Lecture 9 : Design Objectives and Sensitivity Functions

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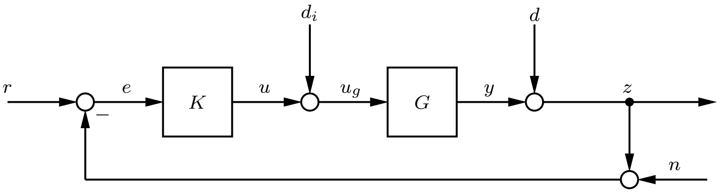
Feedback Structure



Consider the closed-loop system shown above. External inputs are a reference input $r(t)$, an output disturbance $d(t)$ and measurement noise $n(t)$. The controlled output is $y(t)$, and we have

$$y = Gu + d, \qquad u = K(r - y - n)$$

yield $y = (I + GK)^{-1}GK(r - n) + (I + GK)^{-1}d$
Define the transfer functions

$$S(s) = (I + G(s)K(s))^{-1} \quad \text{and} \quad T(s) = (I + G(s)K(s))^{-1}G(s)K(s)$$

Feedback Structure

- The function $S(s)$ is called **sensitivity function** of the feedback system. It is the transfer function from the output disturbance d to the controlled output y.
- the function $T(s)$ is called **complementary sensitivity function** of the system.
- it is the closed-loop transfer function from r to y, and it is also the transfer function from measurement noise to controlled output.
- $S + T = I$.

In terms of S and T, we have

$$y = T(r - n) + Sd$$

The control error $e(t)$ is

$$e = r - n - y = r - n - (GKe + d) \qquad \text{then } e = S(r - n - d)$$

S is also the transfer function from reference input, measurement noise and output disturbance, to the control error.

Control Objectives

We consider the following objectives:

- tracking: the controlled output should track the reference input, ideally $y(t) = r(t)$
- disturbance rejection: the controller should keep the controlled output at its desired value in the presence of a disturbance $d(t) \neq 0$. Ideally the transfer function from d to y would be zero.
- noise rejection: the controller should suppress the effect of measurement noise on the control input
- reasonable control effort: the above design objectives must be achieved within given constraints on the actuators, which means the magnitude of the control input must not exceed given limits.
- the perfect tracking and disturbance rejection require $T(s) = I$, this implies $S(s) = 0$ and the control error should be zero.
- the prefect noise rejection requires $T(s) = 0$ or $S(s) = I$
- Clearly, tracking and disturbance rejection on one hand and the suppression of measurement noise are conflicting design objectives.

Control Objectives

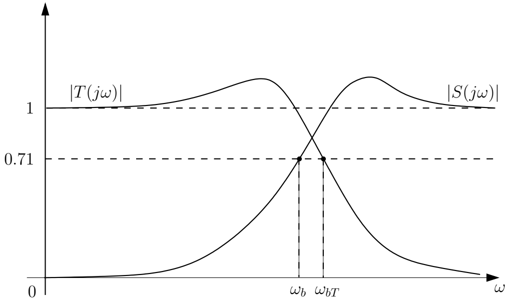
The control effort is related to the sensitivity because

$$u = Ke = KS(r - n - d)$$

To achieve the design objectives with reasonable control effort, the transfer function $K(s)S(s)$ should not become "too large". The function KS is called the **control sensitivity**.

- In practice, the design objectives are interpreted such that good tracking and disturbance rejection properties should be achieved for signals within the closed-loop bandwidth.
- noise rejection for high frequency measurement noise beyond the bandwidth.

Mixed Sensitivity Design for SISO Systems



- At low frequencies, the sensitivity S is close to zero, which means good tracking and disturbance rejection.
- At higher frequencies, the magnitude of S increases, and at the frequency $\omega = \omega_b$, it is 0.71 (-3dB). We will consider **control to be effective** if $|S| < 0.71$, and define the closed-loop bandwidth as this frequency.

Mixed Sensitivity Design for SISO Systems

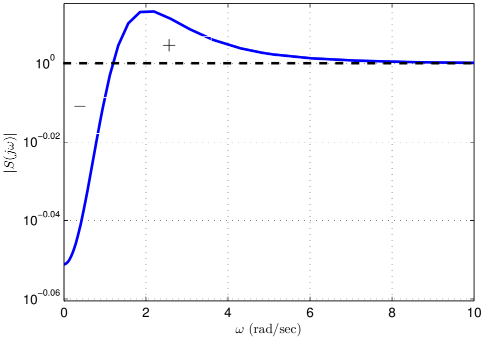
- At low frequencies, T is approximately 1, which is consistent with $|S| \approx 0$ and indicates good tracking properties, because T is the transfer function from r to y.
- At high frequencies, $|T|$ rolls off and approaches zero. This is required for the rejection of high frequency measurement noise, and implies that $|S|$ approaches 1.
- The closed-loop bandwidth is take as the frequency ω_{bT} where $|T| = 0.71$.
- the two frequencies ω_b and ω_{bT} are usually not equal, but also not far from each other.

The Waterbed Effect

- From the previous Figure, both $|S|$ and $|T|$ are shown to have peak values greater than 1. This happens often and is in many cases unavoidable.
- A peak of $|T|$ greater than 1 indicates a resonant peak with overshoot in the step response.
- $|S| > 1$ means that disturbances in this frequency range are not suppressed but actually amplified.
- These peaks are unavoidable if
  - the pole excess of the plant is greater than one, or
  - if the plant has one or more zeros in the right half plane
- If the plant is minimum-phase, one can show that

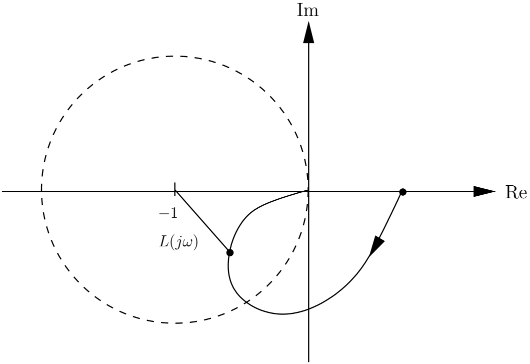
$$\int_0^\infty \ln |S(j\omega)| d\omega = 0$$

The Waterbed Effect



- The area below 1 and above $|S(j\omega)|$ is equal to the area above 1 and below $|S(j\omega)|$, when $|S(j\omega)|$ is plotted on a logarithmic scale and ω is plotted on a linear scale.

The Waterbed Effect



The Nyquist polot of the loop transfer function $L(s) = G(s)K(s)$ is shown for the above control system with a SISO plant.

The Waterbed Effect

- If $L(s)$ has pole excess of 2, then the Nyquist plot of $L(s)$ will penetrate a disc of radius 1 around the critical point -1
- Since the distance between $L(j\omega)$ and -1 is $|1 + L(j\omega)|$, we have inside the disc

$$|1 + L(j\omega)| < 1 \quad \text{and} \quad |S(j\omega)| > 1$$

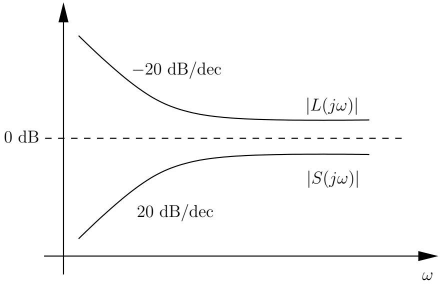
- if the plant has right half plane zeros, the resulting phase lag will lead to $L(j\omega)$ penetrating the disc even if the pole excess is only 1.

Low Frequency Design - Shaping the Sensitivity

the $\mathcal{H}_\infty$ norm can be used to express constraints on the sensitivity that are required to achieve the design objectives. Properties of the closed-loop system that depend on the shape of $|S(j\omega)|$ include

- the bandwidth ω_b
- peak overshoot
- the system type (capability of tracking step or ramp inputs with zero steady state error)

Low Frequency Design - Shaping the Sensitivity



- The magnitude response of a type 1 system has at low frequencies a slope of -20 dB/dec. Because at low frequencies ($\omega \ll \omega_b$) we have $|L(j\omega)| \gg 1$, we can approximate the sensitivity by

$$S(j\omega) = \frac{1}{1 + L(j\omega)} \approx \frac{1}{L(j\omega)}$$

Low Frequency Design - Shaping the Sensitivity

- At low frequencies the graph of $|S(j\omega)|$ in dB is approximately equal to the mirror image of $|L(j\omega)|$ about the 0 dB line.
- a feedback system has integral action in the loop if the magnitude of the sensitivity has at low frequencies a positive slope of 20 dB/dec.
- Similarly, for a type 2 system we need $|S(j\omega)|$ to have a slope of 40 dB/dec.

One can use the $\mathcal{H}_\infty$ norm to express constraints on the magnitude of the sensitivity,

$$\|S(s)\|_\infty < M_S \quad \Rightarrow \quad |S(j\omega)| < M_S \qquad \forall\omega$$

The constraint is unrealistic, since we cannot keep the sensitivity small at all frequencies. A weighting function $W_S(s)$ is used, yield

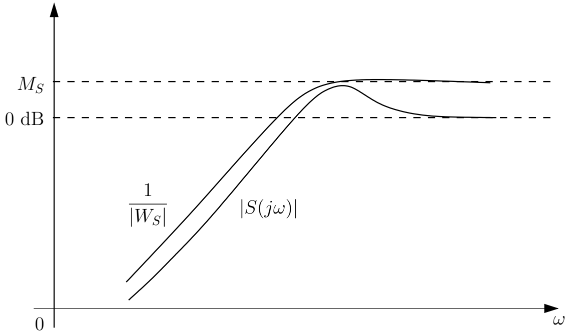
$$\|W_S(s)S(s)\|_\infty < 1$$

This enforces

$$|W_S(j\omega)S(j\omega)| < 1 \quad \forall\omega \quad \text{or} \quad |S(j\omega)| < \frac{1}{|W_S(j\omega)|} \quad \forall\omega$$

Low Frequency Design - Shaping the Sensitivity

$\mathcal{H}_\infty$ norm



- If the $\mathcal{H}_\infty$ norm is less than 1, the inverse of the weighting function is an upper bound on the sensitivity.
- the weighting function enforces integral action and an upper bound M_S on the peak value of $|S(j\omega)|$.

High Frequency Design

Shaping the $|T(s)|$ and $|K(s)S(s)|$

At high frequencies ($\omega \ll \omega_b$) the complementary sensitivity is required to roll off in order to suppress measurement noise. This can be expressed by introducing a weighting function $W_T(s)$ as

$$\|W_T(s)T(s)\|_\infty < 1$$

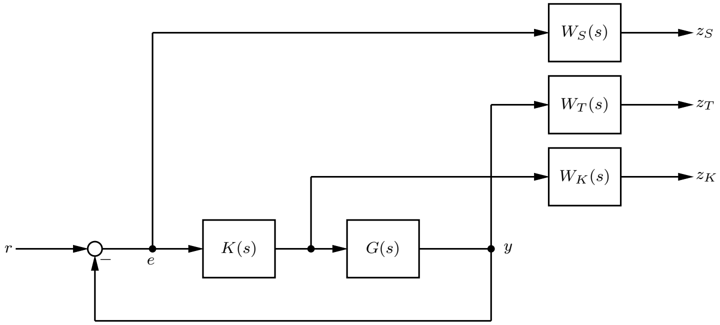
This enforces

$$|W_T(j\omega)T(j\omega)| < 1 \quad \forall\omega \quad \text{or} \quad |T(j\omega)| < \frac{1}{|W_T(j\omega)|} \quad \forall\omega$$

In order to maintain a reasonable control effort, the control sensitivity $K(s)S(s)$ should not become "too large". We can be more specific and impose

$$\|W_K(s)K(s)S(s)\|_\infty < 1$$

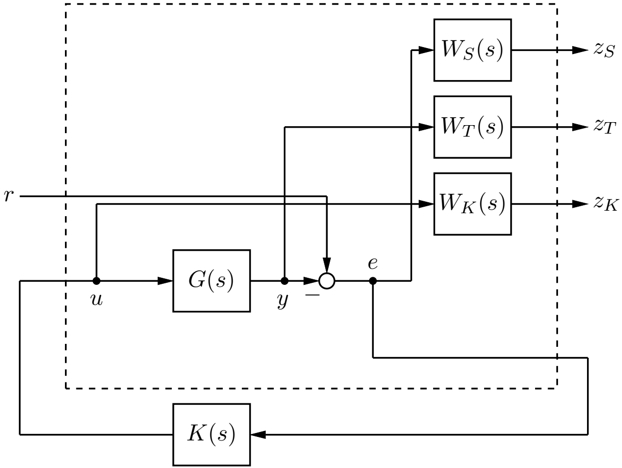
Weighting Filters and Generalized Plant



$$\|W_S(s)S(s)\|_\infty < 1$$
$$\|W_T(s)T(s)\|_\infty < 1$$
$$\|W_K(s)K(s)S(s)\|_\infty < 1$$

Weighting Filters and Generalized Plant

Constraints on S, T and KS in terms of a generalized plat



Weighting Filters and Generalized Plant

Assume we want to design a controller with integral action and given actuator constraints. We choose z_S and z_K as outputs and define the fictitious output vector z of the generalized plant as

$$z = \begin{bmatrix} z_S \\ z_K \end{bmatrix}$$

we have

$$\begin{bmatrix} z_S \\ z_K \end{bmatrix} = \begin{bmatrix} W_S S \\ W_K K S \end{bmatrix} r, \quad \text{therefore} \quad T_{zr}(s) = \begin{bmatrix} W_S(s)S(s) \\ W_K(s)K(s)S(s) \end{bmatrix}$$

Find a controller $K(s)$ such that the closed-loop transfer function satisfies $\|T_{zr}(s)\|_\infty < 1$ or

$$\left\| \begin{bmatrix} W_S S \\ W_K K S \end{bmatrix} \right\|_\infty < 1$$

Weighting Filters and Generalized Plant

The above $\mathcal{H}_\infty$ norm is equivalent to

$$\sup_\omega \bar{\sigma}\left(\begin{bmatrix} W_S(j\omega)S(j\omega) \\ W_K(j\omega)K(j\omega)S(j\omega) \end{bmatrix}\right) < 1$$

In SISO systems, we have

$$\sup_\omega \sqrt{|W_S S|^2 + |W_K K S|^2} < 1$$

The largest approximation error occurs if $|W_S S| = |W_K K S|$, in this case the approximation of above constraint become

$$\sup_\omega |W_S S| < 0.71 \quad \text{and} \quad \sup_\omega |W_K K S| < 0.71$$

Mixed Sensitivity Design for MIMO Systems

For MIMO plats, the sensitivity $S(s)$ is a $l \times l$ transfer function matrix (where l is the number of plant outputs). Accordingly, we need a $l \times l$ weighting matrix. For example,

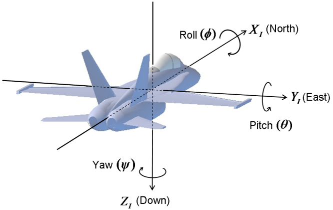
$$W_S(s) = \begin{bmatrix} w_S(s) & & 0 \\ & \ddots & \\ 0 & & w_S(s) \end{bmatrix}$$

where the same scalar shaping filter $w_S(s)$ is used for each output of $S(s)$. With this choice we have

$$\sup_\omega \bar{\sigma}(W_S(j\omega)S(j\omega)) = \sup_\omega \bar{\sigma}(w_S(j\omega)S(j\omega)) = \sup_\omega |w_S(j\omega)|\bar{\sigma}(S(j\omega)) < 1$$

This is equivalent to $\bar{\sigma}(S(j\omega)) < \dfrac{1}{|w_S(j\omega)|} \quad \forall\omega$

Aircraft control



Assuming the state space model represents a linearized model of the vertical-plane dynamics of an aircraft is described below:

$$A = \begin{bmatrix} 0 & 0 & 1.132 & 0 & -1 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.0485 & 0 & -0.8556 & -1.013 \\ 0 & -0.2909 & 0 & 1.0532 & -0.6859 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -0.12 & 1 & 0 \\ 0 & 0 & 0 \\ 4.419 & 0 & -1.665 \\ 1.575 & 0 & -0.0732 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Aircraft control

u_1 spoiler angle (in 0.1 deg)

u_2 forward acceleration (in m s^{-2})

u_3 elevator angle (in deg)

x_1 relative altitude (in m)

x_2 forward speed (in m s^{-1})

x_3 pitch angle (in deg)

x_4 pitch rate (in deg s^{-1})

x_5 vertical speed (in m s^{-1})

The design objectives are:

- fast tracking of step changes for all three reference inputs, with little or no overshoot
- control input must satisfy $|u_3| < 20$.

In this design we will shape the sensitivity S and the control sensitivity KS, to achieve desired properties of the closed-loop transfer function $T_{zw}(s)$ from the reference input r to the fictitious output vector $z = \begin{bmatrix} z_S^T & z_K^T \end{bmatrix}^T$

$$\|T_{zw}\|_\infty = \left\| \begin{bmatrix} W_S S \\ W_K KS \end{bmatrix} \right\|_\infty < 1$$

If there are on other constraints on the closed-loop system, one can compute the controller that minimizes $\|T_{zw}(s)\|_\infty$. Let

$$\gamma_0 = \min_K \|T_{zw}(s)\|_\infty$$

denote the optimal value of the $\mathcal{H}_\infty$ norm.

The γ_0 may or may not be less than one.

- If $\gamma_0 > 1$, the constraints expressed by the weighting functions W_S and W_K are too strong and a controller that satisfies them does not exist. The constraints must be relaxed.
- If $\gamma_0 < 1$, controllers that satisfy the constraints do exist and the constraints can actually be strengthened.

To adjusting the weighting filters is scaling. Assume that the minimum value of $\|T_{zw}(s)\|_\infty$ is $\gamma_0 \neq 1$ and that this value is achieved with the optimal controller $K_0(s)$. Introduce the scaled weighting functions

$$\tilde{W}_S(s) = \frac{1}{\gamma} W_S(s) \quad \text{and} \quad \tilde{W}_K(s) = \frac{1}{\gamma_0} W_K(s)$$

Replace the weighting filters by $\tilde{W}_S$ and $\tilde{W}_K$. The $\mathcal{H}_\infty$ norm is minimized by the same controller $K_0(s)$ as before, and we have

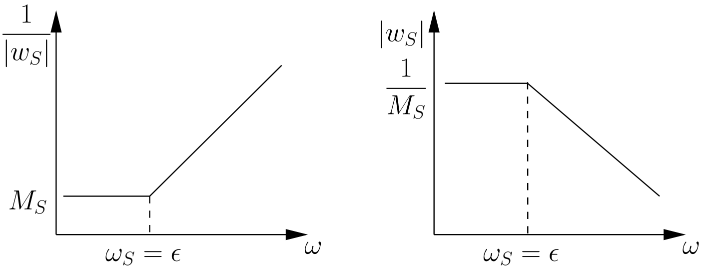
$$\left\| \begin{bmatrix} \tilde{W}_S S \\ \tilde{W}_K K S \end{bmatrix} \right\|_\infty = 1$$

Choice of Weighting Functions – Sensitivity

We begin the design with scalar weighting filters $w_S(s)$ and $w_K(s)$.

- To have integral action, a positive slope of 20 dB/dec of the sensitivity is required at low frequencies. This could be enforced by including a factor $1/s$ in the weighting function $w_S(s)$.
- the weight filters are factors of the closed-loop transfer function $T_{zw}(s)$, and if $w_S(s)$ has a pole at the origin then the same is true for T_{zw}.
- the $\mathcal{H}_\infty$ norm is only defined for proper, stable transfer functions, the weighting filters must therefore also be proper and stable.
- to enforce integral action in the loop, one can choose the weighting function to include a factor $\dfrac{1}{s+\epsilon}$ where $\epsilon > 0$ is a small constant.
- M_S is a small constant that is chosen as an upper bound on the sensitivity at low frequencies.

Choice of Weighting Functions – Sensitivity



- The transfer function of the weighting filter is

$$w_S(s) = \frac{\omega_S / M_S}{s + \omega_S}$$

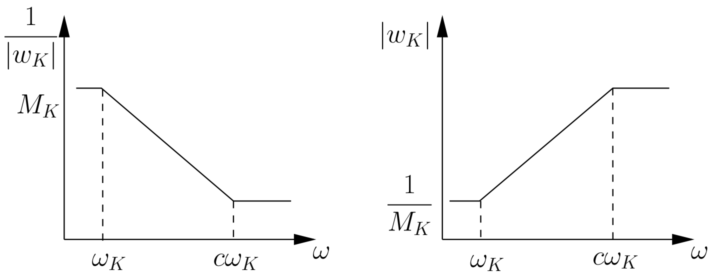
where $w_S = \epsilon$.

Choice of Weighting Functions – Control Sensitivity

The weighting filter $w_K(s)$ can be used to impose an upper bound on the control sensitivity.

- The control sensitivity should roll off at high frequencies, the inverse of this filter should have low-pass behaviour and thus the filter itself should be a high-pass filter.
- the weight filters must be stable and proper, and if we start with a zero at ω_K, an additional pole at a frequency well above the bandwidth is required to make $\omega_K(s)$ proper.
- the pole is placed at $c\omega_K$, where c is a sufficiently large constant.

Choice of Weighting Functions – Sensitivity



- The transfer function of the weighting filter is

$$w_K(s) = \frac{c}{M_K} \frac{s + \omega_K}{s + c\omega_K}$$

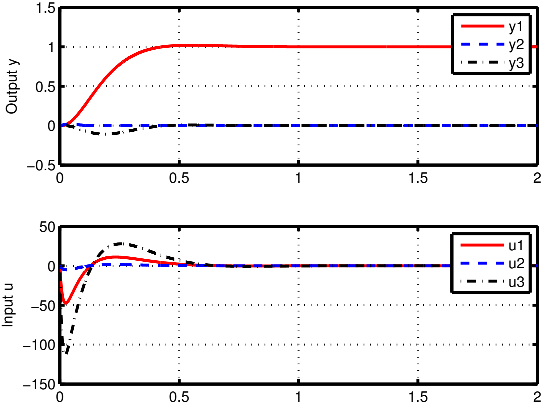
Loop Shaping

- we have made choices concerning the structure and order of the weighting functions W_S and W_K – we use the same scalar, first order filters for all outputs.
- this choices are restrictive and can be changed if required.
- to reduce the design parameters, we fix $c = 10^3$ (placing the pole of w_K three decades above the zero). That leaves us with the design parameters ω_s, M_S, ω_K and M_K.
- the parameter ω_S can be used to determine the closed-loop bandwidth, and M_S can be used to push the steady state error towards zero.
- We are not imposing an upper bound on $|S|$. The reason for this is that we can use the upper bound M_K on the control sensitivity to impose a limit on a sensitivity peak.
- the corner frequency ω_K of W_K should be chosen high enough not to interfere with the bandwidth constraint on $|S|$.

Design 1

response to $r(t) = \begin{bmatrix} u(t) & 0 & 0 \end{bmatrix}^T$

ω_S	M_S	ω_K	M_K
10^{-4}	10^{-4}	10^{2}	10^{3}



Design 1

The aircraft model has three inputs and three outputs, the optimal controller $K_0(s)$ is a three-by-three transfer function matrix.

- the plant is of fifth order
- we have first order weighting filters for each of the three outputs of S and KS.
- we have a generalized plant of order 11, and therefore with an 11th order controller.
- the response is fast, but the control input u_3 violates the actuator constraint.
- we have

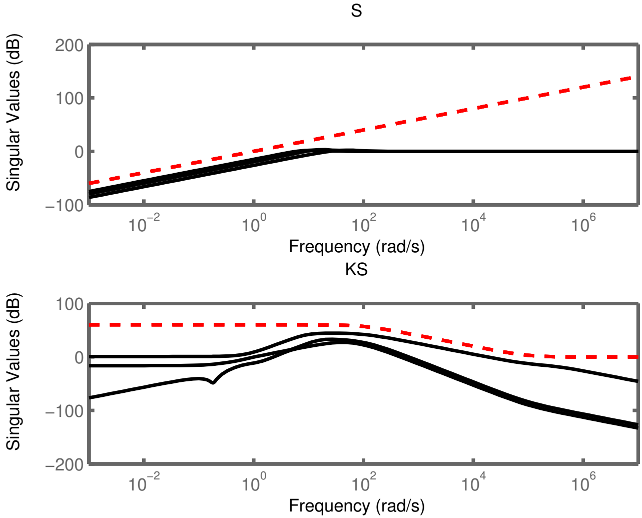
$$\gamma_0 = \min_K \|T_{zw}\|_\infty = 0.1794$$

- Since $\gamma_0 < 1$ means that we can tighten the constraints. Introduce the scaled weighting filters

$$\tilde{W}_S(s) = \frac{1}{\gamma_0} W_S(s) \quad \text{and} \quad \tilde{W}_K(s) = \frac{1}{\gamma_0} W_K(s)$$

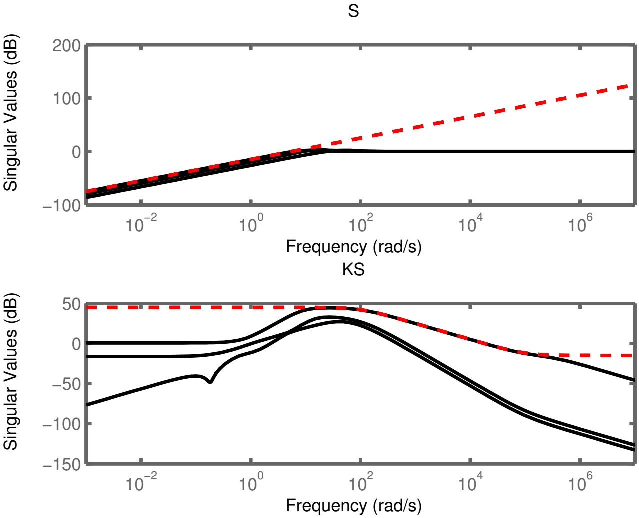
Design 1

Sensitivity S and Control Sensitivity KS



Design 1

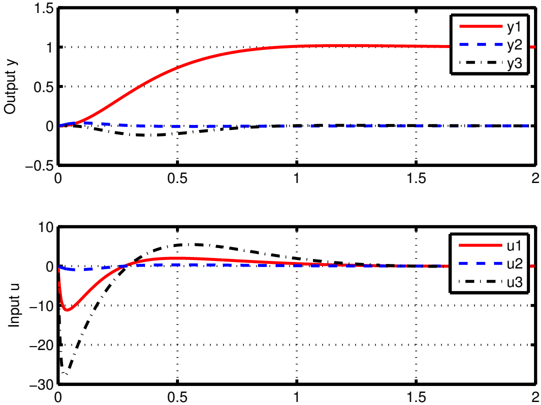
Sensitivity S and Control Sensitivity KS and scaled constraint



Design 2

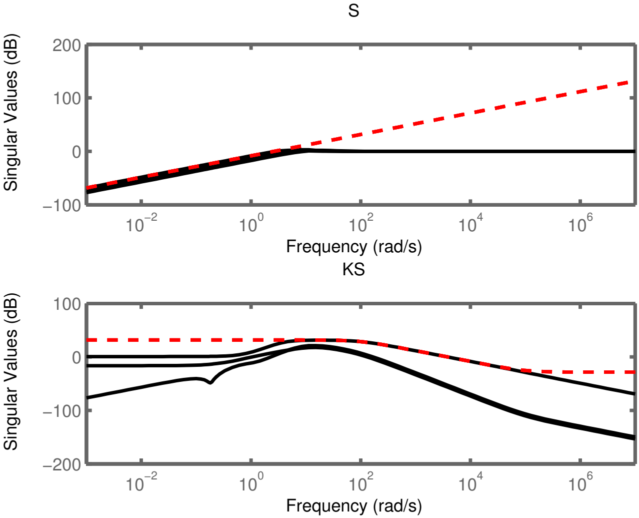
response to $r(t) = \begin{bmatrix} u(t) & 0 & 0 \end{bmatrix}^T$

ω_S	M_S	ω_K	M_K
10^{-4}	10^{-4}	10^2	10^2



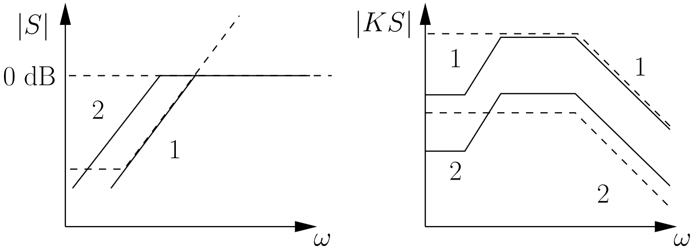
Design 2

Sensitivity S and Control Sensitivity KS and scaled constraint



Design 2

- Because we are minimizing $\|T_{zw}\|_\infty$ rather than rigidly enforcing $\|T_{zw}\|_\infty < 1$, the functions W_S and W_K act as weights – or "soft constraints" – rather than hard constraints.
- the effect of reducing M_K is illustrated in Figure below:



- the curves labelled 1 represent scaled constraint (dashed) and sensitivity shapes (solid) of the previous design.

Design 2

- the dashed line labelled 2 shows the constraint on KS when M_K is reduced.
- the new design were scaled for the previous design, the new design will give a controller with $\gamma_0 > 1$, which means the constraints will be violated.
- The violation of the constraints is spread equally over S and KS, indicated by the solid lines labelled 2.

Design 3

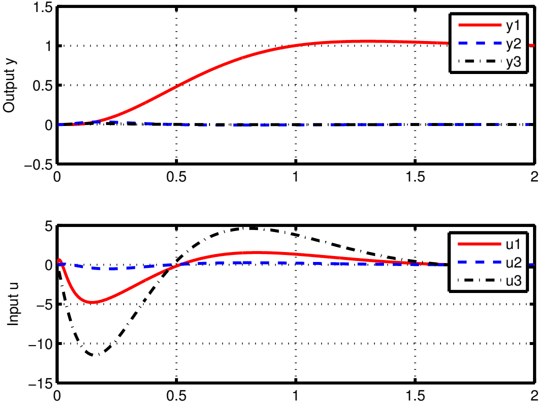
- we needs to reduced the control effort to meet the design specification.
- the response to a reference step change in y_1 leads to a perturbation in y_3.
- To improve tracking of r_3, we increase the weight on the corresponding channel in W_S and make the following adjustment to obtain Design 3:

$$W_S(s) = \begin{bmatrix} w_S(s) & 0 & 0 \\ 0 & w_S(s) & 0 \\ 0 & 0 & w_S(s) \end{bmatrix} \to \begin{bmatrix} w_S(s) & 0 & 0 \\ 0 & w_S(s) & 0 \\ 0 & 0 & 10w_S(s) \end{bmatrix}$$

Design 3

response to $r(t) = \begin{bmatrix} u(t) & 0 & 0 \end{bmatrix}^T$

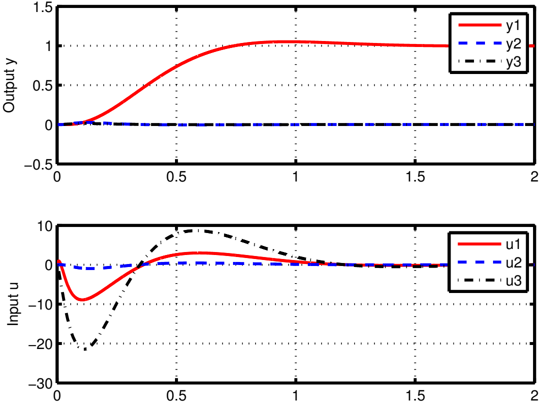
ω_S	M_S	ω_K	M_K
10^{-4}	10^{-4}	10^2	10^2



Design 4

response to $r(t) = \begin{bmatrix} u(t) & 0 & 0 \end{bmatrix}^T$

ω_S	M_S	ω_K	M_K
2.5×10^{-4}	10^{-4}	10^2	10^2



Reference

1. Kemin Zhou and John Doyle "*Essentials of Robust Control*", Prentice Hall, 1998
2. Herbert Werner "Lecture note on *Optimal and Robust Control*", 2012