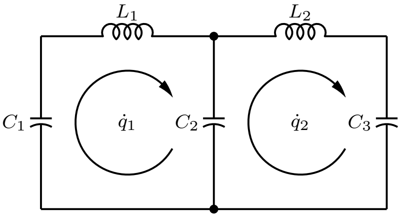
$$T = \frac{1}{2}\left((m_1+m_2)\dot{q}_1^2 + m_2\dot{q}_2^2\right)$$
$$V = (m_1+m_2)gq_1$$
$$\mathcal{L} = \frac{1}{2}\left(m_1\dot{q}_1^2 + m_2\dot{q}_2^2\right) - (m_1+m_2)gq_1$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} = (m_1+m_2)\ddot{q}_1 + (m_1+m_2)g = 0$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} = m_2\ddot{q}_2 = 0$$

Lagrangian Method Applied to Electrical Circuits

- the generalized coordinates in a mechanical system are the *position* variables that are consistent with the constraints.
- in electrical systems, the variables should be *charge* and have to be defined depending on the circuit connection.
- the configuration coordinates q_i in an electrical circuit are defined as the charges flowing in the meshes.
- the configuration coordinates $\dot{q}_i$ are the mesh currents.
- the equivalences between mechanical and electrical systems are:
  - an inductor is equivalent to an inertial element like a mass.
  - a capacitor is equivalent to a compliant element like a spring.

Lagrangian Method Applied to Electrical Circuits



The configuration of the system can be completely defined by the q_1 and q_2 coordinates.

$$T = \frac{1}{2} L_1 \dot{q}_1^2 + \frac{1}{2} L_2 \dot{q}_2^2$$
$$V = \frac{1}{2C_1} q_1^2 + \frac{1}{2C_2} (q_1 - q_2)^2 + \frac{1}{2C_3} q_2^2$$

Then, the Lagrangian equation for the q_1 and q_2 coordinates are

$$L_1 \ddot{q}_1 + \frac{q_1}{C_1} + \frac{1}{C_2}(q_1 - q_2) = 0$$
$$L_2 \ddot{q}_2 + \frac{q_2}{C_3} - \frac{1}{C_2}(q_1 - q_2) = 0$$

Note that the equations are second order differential equations.

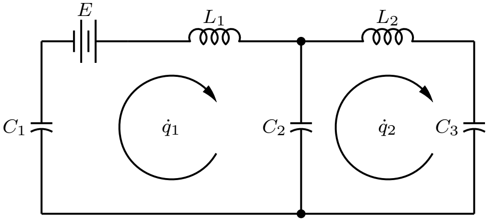
Systems with External Forces or Electromotive Forces

- So far we have considered only conservative systems with no externally applied force or voltage source.
- if there are external forces or voltage sources $F_i(t)$ present in the system, the same general framework can be used to formulate the differential equations.
- this can be done by adding a term to V.

$$V \quad \text{due to external force} \ = -F_i q_i$$

- This term, when differentiated with respect to that generalized coordinate, will yield the external force. Therefore, the system is still conservative, and the same Lagrangian equation can be applied.

Systems with External Forces or Electromotive Forces



Here

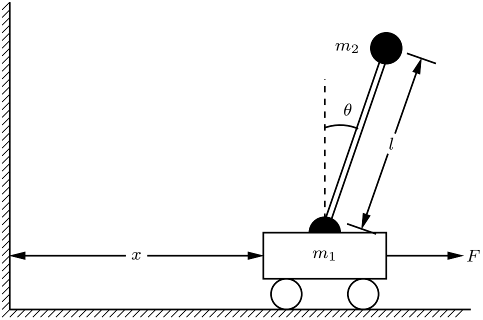
$$T = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2, \qquad V = \frac{1}{2C_1}q_1^2 + \frac{1}{2C_3}q_2^2 + \frac{1}{2C_2}(q_1 - q_2)^2 - Eq_1$$

and the Lagrangian equation will be modified to

$$L_1\ddot{q}_1 + \frac{q_1}{C_1} + \frac{1}{C_2}(q_1 - q_2) - E = 0$$

$$L_2\ddot{q}_2 + \frac{q_2}{C_3} - \frac{1}{C_2}(q_1 - q_2) = 0$$

Inveted pendulumn on a cart



- kinetic energy of the cart is $\frac{1}{2}m_1\dot{x}^2$
- the horizontal component of the kinetic energy of the bob is the kinetic energy of the summation of horizontal and vertical velocity vectors $\frac{1}{2}m_2(l^2\dot{\theta}^2 + \dot{x}^2 + 2l\dot{\theta}\dot{x}\cos\theta)$
- the vertical component of the kinetic energy of the bob is $\frac{1}{2}m_2(l\dot{\theta}\sin\theta)^2$
- the potential has two parts: one is due to the height of the bob from the level of contact $m_2gl\cos\theta$, and due to the applied force $-Fx$.

Inveted pendulumn on a cart

The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\theta}^2 + m_2 l\dot{\theta}\dot{x}\cos\theta - m_2 gl\cos\theta + Fx$$

We obtain the partial derivatives as

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{x}} &= m_1\dot{x} + m_2\dot{x} + m_2 l\dot{\theta}\cos\theta, \\
\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= m_2 l^2\dot{\theta} + m_2 l\dot{x}\cos\theta, \\
\frac{\partial\mathcal{L}}{\partial x} &= F, \\
\frac{\partial\mathcal{L}}{\partial\theta} &= -m_2 l\dot{\theta}\dot{x}\sin\theta + m_2 gl\sin\theta
\end{aligned}$$

Finally we get the equations for x and θ as

$$\begin{aligned}
m_1\ddot{x} + m_2\ddot{x} + m_2 l\cos\theta\ddot{\theta} - m_2 l\dot{\theta}^2\sin\theta - F &= 0 \\
l\ddot{\theta} + \ddot{x}\cos\theta - g\sin\theta &= 0
\end{aligned}$$

System with Resistances or Frictions

- From Newton-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i = 0$$

- Q_i are dissipative elements, belong to resistances or frictions, depending on the velocity. (think about force from friction is $k\dot{x}$ and a voltage across a resistor is Ri.
- As known as Rayleigh potential energy, we have

$$\mathcal{R} = \sum \frac{1}{2} R_i \dot{q}_i^2$$

  We also consider the friction coefficient b as the resistance R of the movement.
- Then

$$Q_i = \underbrace{-\frac{\partial V}{\partial q_i}}_{\text{conservative}} \underbrace{-\frac{\partial \mathcal{R}}{\partial \dot{q}_i}}_{\text{dissipative}}$$

System with Resistances or Frictions

- Finally we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial \mathcal{R}}{\partial \dot{q}_i} = 0$$

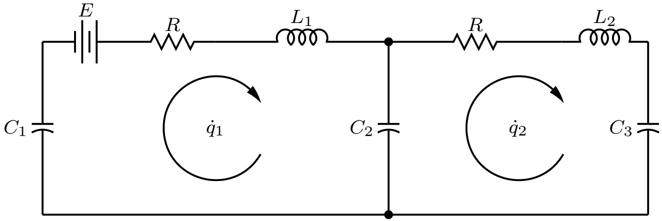
$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_i}\right) - \frac{\partial (T-V)}{\partial q_i} + \frac{\partial \mathcal{R}}{\partial \dot{q}_i} = 0$$

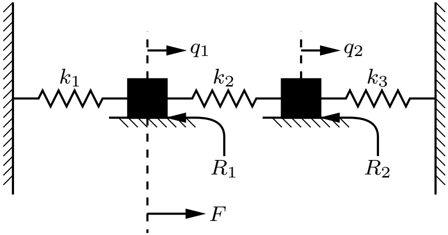
Since $\partial V/\partial \dot{q}_i = 0$, hence

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathcal{R}}{\partial \dot{q}_i} = 0$$

System with Resistances or Frictions

RLC and spring-mass system





System with Resistances or Frictions

RLC and spring-mass system

$$T = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2 \qquad \text{electrical domain}$$

$$= \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 \qquad \text{mechanical domain}$$

$$V = \frac{1}{2C_1}q_1^2 + \frac{1}{2C_2}q_2^2 + \frac{1}{2C_2}(q_1 - q_2)^2 - Eq_1 \qquad \text{electrical domain}$$

$$= \frac{1}{2}k_1q_1^2 + \frac{1}{2}k_3q_2^2 + \frac{1}{2}k_2(q_1 - q_2)^2 - Fq_1 \qquad \text{mechanical domain}$$

$$\mathcal{R} = \frac{1}{2}R_1\dot{q}_1^2 + \frac{1}{2}R_2\dot{q}_2^2$$

Then the dynamic equations are

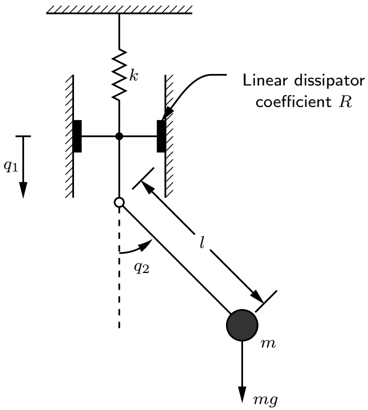
$$L_1\ddot{q}_1 + \frac{1}{C_1}q_1 + \frac{1}{C_2}(q_1 - q_2) - E + R_1\dot{q}_1 = 0$$

$$L_2\ddot{q}_2 + \frac{1}{C_3}q_2 - \frac{1}{C_2}(q_1 - q_2) + R_2\dot{q}_2 = 0$$

For mechanical system: $L \to m$, and $1/C \to k$.

System with Resistances or Frictions

Spring-pendulum system



The velocity for the spring-pendulum is the summation of a vector $l\dot{q}_2$ and a vecotr $\dot{q}_1$ then we have

$$T = \frac{1}{2}ml^2\dot{q}_2^2 + \frac{1}{2}m\dot{q}_1^2 - ml\dot{q}_1\dot{q}_2\sin q_2$$

$$V = -mgl\sin q_2 - mgq_1 + \frac{1}{2}kq_1^2$$

The system Lagrangian is thus

$$\mathcal{L} = \frac{1}{2}ml^2\dot{q}_2^2 + \frac{1}{2}m\dot{q}_1^2 - \frac{1}{2}kq_1^2 - ml\dot{q}_1\dot{q}_2\sin q_2 + mgl\sin q_2 + mgq_1$$

$$\mathcal{R} = \frac{1}{2}R\dot{q}_1^2$$

System with Resistances or Frictions

Spring-pendulum system

For the coordinate q_1, the Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} + \frac{\partial \mathcal{R}}{\partial \dot{q}_1} = 0$$

$$m\ddot{q}_1 - ml\ddot{q}_2 \sin q_2 - ml\dot{q}_2^2 \cos q_2 + kq_1 - mg + R\dot{q}_1 = 0$$

For the coordinate q_2, the Lagrange equation is

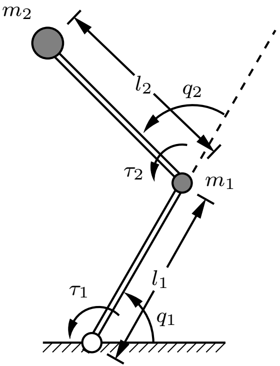
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} + \frac{\partial \mathcal{R}}{\partial \dot{q}_2} = 0$$

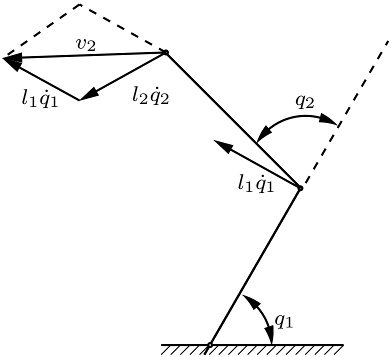
$$ml^2\ddot{q}_2 - ml\ddot{q}_1 \sin q_2 - ml\dot{q}_1\dot{q}_2 \cos q_2 + ml\dot{q}_1\dot{q}_2 \cos q_2 + mgl \sin q_2 = 0$$

$$l\ddot{q}_2 - \ddot{q}_1 \sin q_2 + g \sin q_2 = 0$$

Mechanical Arm system

A mechanical arm system consists of two light rods moved in horizontal plane so that gravity can be neglected. The arm is driven by two motors which are represented by torque sources τ_1 and τ_2.





Mechanical Arm system

The system moves in a horizontal plane and there are no compliant members, therefore the system potential energy is zero. There are two force from τ_1 and τ_2. Then

$$V = -\tau_1 q_1 - \tau_2 q_2$$

The kinetic energy is the sum of the kinetic energies of the masses m_1, m_2 and

$$T = \frac{1}{2} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{q}_1^2 + l_2^2 \dot{q}_2^2 + 2 l_1 l_2 \dot{q}_1 \dot{q}_2 \cos q_2 \right)$$

For the coordinated q_1, q_2, the Lagrange equation are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} = 0$$

$$(m_1 + m_2) l_1^2 \ddot{q}_1 + m_2 l_1 l_2 \ddot{q}_2 \cos q_2 - m_2 l_1 l_2 \dot{q}_2^2 \sin q_2 - \tau_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} = 0$$

$$m_2 l_2^2 \ddot{q}_2 + m_2 l_1 l_2 \ddot{q}_1 \cos q_2 - 2 m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \sin q_2 - \tau_2 = 0$$

Reference

1. Wellstead, P. E. *Introduction to Physical System Modelling*, Electronically published by: `www.control-systems-principles.co.uk`, 2000
2. Banerjee, S., *Dynamics for Engineers*, John Wiley & Sons, Ltd., 2005
3. Rojas, C. , *Modeling of Dynamical Systems*, Automatic Control, School of Electrical Engineering, KTH Royal Institute of Technology, Sweeden
4. Fabien, B., *Analytical System Dynamics: Modeling and Simulation* Springer, 2009
5. Spong, M. W. and Hutchinson, S. and Vidyasagar, M., *Robot Modeling and Control*, John Wiley & Sons, Inc.