INC 693, 481 Dynamics System and Modelling: Unified System Representation

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Outline

- Kinematic variables
- Kinetic variables
- Work, power and energy
- System Component
    - Ideal inductors
    - Ideal capacitors
    - Constraint elements
    - Source elements

Generalized Variables and System Elements

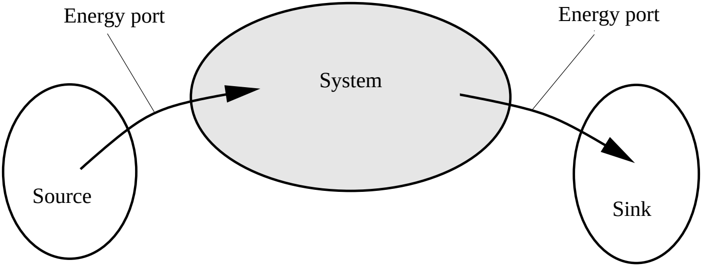
The aims of system models:

- to obtain a description of the dynamic behavior of a system in terms of some physically significant variables in mathematic form.
- in electrical systems are voltage and current
- in mechanical systems are force and velocity
- in fluid systems are pressure and volumetric flow rate

A suitable unifying concept is energy :

- A physical system can be thought of as operating upon a pair of variables whose product is power.

Generalized Variables and System Elements



- energy can be considered as begin injected into a system via an energy port
- energy port applied to read out the system response.

System Variables

In each engineering discipline (i.e. , mechanical, electrical, fluid and thermal) the state of the system is defined using a pair of *kinematic* variable and a pair of *kinetic* variables.

- the kinematic variables is used to describe the motion of objects without considering the course of motion–external force. They are *displacement* $q(t)$ and *flow* $f(t)$.
- the kinetic variables is used to describe the motion of objects with considering the course of motion only. They are *momentum* $p(t)$ and *effort* $e(t)$.

Kinematic Variables

The kinematic variables: displacement $q(t)$ and flow $f(t)$

$$f(t) = \frac{dq(t)}{dt} \qquad \text{and} \qquad q(t) = \int f(t)dt$$

- **Mechanical Translation:** displacement is a linear displacement $x(t)$ and flow is a linear velocity $v(t)$, and

$$v(t) = \frac{dx(t)}{dt} \qquad \text{and} \qquad x(t) = \int v(t)dt$$

- **Mechanical Rotation:** displacement is a angular displacement $\theta(t)$ and flow is a angular velocity $\omega(t)$, and

$$\omega(t) = \frac{d\theta(t)}{dt} \qquad \text{and} \qquad \theta(t) = \int \omega(t)dt$$

Kinematic Variables

- **Electrical:** the displacement is corresponding to the charge $q(t)$ and the flow variable is the current $i(t)$, thus

$$i(t) = \frac{dq(t)}{dt} \qquad \text{and} \qquad q(t) = \int i(t)dt$$

- **Fluid:** the displacement is a volume variable $V(t)$ and the flow variable is the volumetric flow rate $Q(t)$, thus

$$Q(t) = \frac{dV(t)}{dt} \qquad \text{and} \qquad V(t) = \int Q(t)dt$$

- **Thermal:** the displacement is the entropy $S(t)$ and the flow variable is the entropy flow rate $\dot{S}(t)$, then

$$\dot{S} = \frac{dS(t)}{dt} \qquad \text{and} \qquad S(t) = \int \dot{S}(t)dt$$

Kinetic Variables

The kinetic variables: momentum $p(t)$ and effort $e(t)$

$$e(t) = \frac{dp(t)}{dt} \qquad \text{and} \qquad p(t) = \int e(t)dt$$

- **Mechanical Translation:** the momentum variable is the linear momentum $p(t)$ and the effort variable is the force $F(t)$, then we have the Newton's second law of motion

$$F(t) = \frac{dp(t)}{dt} = \frac{dmv(t)}{dt} = ma(t)$$

- **Mechanical Rotation:** the momentum variable is the angular momentum $H(t)$ and the effort variable is the torque $\tau(t)$, and we have the Euler's equation of motion

$$\tau(t) = \frac{dH(t)}{dt}$$

Kinetic Variables

- **Electrical:** the momentum variable is the flux linkage $\lambda(t)$ and the effort variable is the voltage $v(t)$, thus we have the Faraday's law

$$v(t) = \frac{d\lambda(t)}{dt}$$

- **Fluid:** the momentum is the pressure momentum variable $\Gamma(t)$ and the effort variable is the pressure $P(t)$, thus

$$P(t) = \frac{d\Gamma(t)}{dt}$$

- **Thermal:** There is no momentum for this system.

Work, power and energy

The *increment in work* done by the effort in displacing the element an amount $dq(t)$ is

$$\delta\mathcal{W}(t) = e(t)dq(t)$$

$e(t)$ is the effort in the direction of the displacement $q(t)$

$$\begin{aligned}\delta\mathcal{W}(t) = e(t)dq(t) &= \frac{dp(t)}{dt}dq(t) \\ &= \frac{dq(t)}{dt}dp(t) = f(t)dp(t)\end{aligned}$$

the $\delta\mathcal{W}(t)$ can also be described as the increment in work done by the flow $f(t)$ in changing the momentum an amount $dp(t)$

Work, power and energy

The *power* is the rate at which work is performed and given by

$$\mathcal{P}(t) = \frac{d\mathcal{W}(t)}{dt} = e(t)\frac{q(t)}{dt} = f(t)\frac{dp(t)}{dt} = e(t)f(t)$$

Discipline	Work		Power
	edq	fdp	ef
Mechanical Translation	Fdx	vdp	Fv
Mechanical Rotation	$\tau d\theta$	ωdH	$\tau\omega$
Electrical	vdq	$id\lambda$	vi
Fluid	PdV	$Qd\Gamma$	PQ
Thermal	TdS	–	$T\dot{S}$

Work, power and energy

Energy is the capacity to do work, and is defined as is the time integral of the power.

$$\mathcal{E}(t) = \int e(t)f(t)dt = \int_{C_q} e(t)dq(t) = \int_{C_p} f(t)dp(t)$$

- $\int_{C_q}$ is the integral along the displacement path C_q
- $\int_{C_q}$ is the integral along the momentum path C_p

System Components

The system components are classified as one of the following:

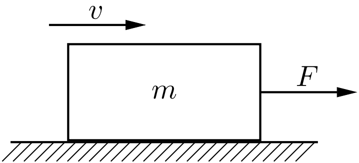
- **Energy storage components** which are represented by ideal *inductors*, and the ideal *capacitors*
- **Energy dissipation components** which are represented by ideal *resistors*
- **Energy transforming components** which are represented by *constraint* elements
- **Energy sources** which provide energy to the systems.

Ideal Inductors

Ideal Inductor is an energy storage components whose behavior is determined by expressions that relate the momentum $p(t)$ and flow variables $f(t)$.

- the constitutive relation $p(t) = \phi_L(f(t))$
- the constitutive relation need not to be linear but $\phi_L(0) = 0$ and $\phi_L^{-1}(t)$ is exists.
- the flow can be determined in terms of the momentum, $f(t) = \phi_L^{-1}(p)$.

Ideal Inductor: Translational Mass



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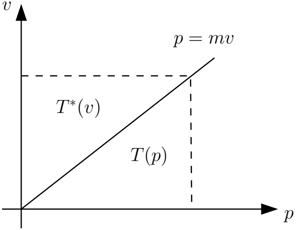

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The momentum p of the mass m is linearly related to the object's velocity v:

$$p = mv$$

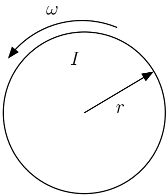
The quantity p , the momentum, is defined by

$$p = \int_{t_0}^{t} F dt + p(t_0) \quad \text{or} \quad F = \frac{dp}{dt}$$

The stored energy $T(p)$ (kinetic energy) and the co-energy $T^*(v)$ are equal:

$$T = \frac{p^2}{2m} = T^* = \frac{1}{2} m v^2$$

Ideal Inductor: Rotational Mass



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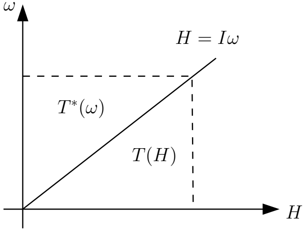

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The angular momentum H of the mass moment of inertia of the rotor I is linearly related to the angular velocity ω:

$$H = I\omega$$

The quantity H , the angular momentum, is defined by

$$H = \int_{t_0}^{t} \tau dt + H(t_0) \quad \text{or} \quad \tau = \frac{dH}{dt}$$

The stored energy $T(H)$ (kinetic energy) and the co-energy $T^*(\omega)$ are equal:

$$T(H) = \frac{H^2}{2I} = T^*(\omega) = \frac{1}{2}\omega^2 I$$

Ideal Inductor: Electrical Inductance

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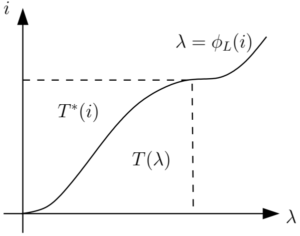


(b)

The *flux linkages* λ is the total magnetic flux linked by the electrical circuit.

$$\lambda = \phi_L(i), \qquad v = \frac{d\lambda}{dt},$$

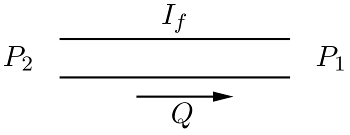
Energies stored in an inductive field are

$$T(\lambda) = \int_0^{\lambda} i d\lambda \qquad T^*(i) = \int_0^{i} \lambda di$$

For linear inductor ($\lambda = Li$)

$$T(\lambda) = \frac{1}{2L}\lambda^2 = T^*(i) = \frac{1}{2}Li^2$$

Ideal Inductor: Fluid



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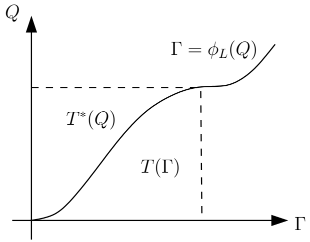


(b)

The *fluid inertia* I_f represents the ideal inductor

$$\Gamma = I_f Q,$$

where Γ is the pressure momentum, I_f is the fluid inertia and Q is the volume flow rate.

$P_{12} = P_1 - P_2 = \dfrac{d\Gamma}{dt}$ and $T(\Gamma) = \displaystyle\int_0^{\Gamma} Q d\Gamma$

The stored energy $T(\Gamma)$ and the co-energy $T^*(Q)$ are

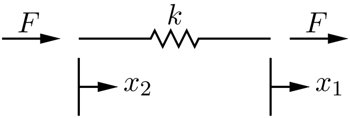
$$T(\Gamma) = \frac{1}{2I_f}\Gamma^2 = T^*(Q) = \frac{1}{2}I_f Q^2$$

Ideal Capacitors

Ideal Capacitor is an energy storage components whose behavior is determined by expressions that relate the displacement $q(t)$ and effort variables $e(t)$.

- the constitutive relation $q(t) = \phi_C(e(t))$
- the constitutive relation need not to be linear but $\phi(0) = 0$ and $\phi_C^{-1}(t)$ is exists.
- the effort can be determined in terms of the displacement, $e(t) = \phi_C^{-1}(q)$.

Ideal Capacitor: Translational Spring



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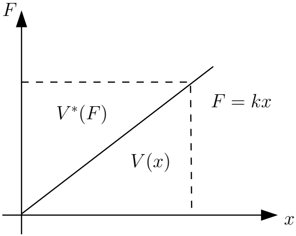


(b)

For a linear spring, the relationship between the applied force $f(t)$ and the deflection $x(t)$ of the spring is given by Hooke's law

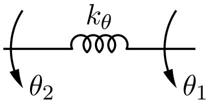
$$F(t) = k(x_2 - x_1) = kx,$$

where $F(t)$ is the force applied to the spring, k is the spring stiffness.

The stored energy (potential energy) $V(x)$ and the co-energy $V^*(F)$

$$V(x) = \int_0^x F(t)dx = \frac{1}{2}kx^2 = V^*(F) = \frac{F^2}{2k}$$

Ideal Capacitor: Rotational Spring



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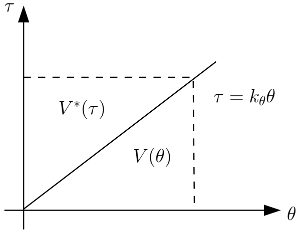


(b)

the torsional spring is a capacitor element. The relationship between the applied torque $\tau(t)$ and the net angular deflection $\theta(t)$

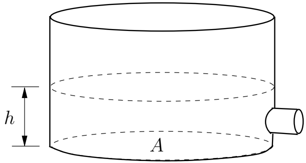
$$\tau(t) = k_\theta(\theta_2 - \theta_1) = k_\theta\theta,$$

where $\tau(t)$ is the torque applied to the spring, k_θ is the torsional spring stiffness.

The stored energy (potential energy) $V(\theta)$ and the co-energy $V^*(\tau)$

$$V(\theta) = \int_0^x \tau(t)d\theta = \frac{1}{2}k_\theta\theta^2 = V^*(\tau) = \frac{\tau^2}{2k_\theta}$$

Ideal Capacitor: Fluid



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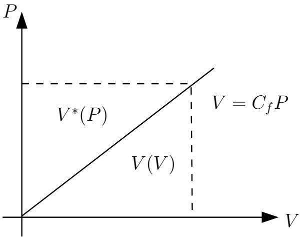


(b)

Tank is a capacitor elements in fluid systems. Linear fluid capacitors satisfies the equation

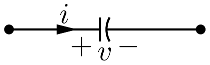
$$P = \frac{V}{C_f}$$

where P is the pressure, V is the volume of the fluid and C_f is the fluid capacitance.

$$P = \rho g h = \frac{\rho g V}{A}, \qquad C_f = \frac{A}{\rho g}$$

$$V(V) = \int_0^V P dV = \frac{V^2}{2C_f} = V^*(P) = \frac{C_f P^2}{2}$$

Ideal Capacitor: Electrical Capacitance



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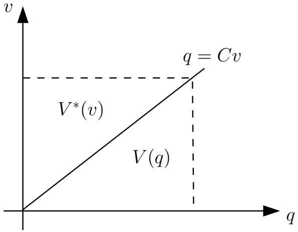


(b)

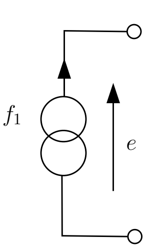
the amount of charge q is determined by the voltage across the conductor.

$$q = \phi(v), \qquad i = \frac{dq}{dt},$$

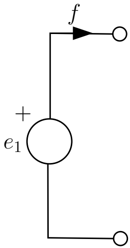
Energies stored in a linear capacitor $(q = Cv)$ are

$$V(q) = \int_0^q v dq = \frac{q^2}{2C}$$
$$V^*(v) = \int_0^v q dv = \frac{1}{2} C v^2$$

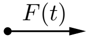
Ideal Energy Sources



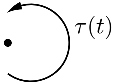
(c) Flow source



(d) Effort source

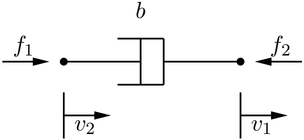


(e) Force



(f) Torque

Ideal Resistance: Mechanical Translation



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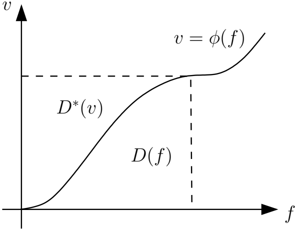


(b)

The damper behaves like

$$f = \phi(v)$$

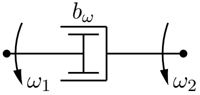
For linear damper,
$f = b(v_2 - v_1) = bv$, the content and co-content energy is given by

$$D(f) = \int_0^f v df = \frac{1}{2b} f^2$$
$$= D^*(v) = \frac{1}{2} bv^2$$

The total power dissipated is

$$P = bv^2 = \frac{f^2}{b}$$

Ideal Resistance: Mechanical Rotational



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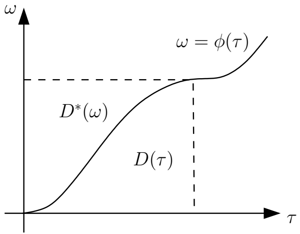


(b)

The damper behaves like

$$\tau = \phi(\omega)$$

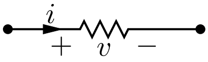
For linear torsional damper, $\tau = b_\omega(\omega_2 - \omega_1) = b_\omega \omega$, the content and co-content energy is given by

$$D(\tau) = \int_0^\tau \omega d\tau = \frac{1}{2b_\omega}\tau^2 = D^*(\omega) = \frac{1}{2}b_\omega \omega^2$$

The total power dissipated is

$$P = b_\omega v^2 = \frac{\tau^2}{b_\omega}$$

Ideal Resistance: Electrical Dissipation



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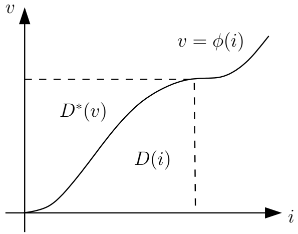


(b)

The general constitutive relation for a resistance is

$$v = \phi(i)$$

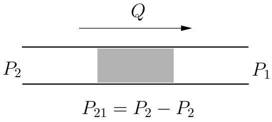
For linear device, $v = Ri$, the content and co-content energy is given by

$$D(i) = \int_0^i v di = \frac{1}{2R}v^2 = D^*(v) = \frac{1}{2}Ri^2$$

The total power dissipated is

$$P = i^2 R = \frac{v^2}{R}$$

Ideal Resistance: Fluid Dissipation



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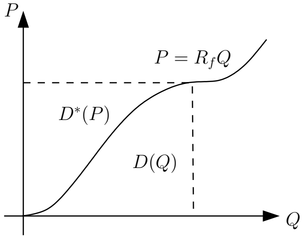


(b)

The general constitutive relation for a fluid resistor is

$$P = \phi(Q), \qquad P = R_f Q$$

where P is the pressure across the terminals, R_f is the fluid resistance, and Q is the volume flow rate. The content and co-content energy is given by

$$D(Q) = \int_0^Q P dQ = \frac{1}{2} R_f Q^2 = D^*(P) = \frac{1}{2R} P^2$$

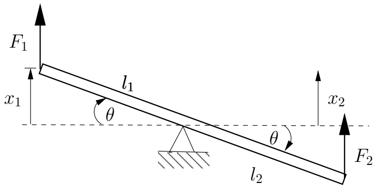
The total power dissipated is

$$P = R_f Q^2 = \frac{P^2}{R_f}$$

Constraint elements

- **Transformers**
  - a transformer transfers energy between the subsystems in the dynamic system model.
  - Ideally, elements do not store, dissipate or generate energy, and ther behave in such a way that the net power into the device is zero.
  - in the case of transformers the energy transfer takes place within the same engineering discipline.
  - these elements give rise to *displacement constraints* or *flow constraints* that do no work on the system.
- **Transducers**
  - the transducer are similar to a transformer
  - however the energy transfer takes place between different engineering disciplines.
  - these elements give rise to displacement or flow constraints the do no work on the ssytem.

Transformer: Mechanical Translation



For small displacements the following kinematic relationship holds,

$$x_1 = l_1\theta \quad \rightarrow \quad \theta = \frac{x_1}{l_1}$$

$$x_2 = -l_2\theta \quad \rightarrow \quad x_2 = -\frac{l_2}{l_1}x_1$$

In terms of the velocities these relations become

$$\frac{l_2}{l_1}v_1 + v_2 = 0, \quad v_1 = \frac{dx_1}{dt}, v_2 = \frac{dx_2}{dt}$$

Transformer: Mechanical Translation

Summing moments about the pivot (counterclockwise positive) gives,

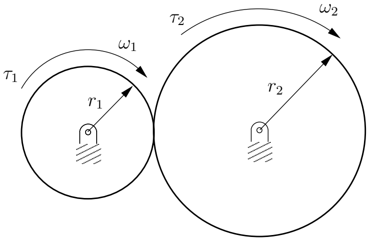
$$-F_1 l_1 + F_2 l_2 = 0$$

which is a constraint on the effort variables of the device. the power balance yields

$$\begin{aligned}\text{Power input} + \text{Power output} &= F_1 v_1 + F_2 v_2 \\ &= \left(F_1 - \frac{l_2}{l_1} F_2\right) v_1 = 0\end{aligned}$$

Hence, no energy is stored or dissipated in the device.

Transformer: Mechanical Rotational



The simple gear train represents a transformer for mechanical systems in rotation. Since there is no slipping or backlash the velocity at the point of contact is

$$r_1\omega_1 = -r_2\omega_2 \quad \text{and} \quad \frac{r_1}{r_2}\omega_1 + \omega_2 = 0$$

Transformer: Mechanical Rotational

This is the flow constraint that the simple gear train must satisfy. Force at the point of contact satisfies

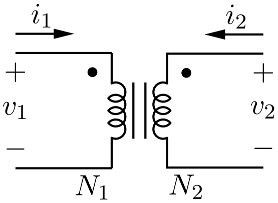
$$\frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}$$

which is an effort constraint for the simple gear train. As a result, the power balance satisfies

$$\begin{aligned}\text{Power input} + \text{Power output} &= \tau_1\omega_1 + \tau_2\omega_2 \\ &= \left(\tau_1 - \frac{r_1}{r_1}\tau_2\right)\omega_1 = 0\end{aligned}$$

Transformer: Electrical Transformer

The coil on the left has N_1 turns, applied voltage v_1 and current i_1. The current in the primary coil creates a magnetic flux $\phi = N_1 i_1$ that induces a voltage v_2



Assume that there is no flux leakage and the inductance of the coils can be neglected. The magnetomotive force balance given,

$$N_1 i_1 + N_2 i_2 = 0$$

which is the flow constraint for the device.

Transformer: Electrical Transformer

From Faraday's law we have

$$v_1 = \frac{d\lambda_1}{dt} = \frac{d(N_1\phi)}{dt} = N_1\frac{d\phi}{dt}$$
$$v_2 = \frac{d\lambda_2}{dt} = \frac{d(N_2\phi)}{dt} = N_2\frac{d\phi}{dt} = \frac{N_2}{N_1}v_1$$

which is the effort constraint that the device must satisfy. The power balance gives

$$\begin{aligned}\text{Power input} + \text{Power output} &= v_1 i_1 + v_2 i_2 \\ &= \left(v_1 - \frac{N_1}{N_2}v_2\right) i_1 = 0\end{aligned}$$

Transformer: Fluid Transformer

This is a fluid transformer

the velocity of the piston is

$$\frac{Q_1}{A_1} = -\frac{Q_2}{A_2}$$

$$\frac{A_2}{A_1}Q_1 + Q_2 = 0$$

which is the flow constraint for the device.

Transformer: Fluid Transformer

If the piston is assumed to be massless, the net force acting on the piston is

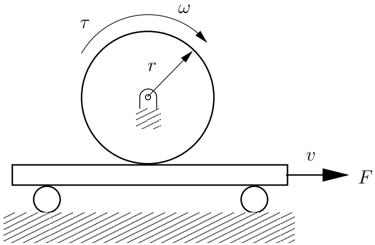
$$P_1 A_1 - P_2 A_2 = 0,$$

which is a constraint on the effort variables. The power balance gives

$$\begin{aligned}\text{Power input} + \text{Power output} &= P_1 Q_1 + P_2 Q_2 \\ &= \left(P_1 - \frac{A_1}{A_2} P_2\right) Q_1 = 0\end{aligned}$$

Transducer:Mechanical Transducer

The rack and pinion system:



the torque associated with the gear is τ, and ω is the corresponding angular velocity. The rack has a force F and velocity v. If there is no slipping or backlash the system satisfies the flow constraint

$$v + r\omega = 0.$$

Transducer:Mechanical Transducer

If the system is inertialess, summing the moments about the center of the gear gives

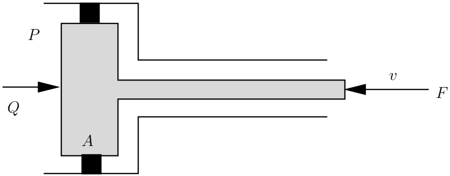
$$Fr - \tau = 0,$$

which is the effort constraint that must satisfied. The power balance gives

$$\begin{aligned}\text{Power input} + \text{Power output} &= Fv + \tau\omega \\ &= \left(F - \frac{1}{r}\tau\right)v = 0\end{aligned}$$

Transducer:MechaniFluid-mechanical Transducer

The hydraulic press is an example of a fluid-mechanical transducer.



An incompressible fluid with volume flow rate Q and pressure P acts on a massless pistion with area A. The piston has an applied force F and velocity v in the direction shown.

$$v + \frac{Q}{A} = 0.$$

which is the flow constraint that the device must satisfy.

Transducer:MechaniFluid-mechanical Transducer

Since the piston is massless, the net force acting on the piston is

$$-F + PA = 0$$

which is the effort constraint that the device must satisfy. The power balance gives

$$\begin{aligned} \text{Power input} + \text{Power output} &= Fv + PQ \\ &= (F - AP)\,v = 0 \end{aligned}$$

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