Lecture 8 : Signal and System Norms

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Vector Norms

A norm $\|x\|$ is a function mapping a vector x into a real number, that satisfies the following four properties for any $x, y \in \mathcal{X}$

- $\|x\| > 0$ (positivity);
- $\|x\| = 0 \; \Leftrightarrow \; x = 0$ (positive definiteness);
- $\|\alpha x\| = \alpha \|x\|$ for any scalar $\alpha > 0$ (homogeneity);
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

A frequently used norm on vector x is the vector p-norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}},$$

where p is a positive integer. There are three important norms:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}, \qquad \|x\|_\infty = \max_i |x_i|$$

Vector Norms

The vector 2-norm can also be written as

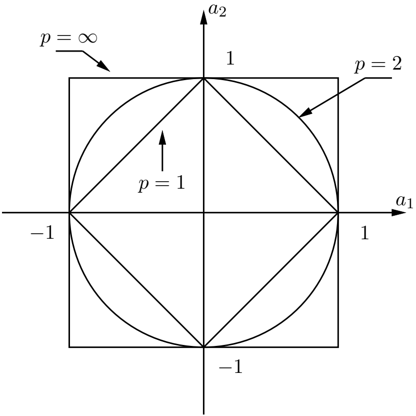
$$\|x\|_2 = \sqrt{x^T x} \qquad \text{if } x \in \mathbb{R}^n$$
$$\|x\|_2 = \sqrt{x^* x} \qquad \text{if } x \in \mathbb{C}^n$$

Here A^* denotes the *Hermitian* of a matrix A:

$$A^* = \bar{A}^T,$$

where $\bar{A}$ is the complex conjugate of A. For convenient, we will drop the subscript and write $\|x\|$ for the vector 2-norm of x.

Vector Norms



Contours for the vector p-norm, $\|a\|_p = 1$ for $p = 1, 2, \infty$.

Signal Norms

The p-norm of a signal is defined as

$$\|x(t)\|_p = \left(\int_{-\infty}^{\infty} |x(\tau)|^p d\tau\right)^{\frac{1}{p}}$$

One of the most used signal norm is the signal 2-norm

$$\|x(t)\|_2 \text{ or } \|x\| = \sqrt{\int_{-\infty}^{\infty} |x(\tau)|^2 d\tau}$$

It should note that *energy signals* have finite 2-norm while the 2-norm does not exist for *power signals*.

A real or complex valued signal vector $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_n(t) \end{bmatrix}^T$ Its signal 2-norm is defined as

$$\|x(t)\|_2 = \sqrt{\int_{-\infty}^{\infty} \|x(\tau)\|^2 d\tau}$$

Matrix Norm

Consider two complex vectors $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^m$, and a linear mapping $y = Ax$. The matrix p-norm induced by the vector norm is defined as

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

Since $y = Ax$:

- $\|A\|_p$ is a ratio or *gain* of the vector norms $\|y\|_p$ and $\|x\|_p$.
- This ratio is the maximum value of $\|y\|_p / \|x\|_p$ over all nonzero $x \in \mathbb{C}^n$ (the *maximum gain* of A).
- It is a positive real number, which is a norm. It depends on the choice of vector 2-norm, it is called an *induced norm*

The matrix 2-norm induced by the vector 2-norm is defined as

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

Matrix 2-Norm

By dropping the subscript the $\|A\|$ is usually known as matrix 2-norm. To find the value of $\|A\|$, we take squares on both sides to get

$$\|A\|^2 = \max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \max_{x \neq 0} \frac{x^* A^* A x}{x^* x} = \max_{x \neq 0} \frac{x^* M x}{x^* x}$$

- $M = A^* A$ is called Hermitian matrix. M is positive semi-definite, where $x^* M x \geq 0, \ \forall x \in \mathbb{C}^n$. With $y = Ax$, this property follows from

  $$x^* M x = x^* A^* A x = y^* y \geq 0$$

  Note that this implies that $x^* M x$ is real even if x is complex.
- the eigenvalues of M are real. It can be shown by letting λ be an eigenvalue and v be an eigenvector of M, and consider

  $$Mv = \lambda v$$

  Multiplying with v^* from the left yields $v^* M v = \lambda v^* v$. We established already that the left hand side of this equation is real, and same on the right hand side.

Matrix 2-Norm

- the eigenvalues of M are orthogonal (two vectors x and y are orthogonal if $x^*y = 0$). To show that two eigenvectors of M belonging to different eigenvalues are orthogonal, consider

$$Mv_1 = \lambda_1 v_1, \qquad Mv_2 = \lambda_2 v_2, \qquad \lambda_1 \neq \lambda_2$$

  We have

$$(\lambda_1 v_1)^* v_2 = (Mv_1)^* v_2 = v_1^* M v_2 = v_1^* \lambda_2 v_2$$

  thus $\lambda_1 v_1^* v_2 = \lambda_2 v_1^* v_2$, and from the assumption $\lambda_1 \neq \lambda_2$ it then follows that $v_1^* v_2 = 0$.
- If all eigenvectors v_i of M are normalized such that $\|v_i\| = 1$, $i = 1, \ldots, n$, the eigenvector matrix V is unitary, i.e. $V^*V = I$, or $V^{-1} = V^*$.

Matrix 2-Norm

Now we can find the value of $\|A\|$ by introducing $A^*A = V\Lambda V^*$

$$\max_{x\neq 0} \frac{x^*A^*Ax}{x^*x} = \max_{x\neq 0} \frac{x^*V\Lambda V^*x}{x^*x}$$

and letting $y = V^*x$ and thus $x = Vy$ (using orthonormality of V), we obtain

$$\max_{y\neq 0} \frac{y^*\Lambda y}{y^*V^*Vy} = \max_{y\neq 0} \frac{y^*\Lambda y}{y^*y} = \max_{y\neq 0} \frac{\lambda_1|y_1|^2 + \lambda_2|y_2|^2 + \ldots + \lambda_n|y_n|^2}{|y_1|^2 + |y_2|^2 + \ldots + |y_n|^2},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A^*A. Assume that the eigenvalues are ordered such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then it is easy to see that the maximum value of the above value is λ_n, which is achieved if we choose $y = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T$, and the minimum value is λ_n, achieved by choosing $y = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}^T$.

Matrix 2-Norm

Because the above expression is the square of the matrix 2-norm of A, we have thus established that

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sqrt{\lambda_{\mathsf{max}}(A^* A)}$$

and we also found that

$$\min_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sqrt{\lambda_{\mathsf{min}}(A^* A)}$$

The square roots of the eigenvalues of $A^* A$ are called the *singular values* of A.

Frobenius norm

- This norm is called the Frobenius norm
- The Frobenius norm of a matrix $A \in \mathbb{R}^{m \times n}$, denoted by $\|A\|_F$, is defined as

$$\|A\|_F = \|\text{trace}(A^* A)\|_2 = \left(\sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|^2 \right)^{1/2}$$

- the Frobenius norm is not an induced norm.

Lemma

Let A and B be any matrices with appropriate dimensions. Then

- $\rho(A) \leq \|A\|$ *(this is also true for the F-norm and any induced matrix norm).*
- $\|AB\| \leq \|A\|\|B\|$. *In particular, this gives $\|A^{-1}\| \geq \|A\|^{-1}$ if A is invertible. (This is also true for any induced matrix norm).*
- $\|UAV\| = \|A\|$, *and* $\|UAV\|_F = \|A\|_F$, *for any appropriately dimensioned unitary matrices U and V.*
- $\|AB\|_F \leq \|A\|\|B\|_F$ *and* $\|AB\|_F \leq \|B\|\|A\|_F$.

Matrix Norms

- premultiplication or postmultiplication of a unitary matrix on a matrix does not change its induced 2-norm and F-norm, it does change its eigenvalues.
- for example, let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \lambda_1(A) = 1, \lambda_2(A) = 0$$

-

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } UA = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

with $\lambda_1(UA) = \sqrt{2}$, $\lambda_2(UA) = 0$

Vector and Matrix Norms

MATLAB Command

$\|A\|_2$	`norm(A,2)`
$\|A\|_1$	`norm(A,1)`
$\|A\|_\infty$	`norm(A,'inf')`

where A is either a matrix or a vector.

The Singular Value Decomposition

Theorem (Singular Value Decomposition)

For every matrix $A \in \mathbb{C}^{m\times n}$ *there exist unitary matrices* $U \in \mathbb{C}^{m\times m}$ *and* $V \in \mathbb{C}^{m\times n}$ *such that*

$$A = U\Sigma V^*$$

and Σ *is real and diagonal with non-negative entries.*

The matrix Σ has the same size as A. For example if A is a 3×2 or 2×3 matrix, then

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} \qquad \text{or} \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}$$

respectively, where $\sigma_{1,2} \geq 0$. The diagonal entries σ_1 are called the *singular values* of A.

The Singular Value Decomposition

Proof

There exists a unitary matrix V such that

$$A^*A = V\Lambda V^*,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the diagonal eigenvalue matrix of A^*A, and the column v_i of V are the corresponding eigenvectors. Thus

$$A^*Av_i = \lambda_i v_i \qquad \text{and} \qquad v_i^* A^* A v_i = \lambda_i v_i^* v_i = \lambda_i,$$

because V is unitary, and therefore

$$\|Av_i\|^2 = \lambda_i$$

This implies that $\lambda_i \geq 0$. Assume that the eigenvalues $\lambda_1, \ldots, \lambda_r$ are positive and the remaining $n - r$ eigenvalues λ_i and vectors Av_i are zero. Note that $r \leq \min(n, m)$. Define

$$\sigma_i = \sqrt{\lambda_i}, \qquad u_i = \frac{1}{\sigma_i} A v_i, \qquad i = 1, \ldots, r$$

It follows that $\|u_i\| = 1$. Moreover, we have

$$u_i^* u_j = \frac{v_i^* A^* A v_j}{\sigma_i \sigma_j} = \frac{\lambda_i v_i^* v_j}{\sigma_i \sigma_j} = 0, \qquad i \neq j$$

this shows that the vectors $u_1, \ldots, u_r$ defined above have the properties required of column vectors for U to be unitary. If $r < m$, one can fill up the matrix U with $m - r$ further orthogonal vectors to make it into a $m \times m$ unitary matrix.

We remain to show that the matrices U, V as defined above satisfy $U^* A V = \Sigma$ where Σ is diagonal with σ_i as diagonal entries.

The (i, j) entry of $U^* A V$ is

$$(U^* A V)_{i,j} = u_i^* \Lambda v_j = \begin{cases} \sigma_j u_i^* u_j, & j \leq r \\ 0 & j > r \end{cases}$$

Because $\sigma_j u_i^* u_j$ is zero if $i \neq j$ and σ_j if $i = j$, the above shows that the entries of $U^* A V$ are all zero except for the first r entries on the main diagonal, which are the singular values of A.

The Singular Value Decomposition

From the definition of SVD we obtain $AV = U\Sigma$ and thus

$$Av_i = \sigma_i u_i, \qquad i = 1, \ldots, n,$$

where v_i and u_i are the columns of V and U, respectively. We also have

$$AA^* = U\Sigma V^* V \Sigma^T U^* = U\Sigma\Sigma^* U^*$$

and

$$A^* A = V\Sigma^T U^* U \Sigma V^* = V\Sigma^T \Sigma V^*$$

These show that U is the eigenvector matrix of AA^* and V is the eigenvector matrix of A^*A. The eigenvalue matrices are $\Sigma\Sigma^T$ and $\Sigma^T\Sigma$, respectively. Again, if A is 3×2 then

$$\Sigma\Sigma^T = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Sigma^T\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

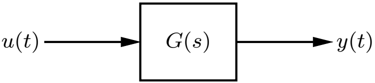
The singular values of A are the square roots of the eigenvalues of AA^* and A^*A.

System Norms

$\mathcal{H}_\infty$ Norm for SISO Systems

For a stable, proper SISO system with a transfer function $G(s)$, the $\mathcal{H}_\infty$ norm is defined as

$$\|G(s)\|_\infty = \sup_\omega |G(j\omega)|$$



- the $\mathcal{H}_\infty$ norm of a SISO system is simply the maximum gain over all frequencies, and can be read off the Bode magnitude plot of the frequency response.
- the $\mathcal{H}_\infty$ norm is equal to the norm induced by the signal 2-norm:

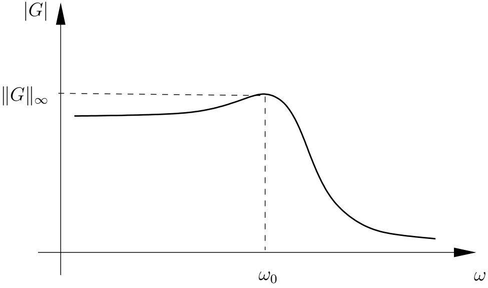
$$\|G(s)\|_\infty = \max_{u \neq 0} \frac{\|y(t)\|}{\|u(t)\|}$$

if $u(t)$ is an energy signal,

- it is

$$\|G(s)\|_\infty = \max_{u \neq 0} \frac{\|y(t)\|_{\text{rms}}}{\|u(t)\|_{\text{rms}}}$$

if $u(t)$ is a power signal.



- in fact the steady state response to an input $u(t) = \sin \omega_0 t$ is

$$y(t) = \|G(s)\|_\infty \sin(\omega_0 t + \phi)$$

The $\mathcal{H}_2$ norm for a SISO system with a stable, strictly proper transfer function $G(s)$ is defined as

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega}$$

- the restriction to strictly proper systems is necessary because otherwise $|G(j\omega)| > 0$ as $\omega \to \infty$ and the integral does not exist.
- this norm is not induced by a signal norm.
- For a stochastic interpretation of the $\mathcal{H}_2$ norm, assume that the input $u(t)$ is white noise with $\mathrm{E}\left[u(t)u(t+\tau)\right] = \delta(\tau)$. In this case, the rms value of the output signal is equal to the $\mathcal{H}_2$ norm of the system

$$\|y(t)\|_{\text{rms}} = \|G(s)\|_2$$

This fact makes it possible to express the LQG problem as the problem of minimizing the $\mathcal{H}_2$ norm of the generalized plant concept.

- A deterministic interpretation is in term of the impulse response

$$\|G(s)\|_2 = \sqrt{\int_0^{\infty} |g(t)|^2 dt} = \|g(t)\|_2$$

  This is by letting $\dot{x} = Ax + bu$ and $y = Cx$ be a state space realization of $G(s)$. And $g(t) = ce^{At}b$ denotes the impulse response of the system. The frequency domain can be changed to time domain by using Parseval's theorem.

- the $\mathcal{H}_2$ norm of the system is equal to the signal 2-norm of its impulse response.
- This interpretation makes it possible to express the deterministic LQR problem as a $\mathcal{H}_2$ optimization problem.

For MIMO system, we consider both size and direction of the signal vectors as the following example:

$$G(s) = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}$$

- the response to a constant input $u(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \sigma(t)$ is $y(t) = \begin{bmatrix} 3 & 4 \end{bmatrix}^T \sigma(t)$, and $\|y(t)\|_{\text{rms}} = 5/\sqrt{2}$.
- the response to $u(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \sigma(t)$ is $y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \sigma(t)$, and $\|y(t)\|_{\text{rms}} = 1/\sqrt{2}$.
- Both cases $\|u(t)\|_{\text{rms}} = 1/\sqrt{2}$, the *gain* in terms of rms values is 5 for the first input signal and 1 for the second input signal.

To define $\mathcal{H}_\infty$ norm for MIMO systems, we need to consider sinusoidal input signals, and find the combination of inputs that maximizes the output signal.

Consider a sinusoidal input with amplitude u_0 and phase ψ can be interpreted as the imaginary part of a complex signal

$$u(t) = u_0 \sin(\omega t + \psi) = \text{Im}\left[u_0 e^{j(\omega t+\psi)}\right]$$
$$\tilde{u}(t) = u_0 e^{j(\omega t+\varphi)} = \hat{u} e^{j\omega t},$$

where $\hat{u}$ is a complex amplitude. The steady state response to the complex input is $\tilde{y} = \hat{y} e^{j\omega t}$, where $\hat{y}$ is the amplitude of the input signal multiplied by the transfer function evaluated at $s = j\omega$

$$\hat{y} = G(j\omega)\hat{u}$$

Applying m inputs to a system that has l outputs. At a given frequency ω, the input and output signals are

$$\tilde{u}(t) = \begin{bmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_m \end{bmatrix} e^{j\omega t}, \qquad \tilde{y}(t) = \begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_l \end{bmatrix} e^{j\omega t} = \hat{y} e^{j\omega t}$$

To find the induced 2-norm of the system, we start with

$$\frac{\|\tilde{y}(t)\|_{\text{rms}}}{\|\tilde{u}(t)\|_{\text{rms}}} = \frac{\|\hat{y}\|}{\|\hat{u}\|}$$

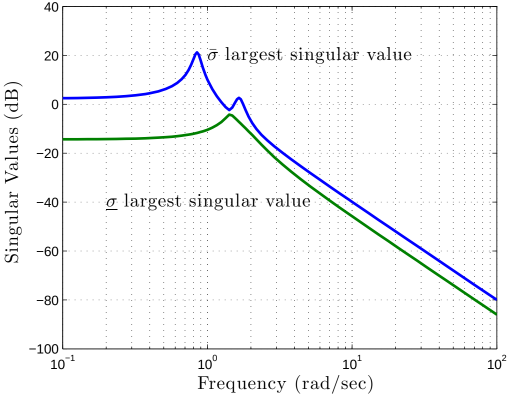
The output amplitude vector at a given frequency ω is obtained by multiplying the input amplitude vector with $G(j\omega)$. For example 2×2 system:

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} G_{11}(j\omega) & G_{12}(j\omega) \\ G_{21}(j\omega) & G_{22}(j\omega) \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}$$

For a given frequency, the transfer function matrix is just a complex matrix $G(j\omega) \in \mathbb{C}^{l \times m}$. The maximum value value of $\|\hat{y}\|/\|\hat{u}\|$ at that frequency is given by the maximum singular value of the transfer function matrix

$$\max_{\hat{u} \neq 0} \frac{\|\hat{y}\|}{\|\hat{u}\|} = \bar{\sigma}(G(j\omega))$$

The $\mathcal{H}_\infty$ norm of the system is defined as the maximum value of this induced matrix norm over all frequencies $\|G(s)\|_\infty = \sup_\omega \bar{\sigma}(G(j\omega))$.



An example of the singular value plot of a mass-spring-damper system, using a command `sigma`.

We used the induced 2-norm for matrices, which is equal to the maximum singular value. By using the Frobenius norm, which is

$$\|A\|_F = \sqrt{\text{trace}(A^* A)}$$

The definition of the $\mathcal{H}_2$ norm of a multivariable system is

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi} \int_0^{\infty} \|g(t)\|_F^2 dt}$$

Again, using Parseval's theorem, one can show that an equivalent definition in time domain is

$$\|G(s)\|_2 = \sqrt{\int_0^{\infty} \|g(t)\|_F^2 dt}$$

where $g(t) = Ce^{At}B$ is the impulse response matrix of the system.

The $\mathcal{H}_2$ norm is

$$\|G(s)\|_2 = \sqrt{\text{trace} \int_0^\infty (g^T(t) g(t)) dt},$$

where $g(t) = Ce^{At}B$ is the impulse response matrix for a system $\dot{x} = Ax + Bu$ and $y = Cx$. Substituting the impulse response in the above equation and taking squares yields

$$\begin{aligned}\|G(s)\|_2^2 &= \text{trace} \int_0^\infty B^T e^{A^T t} C^T C e^{At} B dt \\ &= \text{trace} B^T \int_0^\infty e^{A^T t} C^T C e^{At} dt B\end{aligned}$$

Defining

$$W_0 = \int_0^\infty e^{A^T t} C^T C e^{At} dt \qquad \text{then} \qquad \|G(s)\|_2 = \sqrt{\text{trace} B^T W_0 B}$$

It is straightforward to show that W_0 is the solution to the Lyapunov equaiton

$$A^T W_0 + W_0 A + C^T C = 0$$

Using the fact that trace$MN =$ traceNM for two matrices M and N we have

$$\|G(s)\|_2^2 = \text{trace} \int_0^\infty C e^{At} B B^T e^{A^T t} C^T dt$$

and obtain as an alternative expression for the value of the $\mathcal{H}_2$ norm

$$\|G(s)\|_2 = \sqrt{\text{trace} C W_c C^T}$$

where $W_c = \displaystyle\int_0^\infty e^{At} B B^T e^{A^T t} dt$ is the solution to

$$A W_c + W_c A^T + B B^T = 0$$

Thus, the $\mathcal{H}_2$ norm can be computed by solving a single Lyapunov equation.

The $\mathcal{H}_\infty$ norm is defined for systems with stable, proper transfer functions. Because the $\mathcal{H}_\infty$ norm of $G(s)$ is the maximum of $\bar{\sigma}(G(j\omega))$ over frequency, one can try to compute $\bar{\sigma}(G(j\omega))$ for many values of ω and then search for the maximum.

The more efficient method is an iterative procedure:

- Check whether $\|G(s)\|_\infty$ is less than a given positive constant γ.
- Consider a stable plant with transfer function $G(s) = C(sI - A)^{-1}B$.
- For a given $\gamma > 0$, define the Hamiltonian matrix

$$M_\gamma = \begin{bmatrix} A & \frac{1}{\gamma}BB^T \\ -\frac{1}{\gamma}C^TC & -A^T \end{bmatrix}$$

Theorem (*)

Given a positive constant $\gamma > 0$, the following two statements are equivalent

1. *γ is a singular value of $G(j\omega_0)$ at some frequency ω_0.*
2. *M_γ has at least one eigenvalue on the imaginary axis.*

From the theorem

- To find the largest singular value over all frequencies, we can start with a sufficiently large value of γ and use the above Theorem to check whether it is a singular value of $G(j\omega)$ at some frequency.
- We do not need to know at which frequency. If M_γ has no imaginary eigenvalue, γ was too large and we try with a smaller value of γ.
- If M_γ does have an imaginary eigenvalue, γ was too small and we try with a larger value.
- A bisection method can be used as an efficient way of finding a value of γ that is equal to $\|G(s)\|_\infty$ within a guaranteed accuracy.

Computing System Norms

Proof the theorem (*)

- First show that (1) $\Rightarrow$ (2). Assume that γ is a singular value of $G(j\omega_0)$ (G_0). Let $G_0 = U\Sigma V^*$ be the singular value decomposition of G_0. Then, from $G_0 V = U\Sigma$ and $G_0^* U = V\Sigma^T$, there exist nonzero vectors u and v such that

$$G_0 v = \gamma u, \qquad G_0^* u = \gamma v$$

Substituting $G(s) = C(sI - A)^{-1}B$ at $s = j\omega_0$ for G_0 yields

$$C(j\omega_0 I - A)^{-1}Bv = \gamma u, \qquad \text{and} \qquad B^T(-j\omega_0 I - A^T s)^{-1}C^T u = \gamma v$$

Introducing the vectors

$$p = (j\omega_0 I - A)^{-1}Bv, \qquad \text{and} \qquad q = (-j\omega_0 I - A^T)^{-1}C^T u$$

this becomes

$$Cp = \gamma u, \qquad \text{and} \qquad B^T q = \gamma u$$

or

$$\begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 & \gamma I \\ \gamma I & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}$$

Solving for v and u yields

$$\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} 0 & \gamma I \\ \gamma I & 0 \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}. \tag{1}$$

The above matrix equation guarantees that

$$\begin{bmatrix} p \\ q \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From vectors p and q , they satisfy

$$(j\omega_0 I - A)p = Bv \qquad \text{and} \qquad (-j\omega_0 I - A^T)q = C^T u$$

or

$$\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -C^T \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = j\omega_0 \begin{bmatrix} p \\ q \end{bmatrix}$$

Substituting from (1) yields

$$\left(\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -C^T \end{bmatrix} \begin{bmatrix} 0 & \gamma I \\ \gamma I & 0 \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix} \right) = j\omega_0 \begin{bmatrix} p \\ q \end{bmatrix} \qquad (2)$$

which is shows that $j\omega_0$ is an eigenvalue of M_γ.

- To prove (2) ⇒ (1), assume that $j\omega_0$ is an eigenvalue of M_γ. Then there exists a nonzero vector $\begin{bmatrix} p^T & q^T \end{bmatrix}^T$ that satisfies (2). Now use (1) to define $\begin{bmatrix} v^T & u^T \end{bmatrix}^T \neq 0$. Then form (1) and (2) it follows that γ is a singular value of $G(j\omega_0)$.
- If $D \neq 0$, the associated Hamiltonian matrix to check is

$$M_\gamma = \begin{bmatrix} A - BR^{-1}D^TC & -\gamma BR^{-1}B^T \\ -\gamma C^TS^{-1}C & -(A - BR^{-1}D^TC)^T \end{bmatrix}$$

where R and S are given by $R = D^TD - \gamma^2 I$ and $S = DD^T - \gamma^2 I$.

Reference

1. Herbert Werner "*Lecture Notes on Control Systems Theory and Design*", 2011
2. Mathwork "*Control System Toolbox: User's Guide*", 2014
3. Kemin Zhou and John Doyle "*Essentials of Robust Control*", Prentice Hall, 1998