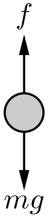
INC 693, 481 Dynamics System and Modelling: Lagrangian Method I

Dr.-Ing. Sudchai Boonto
Assistant Professor

Department of Control System and Instrumentation Engineering
King Mongkut's Unniversity of Technology Thonburi Thailand



by Newton's second law, the equation of motion is

$$m\ddot{y} = f - mg$$

$$m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = \frac{d}{dt}\frac{\partial T}{\partial \dot{y}},$$

where $T = \frac{1}{2}m\dot{y}^2$ is the **kinetic energy**. We can express the gravitational force as

$$mg = \frac{\partial}{\partial y}(mgy) = \frac{\partial V}{\partial y},$$

where $V = mgy$ is the **potential energy** due to gravity.

Let define

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{y}^2 - mgy$$

and note that

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial T}{\partial \dot{y}} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial V}{\partial y}$$

Then we can write the all equations as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = f$$

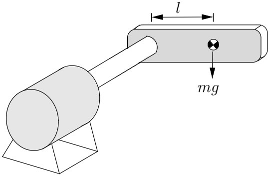
- the function $\mathcal{L}$, which is the difference of the kinetic and potential energy, is called the **Lagrangian** of the system.

The last equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = f$$

is called the **Euler-Lagrange Equation**.

- The Euler-Lagrange equations provide a formulation of the dynamic equations of motion equivalent to those derived using Newton's Second Law.
- However, the Lagrangian approach is advantageous for more complex systems such as multi-link robots.
- From the equivalence of the systems, we can apply such technique to the electrical, fluid, thermal systems.



A single-link robot arm consisting of a rigid link coupled through a gear train to a DC motor. Let θ_l and θ_m denote the angles of the link and motor shaft. Then $\theta_m = r\theta_l$ where $r:1$ is the gear ratio. The kinetic energy of the system is given by

$$T = \frac{1}{2} J_m \dot{\theta}_m^2 + \frac{1}{2} J_l \dot{\theta}_l^2 = \frac{1}{2} \left(r^2 J_m + J_l \right) \dot{\theta}_l^2,$$

where J_m, J_l are the rotational inertias of the motor and link.

The potential energy is given as

$$V = Mgl(1 - \cos\theta_l),$$

where M is the total mass of the link and l is the distance from the joint axis to the link center of mass.

Defining $J = r^2 J_m + J_l$, the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2} J \dot{\theta}_l^2 - Mgl(1 - \cos\theta_l)$$

Substituting the expression into the Euler-Lagrange equations yields

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_l} - \frac{\partial \mathcal{L}}{\partial \theta_l} &= \frac{d}{dt}\left(J\dot{\theta}_l\right) + Mgl\sin\theta_l \\ &= J\ddot{\theta}_l + Mgl\sin\theta_l = \tau_l \end{aligned}$$

The generalized force τ_l represents those external forces and torques that are not derivable from a potential function. τ_l consists of the motor torque $u = r\tau_m$, reflected to the link, and damping torques $b_m\dot{\theta}_m$, and $b_l\dot{\theta}_l$. Reflecting the motor damping to the link is

$$\tau = u - b\dot{\theta}_l,$$

where $b = rb_m + b_l$. The complete expression is

$$J\ddot{\theta}_l + b\dot{\theta}_l + Mgl\sin\theta_l = u$$

General Cases

In the Newtonian approach, there is two kinds of forces: the given force F and the constraint force F^c.

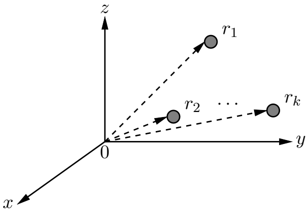
- The given force includes the externally impressed forces, and the forces of interaction between mass points through springs and frictional elements.
- the newtonian equation for each jth mass point is

$$m_j\ddot{r}_j - F_j = F_j^c$$

- for N number of mass points, there will be N equation and the sum of N equations is

$$\sum_{j=1}^{N}(m_j\ddot{r}_j - F_j) = \sum_{j=1}^{N} F_j^c$$

General Cases : System of k particles



- If k particles are move freely, then it is k equations to explain the dynamic.

- If the motion of the particles is constrained in some fashion, one must take into account not only the applied forces F but also the constraint forces F^c.
- For example, two particles are joined by a massless rigid wire of length l. Then the two coordinates r_1 and r_2 must satisfy the constraint

$$\|r_1 - r_2\| = l, \qquad \text{or} \qquad (r_1 - r_2)^T(r_1 - r_2) = l^2$$

General Cases : System of k particles

- If one applies some external forces to each particle, then the particles experience not only these external forces but also the force exerted by the wire, which is along the direction $r_2 - r_1$ and of appropriate magnitude.
- we can analyze the motion of these two particles by:
  1. compute under each set of external forces with the corresponding constraint force.
  2. using a method that does not require the knowledge of the constraint force. This method is more preferable.
- A constraint on the k coordinates $r_1, \dots, r_k$ is called **holonomic** if it has a form

$$g_i(r_1, \dots, r_k) = 0, \qquad i = 1, \dots, l$$

otherwise we call it **nonholonomic**.

Generalized coordinates

Firstly, we need to minimize the number of coordinates.

- If a system is subjected to l holonomic constraints, then the constrained system has l fewer degrees-of-freedom than the unconstrained system.
- For example, it is possible to express the coordinates of the k particles in terms of n **generalized coordinates** $q_1, \ldots, q_n$. We assume that the coordinates of the various particles, subjected to the set of constraints (h constraints), can be expressed in the form

$$r_i = r_i(q_1, \ldots, q_n), \qquad i = 1, \ldots, k, \quad \text{where } n = k - h$$

- the set of coordinates $\{q_1, q_2, q_3, \ldots, q_n\}$ is called a set of generalized coordinates of a system if and only if the number n of its members is necessary and sufficient to define the configuration or positional status of the system uniquely.

Generalized coordinates

Consider r_j in terms of the new n coordinates:

$$
\begin{aligned}
r_1 &= r_1(q_1, q_2, q_3, \ldots, q_n, t), \\
&\vdots \\
r_N &= r_N(q_1, q_2, q_3, \ldots, q_n, t)
\end{aligned}
$$

or, in short

$$r_j = r_j(q_i, t),$$

where $j = 1, \ldots, N$ and $i = 1, \ldots, n$. By differentiation with respect to t and by applying the chain rule, we have

$$\dot{r}_j = \frac{dr_j}{dt} = \sum_{i=1}^{n} \frac{\partial r_j}{\partial q_i}\dot{q}_i + \frac{\partial r_j}{\partial t}.$$

Generalized coordinates

The term $\partial r_j / \partial t$ is independent of $\dot{q}_i$ and $\dot{q}_k$ is also independent of $\dot{q}_i$. Partially differentiating this equation with respect to $\dot{q}_i$ we get

$$\frac{\partial \dot{r}_j}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} \left(\sum_{i=1}^{n} \frac{\partial r_j}{\partial q_i} \dot{q}_i + \frac{\partial r_j}{\partial t} \right) = \frac{\partial r_j}{\partial q_i}$$

Now consider **virtual (admissible) displacements**, which are any set $\delta r_1, \ldots, \delta r_k$ of infinitesimal displacements that are consistent with the constraints. The work done by the constraint force at an virtual displacement is given by $F^c \delta r$. At the equilibrium, the net force on each particle is zero, which implies that the work done by each set of virtual displacements is zero. Hence

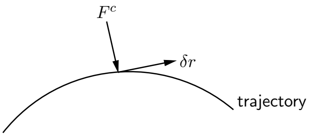
$$\sum_{i=1}^{k} F_i^T \delta r_i = 0, \qquad \text{where } F_i \text{ is the total force on particle } i.$$

Generalized coordinates

The force F_i is the sum of the externally applied force F_i and the constraint force F_i^c . Suppose that the total work done by the constraint force corresponding to any set of virtual displacements is zero, that is

$$\sum_{i=1}^{k}(F_i^c)^T \delta r_i = 0$$

This is from the principle of virtual work.



Principle of virtual work

The work done by external forces corresponding to any set of virtual displacements is zero.

Generalized coordinates

The δr_j in the old set of coordinates is related to virtual displacements δq_i in the new generalized coordinates by

$$\delta r_j = \sum_{i=1}^{n} \frac{\partial r_j}{\partial q_i} \delta q_i, \qquad i = 1, \ldots, k$$

There is no time variation term δt because the virtual displacement involves changes in space coordinates only.

Dyanmical equations in terms of energies

From,

$$\sum_{j=1}^{k} (m_j \ddot{r}_j - F_j)\, \delta r_j = 0$$

Substituting the expression for virtual displacement, we have

$$\sum_{j=1}^{k} (m_j \ddot{r}_j - F_j) \sum_{i=1}^{n} \frac{\partial r_j}{\partial q_i} \delta q_i = 0$$

$$\sum_{i=1}^{n} \left[\sum_{j=1}^{k} (m_j \ddot{r}_j - F_j) \frac{\partial r_j}{\partial q_i} \right] \delta q_i = 0$$

$$\sum_{i=1}^{n} \left[\sum_{j=1}^{k} \left(m_j \ddot{r}_j \frac{\partial r_j}{\partial q_i} - F_j \frac{\partial r_j}{\partial q_i} \right) \right] \delta q_i = 0$$

Consider the time derivative

$$\frac{d}{dt}\left(\dot{r}_j \frac{\partial r_j}{\partial q_i} \right) = \ddot{r}_j \frac{\partial r_j}{\partial q_i} + \dot{r}_j \frac{\partial \dot{r}_j}{\partial q_i} \text{ or } \ddot{r}_j \frac{\partial r_j}{\partial q_i} = \frac{d}{dt}\left(\dot{r}_j \frac{\partial r_j}{\partial q_i} \right) - \dot{r}_j \frac{\partial \dot{r}_j}{\partial q_i}$$

Dyanmical equations in terms of energies

we get

$$\sum_{i=1}^{n}\left[\sum_{j=1}^{k} m_j \frac{d}{dt}\left(\dot{r}_j \frac{\partial r_j}{\partial q_i}\right) - \sum_{j=1}^{k} m_j \dot{r}_j \frac{\partial \dot{r}_j}{\partial q_i} - \sum_{j=1}^{k} F_j \frac{\partial r_j}{\partial q_i}\right] \delta q_i = 0 \qquad (1)$$

The kinetic energy T is

$$T = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{r}_j^2$$

The differentiating with respect to the generalized velocities, we get

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{k} m_j \dot{r}_j \frac{\partial \dot{r}_j}{\partial \dot{q}_i}$$

Since $\partial \dot{r}_j / \partial \dot{q}_i = \partial r_j / \partial q_i$, we get

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{k} m_j \dot{r}_j \frac{\partial r_j}{\partial q_i}$$

Dyanmical equations in terms of energies

Then the 1st term in (1) can be consider as

$$\sum_{j=1}^{k} m_j \frac{d}{dt}\left(\dot{r}_j \frac{\partial r_j}{\partial q_i}\right) = \frac{d}{dt}\left(\sum_{j=1}^{k} m_j \dot{r}_j \frac{\partial r_j}{\partial q_i}\right)$$

Differentiating the expression for kinetic energy with respect to the generalized coordinates, we get

$$\frac{\partial T}{\partial q_i} = \sum_{j=1}^{k} m_j \dot{r}_j \frac{\partial \dot{r}_j}{\partial q_i} \quad \text{which is the 2}^{\text{nd}} \text{ term in } (1)$$

The 3nd is the given forces on the mass points (F_j) along the generalized coordinates as

$$Q_i = \sum_{j=1}^{k} F_j \frac{\partial r_j}{\partial q_i}$$

This forces are called the **generalized forces**.

Dyanmical equations in terms of energies

Then we have

$$\sum_{i=1}^{n} \left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} - Q_i \right\} \delta q_i = 0$$

Since the generalized coordinates are independent, any virtual displacement along the i^{th} coordinate (δq_i) would be independent of virtual displacements. We can conclude that each coefficient in curry bracket is zero, that

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} - Q_i = 0$$

Here Q_i contains all the given forces in the system acting along the i^{th} coordinate:

- forces happen by means of springs or elements of spring-like characteristics.
- external forces, including the force due to gravity.
- forces due to friction.

The first two categories can be derived from a scalar **potential function** denoted as V. The generalized forces are derived from the potential as $Q_i = -\partial V / \partial q_i$.

Dyanmical equations in terms of energies

For the systems where forces are only of the first and second categories (conservative) we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

Defining a new function called the **Lagrangian** function as

$$\mathcal{L} = T - V$$

Since for most systems V is independent of the generalized velocities $\dot{q}_i$,

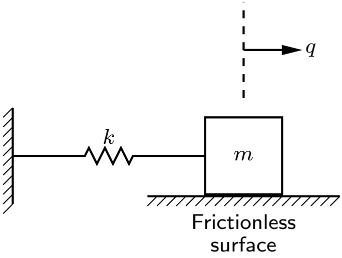
$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

Finally we have

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

This is the Lagrangian equation of conservative systems (no frication or resistive elements) .

Spring mass system



There is only one generalized coordinate q. By Newton's law we have

$$m\ddot{q} = -kq$$

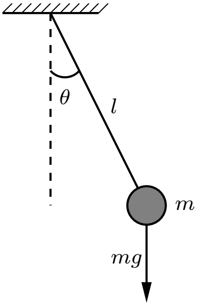
We can show how to use the Lagrangian method:

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}(m\dot{q}) - kq$$

$$m\ddot{q} + kq = 0$$

Simple pendulum



The speed of the pendulum in the tangential direction is $l\dot{\theta}$, and the knetic energy becomes

$$T = \frac{1}{2}m(\dot{\theta}l)^2$$

The potential energy measured with respect to the point of suspension is

$$V = -mgl\cos\theta$$

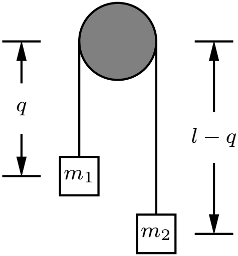
The Lagrangian function becomes

$$\mathcal{L} = \frac{1}{2}m\dot{\theta}^2l^2 + mgl\cos\theta$$

Then

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl\sin\theta$$
$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Motion of Pulley



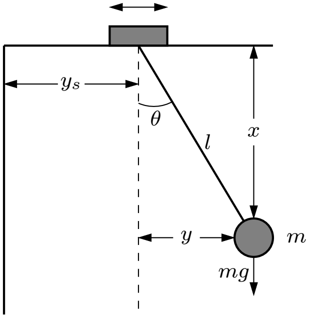
The configuration can be defined with only one generalized coordinate q. The position of the other mass is $(l-q)$ where l is the length of the string.

$$V = -m_1 g q - m_2 g(l-q),$$
$$T = \frac{1}{2}(m_1 + m_2)\dot{q}^2$$
$$\mathcal{L} = T - V = \frac{1}{2}\left(m_1 + m_2\right)\dot{q}^2 + m_1 g q + m_2 g(l-q)$$

Then the Lagrangian equation gives

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\left((m_1+m_2)\dot{q}\right) + (m_1 - m_2)g$$
$$(m_1 + m_2)\,\ddot{q} - (m_1 - m_2)g = 0$$

Motion of Pulley



The coordinates of the mass is given by

$$x = l\cos\theta, \quad y = l\sin\theta, \quad \dot{x} = -l\sin\theta\dot{\theta}, \quad \dot{y} = l\cos\theta\dot{\theta}$$

The position of the mass is $(x, y + y_s)$. Therefore, the kinetic energy is given by

$$T = \frac{1}{2}m\left[(\dot{y} + \dot{y}_s)^2 + \dot{x}^2\right] = \frac{1}{2}m\left(l^2\dot{\theta}^2 + 2l\cos\theta\dot{\theta}\dot{y}_s + \dot{y}_s^2\right)$$

$$V = -mgl\cos\theta$$

$$\mathcal{L} = T - V = \frac{1}{2}m\left(l^2\dot{\theta}^2 + 2l\cos\theta\dot{\theta}\dot{y}_s + \dot{y}_s^2\right) + mgl\cos\theta$$

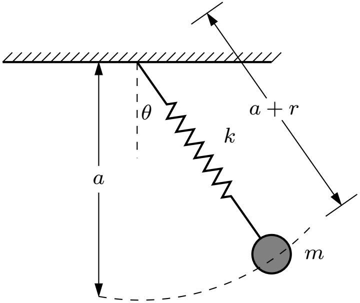
Then the Lagrangian equation gives

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\left(ml^2\dot{\theta} + ml\cos\theta\dot{y}_s\right) + ml\dot{\theta}\dot{y}_s\sin\theta + mgl\sin\theta$$

$$ml^2\ddot{\theta} + ml\cos\theta\ddot{y}_s - ml\dot{\theta}\dot{y}_s\sin\theta + ml\dot{\theta}\dot{y}_s\sin\theta + mgl\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta + \frac{1}{l}\cos\theta\ddot{y}_s = 0$$

Spring-pendulum



The spring pendulum requires two configuration coordinates. One coordinate is θ and the other is dependent on the radial distance from the suspension to the mass.

$$T = \frac{1}{2}m\left[\dot{r}^2 + (a+r)^2\dot{\theta}^2\right]$$

The potential energy has two components – one due to the spring and the other due to the gravitational potential.

$$V = \frac{1}{2}k\left(r + \frac{mg}{k}\right)^2 - mg(a+r)\cos\theta + mga.$$

Then

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(a+r)^2\dot{\theta}^2 - \frac{1}{2}k\left(r + \frac{mg}{k}\right)^2 + mg(a+r)\cos\theta - mga$$

Spring-Pendulum

The Lagrangian equation is

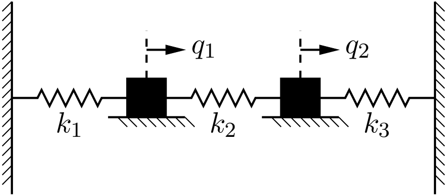
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}(m\dot{r}) - m\dot{\theta}^2(a+r) + k\left(r + \frac{mg}{k}\right) - mg\cos\theta$$

$$m\ddot{r} - m\dot{\theta}^2(a+r) + k\left(r + \frac{mg}{k}\right) - mg\cos\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(m(a+r)^2\dot{\theta}\right) + mg(a+r)\sin\theta$$

$$m(a+r)\ddot{\theta} + 2m\dot{\theta}\dot{r} + mg(a+r)\sin\theta = 0$$

Double spring-mass



The configuration of the system can be completely defined by the q_1 and q_2 coordinates.

$$T = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2$$
$$V = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2(q_1 - q_2)^2 + \frac{1}{2}k_3 q_2^2$$

Then, the Lagrangian equation for the q_1 and q_2 coordinates are

$$m_1\ddot{q}_1 + k_1 q_1 + k_2(q_1 - q_2) = 0$$
$$m_2\ddot{q}_2 + k_3 q_2 - k_2(q_1 - q - 2) = 0$$

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