INC 693, 481 Dynamics System and Modelling: Linear Graph Modeling II

Dr.-Ing. Sudchai Boonto
Assistant Professor

Department of Control System and Instrumentation Engineering
King Mongkut's Unniversity of Technology Thonburi Thailand

State-Space System Representation

The modeling system is written in the form:

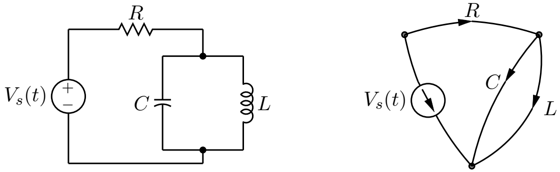
$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx + Du \tag{2}$$

The matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n_u}$ are properties of the system. The output equation matrices $C \in \mathbb{R}^{n_y \times n}$ and $D \in \mathbb{R}^{n_y \times n_u}$ are determined by the particular choice of output variables.

- System order n and selection of a set of state variables from the linear graph.
- Generation of a set of state equations and the system A and B matrices.
- Determination of a suitable set of output equation and derivation of the appropriate C and D.

Simple RLC example



- There are 6 possible variables : $i_c, v_c, i_L, v_L, i_R, v_R$ and i_s
- There are 3 constitute relations:

$$\frac{dv_C}{dt} = \frac{i}{C} i_C, \quad \frac{i_L}{dt} = \frac{1}{L} v_L, \quad i_R = \frac{1}{R} v_R$$

Using a continuity equation and two compatibility equations

$$i_C = i_R - i_L, \quad v_L = v_C, \quad v_R = V_s - v_C$$

Simple RLC example

We have

$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L + \frac{1}{RC}V_s$$

$$\frac{di_L}{dt} = \frac{1}{L}v_C$$

$$A = \begin{bmatrix} -1/RC & -1/C \\ 1/L & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1/RC \\ 0 \end{bmatrix}$$

If the variables i_R, v_R, v_L and i_C are of interest as output variables:

$$i_R = -\frac{1}{R}v_C + \frac{1}{R}V_s, \qquad v_R = -v_C + V_s$$

$$v_L = v_C, \qquad i_C = -\frac{1}{R}v_C - i_L + \frac{1}{R}V_s$$

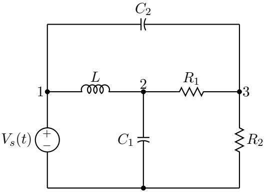
Simple RLC example

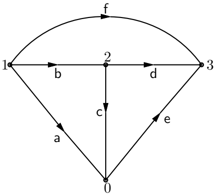
So if the output vector is defined to be

$$y = \begin{bmatrix} i_R & v_R & v_L & i_C \end{bmatrix}^T$$

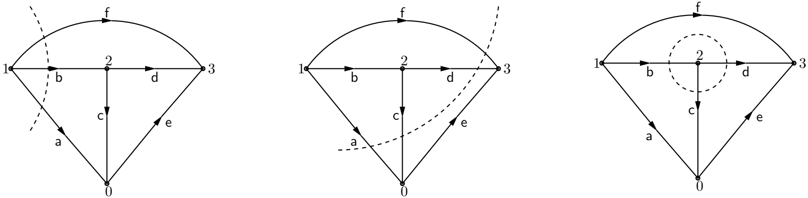
The C and D matrices are

$$C = \begin{bmatrix} -1/R & 0 \\ -1 & 0 \\ 1 & 0 \\ -1/R & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1/R \\ 1 \\ 0 \\ 1/R \end{bmatrix}$$





cutset is a set of branches in a graph, which when cut off, will divide the graph into two disconnected pieces.



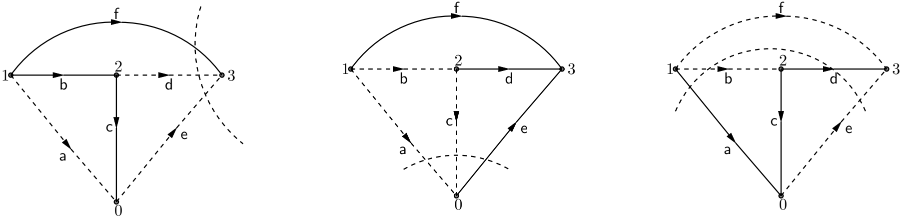


Figure: three possible basic cutsets

- **basic cutset** is the cutset that contains only one tree branch and several co-tree links.
- the continuity equations are corresponding to the basic cutsets are independent.

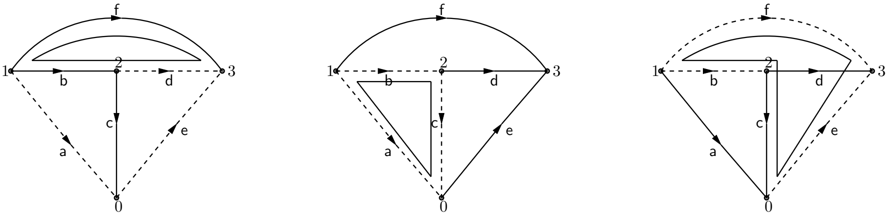


Figure: three possible basic loopsets

- **basic loopset** is a loop that contains only one co-tree link and several tree branches.
- the compatibility equations are corresponding to the basic loopsets are independent.

Linear Graph

Identify the particular tree

A normal tree for a connected system graph is formed by the following steps:

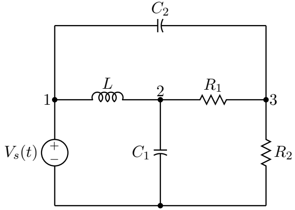
1. Draw the system graph nodes.
2. The tree should include *all* effort sources as tree branches.
3. The tree should include a maximum number of capacitors elements.
4. The tree should include a maximum possible number of resistor elements.
5. The tree may then include the necessary number of inductor elements to complete the tree.

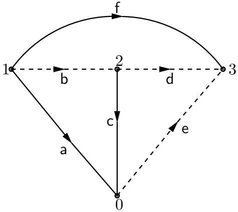
- if the variable is a capacitor element voltage, identify the basic cutset containing that capacitor element voltage. The differential equation is given by the continuity equation for that basic cutset.
- if the variable is an inductor element current, identify the basic loop containing that inductor element current. The compatibility equation for that basic loop will yield the desired differential equation.

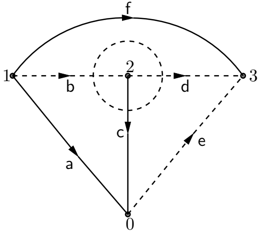
Or

- Select the state variables as effort variables on capacitor energy storage elements in the normal tree branches, and flow variables on inductor energy storage elements in the links.

Linear Graph
Example







The continuity equation

$$i_b - i_c - i_d = 0$$

$$\frac{dv_{C_1}}{dt} = \frac{1}{C_1}(i_b - i_d)$$

First $i_b = i_L$, which is a state variable. And i_d can be expressed as follows:

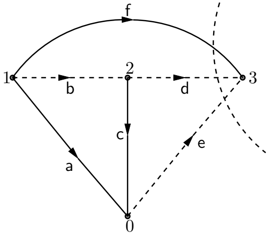
$$i_d = v_d / R_1$$

The basic loop that involves R_1 (loop cafd)

$$v_d = v_c - v_a + v_f = v_{C_1} - E + v_{C_2}$$

Then

$$\begin{aligned}\frac{dv_{C_1}}{dt} &= \frac{1}{C_1}(i_L - \frac{v_d}{R_1}) \\ &= \frac{1}{C_1} i_L - \frac{1}{R_1 C_1} v_{C_1} - \frac{1}{R_1 C_1} v_{C_2} + \frac{1}{R_1 C_1} E\end{aligned}$$



The continuity equation gives

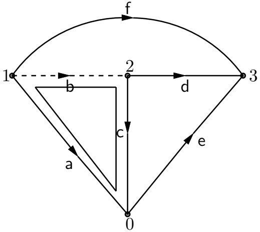
$$i_f + i_d + i_e = 0$$

which in turn gives,

$$\begin{aligned}\frac{dv_{C_2}}{dt} &= \frac{1}{C_2}(-i_d - i_e) \\ &= \frac{1}{C_2}\left(-\frac{v_d}{R_1} - \frac{v_e}{R_2}\right) = \frac{1}{C_2}\left[-\frac{v_{C_1} - E + v_{C_2}}{R_1} - \frac{v_{C_2} - E}{R_2}\right] \\ &= -\frac{1}{R_1 C_2} v_{C_1} - \frac{1}{C_2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) v_{C_2} + \frac{1}{C_2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) E\end{aligned}$$

Linear Graph

Example



The compatibility equation gives

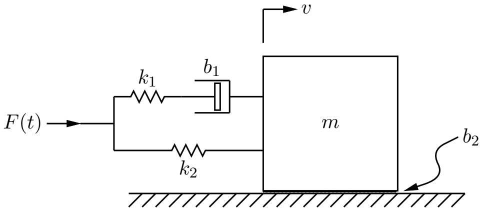
$$v_b + v_c - v_a = 0$$

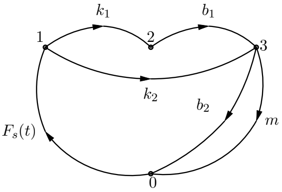
which in turn gives,

$$\begin{aligned}\frac{di_L}{dt} &= \frac{1}{L}(-v_c + v_a) \\ &= -\frac{1}{L}v_{C_1} + \frac{1}{L}E\end{aligned}$$

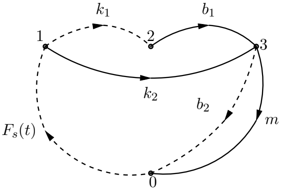
Linear Graph

Mechanical System 1





(a) linear graph



(b) normal tree

the state variables are v_m (capacitor element of the branch) and F_{k_1} (inductor element of the link) and

$$\frac{dv_m}{dt} = \frac{1}{m}F_m, \qquad \frac{dF_{k_1}}{dt} = k_1 v_{k_1}, \qquad F_{b_2} = b_2 v_{b_2}$$
$$v_{b_1} = \frac{1}{b_1}F_{b_1}, \qquad v_{k_2} = \frac{1}{k_2}\frac{dF_{k_2}}{dt}$$

There are two compatibility equations:

$$v_{k_1} = v_{k_2} - v_{b_1}, \qquad v_{b_2} = v_m$$

and three continuity equations:

$$F_{k_2} = F_s(t) - F_{k_1}, \qquad F_{b_1} = F_{k_1}, \qquad F_m = F_s(t) - F_{b_2}$$

The result is

$$\begin{bmatrix} \dot{v}_m \\ \dot{F}_{k_1} \end{bmatrix} = \begin{bmatrix} -\frac{b_2}{m} & 0 \\ 0 & -\frac{k_1 k_2}{b_1(k_1+k_2)} \end{bmatrix} \begin{bmatrix} v_m \\ F_{k_1} \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} F_s(t) + \begin{bmatrix} 0 \\ \frac{k_1}{k_1+k_2} \end{bmatrix} \dot{F}_s(t)$$

The result showing the dependence on the derivative of the input $F_s(t)$. The output equation for v_{b_1} is:

$$v_{b_1} = \frac{1}{b_1} F_{k_1}$$

or in matrix form

$$v_{b_1} = \begin{bmatrix} 0 & \frac{1}{b_1} \end{bmatrix} \begin{bmatrix} v_m \\ F_{k_1} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} F_s(t)$$

The state variables may be transformed as $\dot{\tilde{x}} = A\tilde{x} + (AE + B)u$ or:

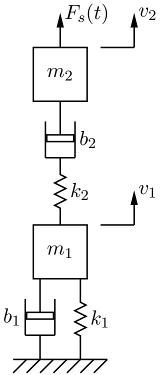
$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -\frac{b_2}{m} & 0 \\ 0 & -\frac{k_1 k_2}{b_1(k_1+k_2)} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ \frac{k_1^2 k_1}{b_1(k_1+k_2)^2} \end{bmatrix} F_s(t)$$

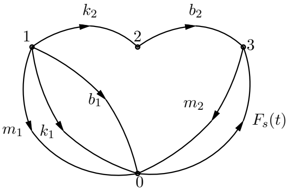
the corresponding output $v_{b_1} = C\tilde{x} + (CE + D)u + F\dot{u}$

$$v_{b_1} = \frac{1}{b_1}\tilde{x}_2 + \frac{k_1}{b_1(k_1 + k_2)} F_s(t)$$

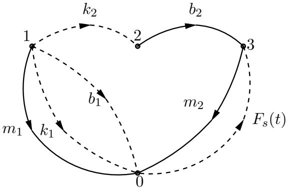
Linear Graph

Mechanical System (In class work)

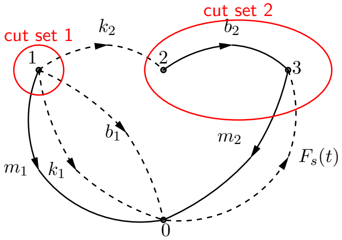




(c) linear graph



(d) normal tree



At a basic cutset 1:

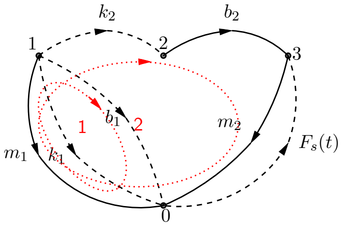
$$i_{m_1} + i_{b_1} + i_{k_1} + i_{k_2} = 0$$

$$\frac{dv_{m_1}}{dt} = -\frac{b_1}{m_1} v_{m_1} - \frac{1}{m_1} F_{k_1} - \frac{1}{m_1} F_{k_2}$$

At a basic cutset 2:

$$i_{m_2} - i_{F_s} - i_{F_{k_2}} = 0$$

$$\frac{dv_{m_2}}{dt} = \frac{1}{m_2} F_{k_2} + \frac{1}{m_2} F_s$$



At a basic loop set 1:

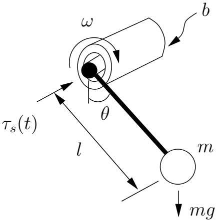
$$v_{k_1} = v_{m_1}$$
$$\frac{dF_{k_1}}{dt} = k_1 v_{m_1}$$

At a basic loop set 2:

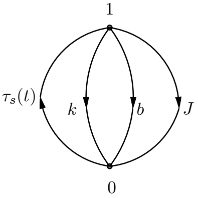
$$-v_{m_1} + v_{k_2} + v_{b_2} + v_{m_2} = 0$$
$$-v_{m_1} + \frac{1}{k_2}\frac{dF_{k_2}}{dt} + \frac{1}{b_2}F_{b_2} + v_{m_2} = 0$$
$$F_{b_2} = F_{k_2}$$
$$\frac{dF_{k_2}}{dt} = -\frac{k_2}{b_2}F_{k_2} - k_2 v_{m_2} + k_2 v_{m_1}$$

The state-space system is

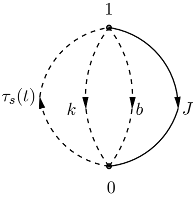
$$\begin{bmatrix} \dot{v}_{m_1} \\ \dot{v}_{m_2} \\ \dot{F}_{k_1} \\ \dot{F}_{k_2} \end{bmatrix} = \begin{bmatrix} -\frac{b_1}{m_1} & 0 & -\frac{1}{m_1} & -\frac{1}{m_1} \\ 0 & 0 & 0 & \frac{1}{m_2} \\ k_1 & 0 & 0 & 0 \\ k_2 & -k_2 & 0 & -\frac{k_2}{b_2} \end{bmatrix} \begin{bmatrix} v_{m_1} \\ v_{m_2} \\ F_{k_1} \\ F_{k_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_2} \\ 0 \\ 0 \end{bmatrix} F_s$$



(a) Nonlinear pendulum



(b) Linear graph



(c) Normal graph

The state-space variables are τ_k, ω_J. By continuity equation, we have

$$i_k + i_b + i_J = i_s \qquad \Rightarrow \qquad \tau_k + \tau_b + \tau_J = T_s$$

$$\tau_k + b\omega_b + J\frac{d\omega_J}{dt} = T_s \qquad \Rightarrow \qquad \frac{d\omega_J}{dt} = -\frac{b}{J}\omega_J - \frac{1}{J}\tau_k + \frac{1}{J}T_s$$

Note: $\omega_J = \omega_b$

The compatibility equation in the basic loop containing τ_k we have

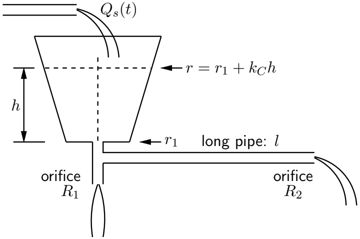
$$v_k = v_J \qquad \Rightarrow \omega_k = \omega_J$$

Since $\tau_k = mgl\sin\theta$ then

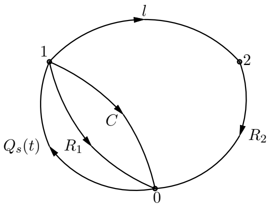
$$\frac{d\tau_k}{dt} = mgl\cos\theta\frac{d\theta_k}{dt} = (mgl\cos\theta)\omega_J$$

The nonlinear state-space equation is

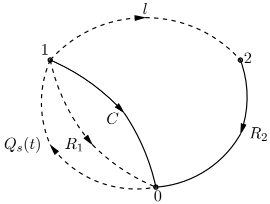
$$\dot{\omega}_J = -\frac{b}{J}\omega_J - \frac{1}{J}\tau_k + \frac{1}{J}T_s$$
$$\dot{\tau}_k = (mgl\cos\theta)\omega_J$$



(a) Fluid distribution system



(b) Linear graph



(c) Normal graph

The fluid flow through an orifice is $Q = C_0\sqrt{|\Delta P|}\text{sgn}(\Delta P)$. The tank is shaped as the frustum of a cone, therefore has a volume which is a nonlinear functin of the height of the fluid in the tank.

$$
\begin{aligned}
V &= \int_0^h \pi r^2 dh = \int_0^h \pi (r_1 + k_C h)^2 dh = \int_0^h \pi \left(r_1^2 + 2r_1 k_C h + k_C^2 h^2\right) dh \\
&= \int_0^h \pi \left(r_1^2 + 2r_1 k_C h + k_C^2 h^2\right) dh = \pi \left(r_1^2 h + r_1 k_C h^2 + k_C^2 \frac{h^3}{3}\right)
\end{aligned}
$$

For an open tank the pressure at the base is $P_C = \rho g h$, then

$$
\begin{aligned}
V &= \frac{\pi r_1^2}{\rho g} P_C + \frac{\pi r_1 k_C}{(\rho g)^2} P_C^2 + \frac{\pi k_C^2}{3(\rho g)^3} P_C^3 \\
&= K_{t_0} P_C + \frac{K_{t_1}}{2} P_C^2 + \frac{K_{t_2}}{3} P_C^3
\end{aligned}
$$

where

$$
K_{t_0} = \frac{\pi r_1^2}{\rho g}, \quad K_{t_1} = \frac{2\pi r_1 k_C}{(\rho g)^2}, \quad \text{and } \frac{\pi k_C^2}{(\rho g)^3}
$$

There are two state-space variables, Q_l and P_c. The continuity equation around node 1 is

$$i_{Q_s} - i_{R_1} - i_C - i_l = 0 \quad \Rightarrow \quad Q_s - Q_{R_1} - Q_C - Q_l = 0$$

Since $P_{R_1} = P_C$ and

$$\begin{aligned}
\frac{dV}{dt} &= Q_C = C\frac{dP_C}{dt} = \left[K_{t_0} + K_{t_1}P_C + K_{t_2}P_C^2\right]\frac{dP_C}{dt} \\
Q_{R_1} &= K_1\sqrt{|P_{R_1}|}\mathrm{sgn}(P_{R_1}), \\
P_{R_1} &= P_C \\
\frac{dP_C}{dt} &= \left[\frac{1}{K_{t_0} + K_{t_1}P_C + K_{t_2}P_C^2}\right]Q_C \\
&= \left[\frac{1}{K_{t_0} + K_{t_1}P_C + K_{t_2}P_C^2}\right]\left[Q_s - K_1\sqrt{|P_C|}\mathrm{sgn}(P_C) - Q_l\right]
\end{aligned}$$

The compatibility is $P_l = P_C - P_{R_2}$ and $P_l = I_l \dfrac{dQ_l}{dt}$ then

$$\frac{I_l dQ_l}{dt} = P_C - P_{R_2}, \qquad P_{R_2} = \frac{1}{K_2^2} Q_{R_2}|Q_{R_2}|, \qquad Q_{R_2} = Q_l$$

$$\frac{dQ_l}{dt} = \frac{1}{I_l}\left[P_C - \frac{1}{K_2^2} Q_l|Q_l|\right]$$

The nonlinear state-space equation is

$$\frac{dQ_l}{dt} = \frac{1}{I_l}\left[P_C - \frac{1}{K_2^2} Q_l|Q_l|\right]$$

$$\frac{dP_C}{dt} = \left[\frac{1}{K_{t_0} + K_{t_1} P_C + K_{t_2} P_C^2}\right]\left[Q_s - K_1\sqrt{|P_C|}\,\mathrm{sgn}(P_C) - Q_l\right]$$

Reference

1. Wellstead, P. E. *Introduction to Physical System Modelling*, Electronically published by: `www.control-systems-principles.co.uk`, 2000
2. Banerjee, S., *Dynamics for Engineers*, John Wiley & Sons, Ltd., 2005
3. Rojas, C. , *Modeling of Dynamical Systems*, Automatic Control, School of Electrical Engineering, KTH Royal Institute of Technology, Sweeden
4. Fabien, B., *Analytical System Dynamics: Modeling and Simulation* Springer, 2009