Lecture 7 : Generalized Plant and LFT form

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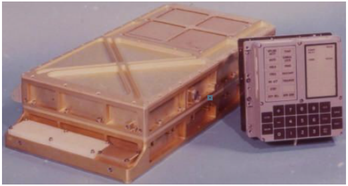
Linear Quadratic Gaussian

- The state space methods for optimal controller design developed in the 1960s
- Linear Quadratic Gaussian (LQG) control was recognized by the Apollo people, and the Kalman filter became the first embedded system.

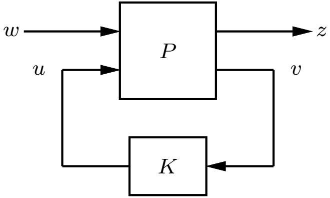


- In 1970s were found to suffer from being sensitive to modelling errors and parameters uncertainty.
- There were a lot of failures of the method in practical application: a Trident submarine caused the vessel to unexpectedly surface in a simulation of rough sea, the same year F-8c crusader aircraft led to disappointing results.
- J. Doyle, "Guaranteed Margins for LQG Regulators", IEEE Transactions on Automatic Control, Vol. 23, No. 4, pp. 756–757, 1978.

Robust Control

- George Zames posed the problem of robust control, also known as $\mathcal{H}_\infty$-synthesis.
- In the 1980s research activities turned to a new approach, where design objectives are achieved by minimizing the $\mathcal{H}_2$ norm or $\mathcal{H}_\infty$ norm of suitable closed-loop transfer functions.
- The new method is closely related to the familiar LQG methods – the computation of both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal controllers involves the solution of two algebraic Riccati equations.
- More efficient methods for such a design have been developed in the 1990s. Instead of solving Riccati equations, one can express $\mathcal{H}_2$ and $\mathcal{H}_\infty$ constraints as linear matrix inequalities (LMI).
- The major problem with modern $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control is the controllers have the same dynamic order as the plant. (This problem has been solved, they claimed, by Pierre Apkarian and Dominikus Noll since 2006.) If the plant to be controlled is of high dynamic order, the optimal design results in controllers that are difficult to implement. Moreover, the high order may cause numerical problems.

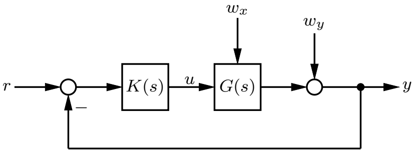
In modern control, almost any design problem is represented in the form shown in the below Figure.



We can show the problem of designing an LQG controller in the generalized plant format. Consider a state space realization of the plant with transfer function $G(s)$ corrupted by process noise w_x and measurement noise w_y.

$$\dot{x} = Ax + Bu + w_x$$
$$y = Cx + w_y,$$

where w_x and w_y are white noise processes.



A regulation problem with $r = 0$. The objective is to find a controller $K(s)$ that minimizes the LQG performance index

$$V = \lim_{T \to \infty} \mathsf{E} \left[\frac{1}{T} \int_0^T \left(x^T Q x + u^T R u \right) dt \right]$$

The state space realization of the generalized P. It has two inputs w and u, and two output z and v:

$$\dot{x} = A_p x + B_w w + B_u u$$

$$z = C_z x + D_{zw} w + D_{zu} u$$

$$v = C_v x + D_{vw} w + D_{vu} u$$

The Concept of a Generalized Plant

LQG control

- The measured output v of the generalized plant to be the control error $e = -y$ in the LQG problem.
- Take the control input u of the generalized plant to be the control input of the LQG problem. Relate the plant model and the generalized plant:

$$A_p = A, \quad B_u = B, \quad C_v = -C, \quad D_{vu} = 0$$

- Select

$$C_z = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}, \qquad D_{zu} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix}$$

- Assume $w = 0$, the square integral of the fictitious output z is

$$\int_0^\infty z^T z dt = \int_0^\infty \left(x^T Q x + u^T R u \right) dt$$

- Assume that w is a white noise process satisfying $\mathsf{E}\left[w(t)w^T(t+\tau)\right] = \delta(\tau)I$, and choose

$$B_w = \begin{bmatrix} Q_e^{1/2} & 0 \end{bmatrix}, \qquad D_{vw} = \begin{bmatrix} 0 & R_e^{1/2} \end{bmatrix}$$

Then

$$w_x = B_w w = \begin{bmatrix} Q_e^{1/2} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = Q_e^{1/2} w_1$$
$$w_y = D_{vw} w = \begin{bmatrix} 0 & R_e^{1/2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = R_e^{1/2} w_2$$

It is easy to see that minimizing

$$\lim_{T \to \infty} \mathsf{E}\left[\frac{1}{T}\int_0^T z^T(t)z(t)dt\right]$$

is equivalent to minimizing the LQG performance index V.

The transfer function of a generalized plant that represents the LQG problem is

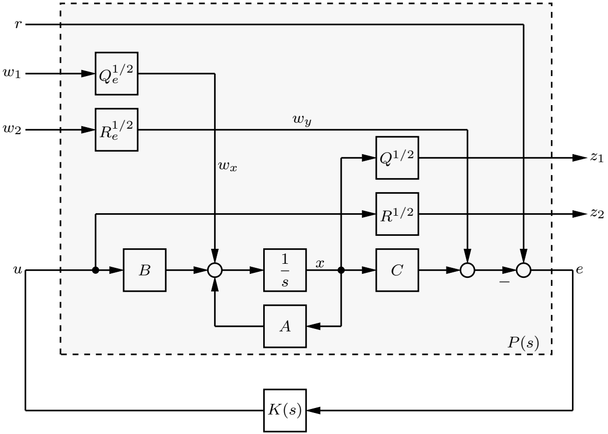
$$P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right] = \left[\begin{array}{c|cc} A & \begin{bmatrix} Q_e^{1/2} & 0 \end{bmatrix} & B \\ \hline \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \\ -C & \begin{bmatrix} 0 & R_e^{1/2} \end{bmatrix} & 0 \end{array}\right]$$

where 0 stand for zero matrix blocks of appropriate dimensions.

In MATLAB, use a command `P = ss(Ap, Bp, Cp, Dp)`

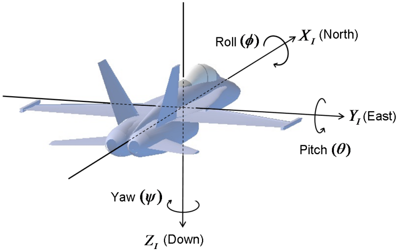
The Concept of a Generalized Plant

LQG control - reference tracking



- the external input $w = \begin{bmatrix} r & w_1 & w_2 \end{bmatrix}^T$
- the fictitious output $z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T$

Aircraft control



Assuming the state space model represents a linearized model of the vertical-plane dynamics of an aircraft is described below:

$$A = \begin{bmatrix} 0 & 0 & 1.132 & 0 & -1 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.0485 & 0 & -0.8556 & -1.013 \\ 0 & -0.2909 & 0 & 1.0532 & -0.6859 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -0.12 & 1 & 0 \\ 0 & 0 & 0 \\ 4.419 & 0 & -1.665 \\ 1.575 & 0 & -0.0732 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Aircraft control

u_1 spoiler angle (in 0.1 deg)

u_2 forward acceleration (in m s^{-2})

u_3 elevator angle (in deg)

x_1 relative altitude (in m)

x_2 forward speed (in m s^{-1})

x_3 pitch angle (in deg)

x_4 pitch rate (in deg s^{-1})

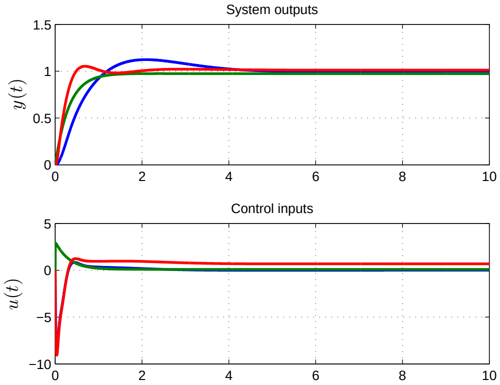
x_5 vertical speed (in m s^{-1})

The design objectives are:

- fast tracking of step changes for all three reference inputs, with little or no overshoot
- control input must satisfy $|u_3| < 20$.
- Hint: use $\mathcal{H}_2$ control synthesis command, `K = h2syn(Gplant, nmeas, ncont)` , of MATLAB
- We will discuss how this function work later.

Aircraft control

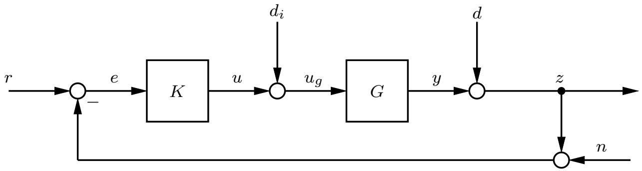
Design example



This result has been done by selecting:

- $R = 1 \times 10^{-5} I$, $Re = 0.1I$, $Q = C$, and $Qe = B$.
- Noting that we did not use an integrator.

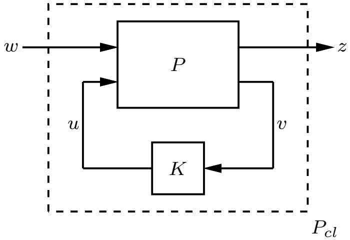
Feedback Structure



The standard feedback configuration is consisted of the interconnected plant P and controller K.

- r is a reference signal
- n is a sensor noise
- d and d_i are plant output disturbance and plant input disturbance
- u_g and y are plant input and output.

Standard Problem: $P - K$-Structure



$$\begin{bmatrix} z \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}}_{P} \begin{bmatrix} w \\ u \end{bmatrix}$$

- external inputs: w
- external outputs: z
- controller input: v
- controller output: u

Transformation into Standard Problem

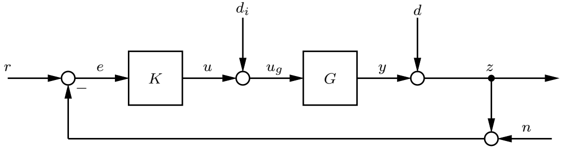
For any control structure, perform the following steps:

- Collect all signals that are evaluated for performance into the performance vector z
- Collect all signals from outside into generalized disturbance vector w
- Collect all signals that are fed to K into generalized measurement vector v
- Denote output of K by u
- Cut out K
- Determine transfer matrix

$$\begin{bmatrix} z \\ v \end{bmatrix} = P \begin{bmatrix} w \\ u \end{bmatrix}$$

Standard Problem – Example

For the classical control



$$z = d + G(d_i + u)$$
$$v = e = r - (n + z) = r - n - d - G(d_i + u)$$

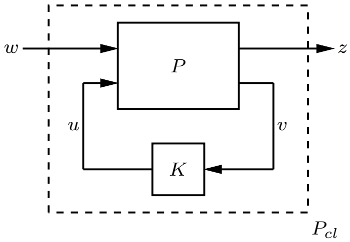
$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} 0 & I & G & 0 & G \\ I & -I & -G & -I & -G \end{bmatrix} \begin{bmatrix} r \\ d \\ d_i \\ n \\ u \end{bmatrix}$$

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] = \left[\begin{array}{cccc|c} 0 & I & G & 0 & G \\ \hline I & -I & -G & -I & -G \end{array}\right]$$

$$w = \begin{bmatrix} r^T & d^T & d_i^T & n^T \end{bmatrix}^T$$

Standard Problem – $P - K$-Structure

Procedure leads to **standard problem** or the $P - K$-Structure:



$$\begin{bmatrix} z \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}}_{P} \begin{bmatrix} w \\ u \end{bmatrix}$$

Closed-loop interconnection described by

$$Z(s) = P_{cl}(s)W(s) \qquad \text{or short } z = P_{cl}w$$

$$\begin{aligned} \text{with } P_{cl} &= P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21} \\ &= P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \\ &= \mathcal{F}(P, K) \end{aligned}$$

Linear Fractional Transformation (LFT)

Consider a mapping $F : \mathbb{C} \mapsto \mathbb{C}$ of the form

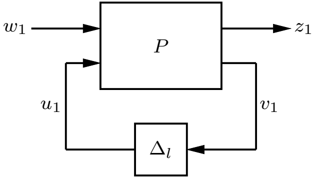
$$F(s) = \frac{a + bs}{c + ds}$$

with $a, b, c,$ and $d \in \mathbb{C}$ is called a **linear fractional transformation**, if $c \neq 0$ the $F(s)$ can also be written as

$$F(s) = \alpha + \beta s(1 - \gamma s)^{-1}$$

for some λ, β and $\gamma \in \mathbb{C}$.

The lower LFT with respect to Δ_l is defined as



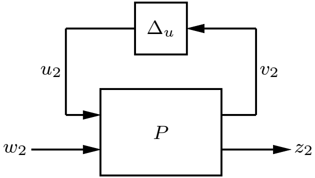
$$\begin{bmatrix} z_1 \\ v_1 \end{bmatrix} = P \begin{bmatrix} w_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} w_1 \\ u_1 \end{bmatrix}$$

$$u_1 = \Delta_l v_1$$

$$\begin{aligned} \mathcal{F}_l(M, \Delta_l) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \star \Delta_l &:= A + B(I - \Delta_l D)^{-1} \Delta_l C \\ &= A + B\Delta_l (I - D\Delta_l)^{-1} C, \end{aligned}$$

provided that the inverse $(I - \Delta_l D)^{-1}$ exists.

The upper LFT with respect to Δ_u is defined as



$$\begin{bmatrix} v_2 \\ z_2 \end{bmatrix} = P \begin{bmatrix} u_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} u_2 \\ w_2 \end{bmatrix}$$

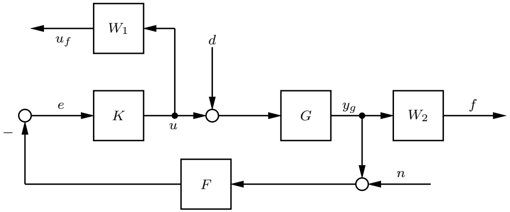
$$u_2 = \Delta_u v_2.$$

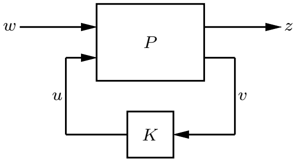
$$\begin{aligned} \mathcal{F}_u(M, \Delta_u) = \Delta_u \star \begin{bmatrix} A & B \\ C & D \end{bmatrix} &:= D + C(I - \Delta_u A)^{-1} \Delta_u B \\ &= D + C\Delta_u (I - A\Delta_u)^{-1} B, \end{aligned}$$

provided that the inverse $(I - \Delta_u A)^{-1}$ exists.

Linear Fractional Transformations

Example





$$w = \begin{bmatrix} d & n \end{bmatrix}^T$$

$$z = \begin{bmatrix} f & u_f \end{bmatrix}^T$$

$$\begin{bmatrix} z \\ v \end{bmatrix} = P \begin{bmatrix} w \\ u \end{bmatrix} = \left[\begin{array}{cc|c} W_2G & 0 & W_2G \\ 0 & 0 & W_1 \\ \hline -FG & -F & -FG \end{array}\right] \left[\begin{array}{c} d \\ n \\ \hline u \end{array}\right]$$

Assuming that the plant G is strictly proper and P, F, W_1, and W_2 have the following state-space realizations:

$$G = \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & 0 \end{array}\right], \qquad F = \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right],$$
$$W_1 = \left[\begin{array}{c|c} A_{w_1} & B_{w_1} \\ \hline C_{w_1} & D_{w_1} \end{array}\right], \qquad W_2 = \left[\begin{array}{c|c} A_{w_2} & B_{w_2} \\ \hline C_{w_2} & D_{w_2} \end{array}\right]$$

That is

$$\begin{aligned} \dot{x}_g &= A_g x_g + B_g(d+u), & y_g &= C_g x_g \\ \dot{x}_f &= A_f x_f + B_f(y_g+n), & -y &= C_f x_f + D_f(y_g+n), \\ \dot{x}_{w_1} &= A_{w_1} x_{w_1} + B_{w_1} u, & u_f &= C_{w_1} x_u + D_{w_1} u, \\ \dot{x}_{w_2} &= A_{w_2} x_{w_2} + B_{w_2} y_g, & f &= C_{w_2} x_{w_2} + D_{w_2} y_g. \end{aligned}$$

Linear Fractional Transformations

Example

Define a new state vector

$$x = \begin{bmatrix} x_g & x_f & x_{w_1} & x_{w_2} \end{bmatrix}^T$$

and elimainate the variable y_g to get a realization of P as

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ v &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

with

$$A = \begin{bmatrix} A_g & 0 & 0 & 0 \\ B_f C_g & A_f & 0 & 0 \\ 0 & 0 & A_{w_1} & 0 \\ B_{w_2} C_g & 0 & 0 & A_{w_2} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_g & 0 \\ 0 & B_f \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_g \\ 0 \\ B_{w_1} \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} D_{w_2} Cg & 0 & 0 & C_{w_2} \\ 0 & 0 & C_{w_1} & 0 \end{bmatrix}, \quad D_{11} = 0, \quad D_{12} = \begin{bmatrix} 0 \\ D_{w_1} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -D_f C_g & -C_f & 0 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & -D_f \end{bmatrix}, \quad D_{22} = 0.$$

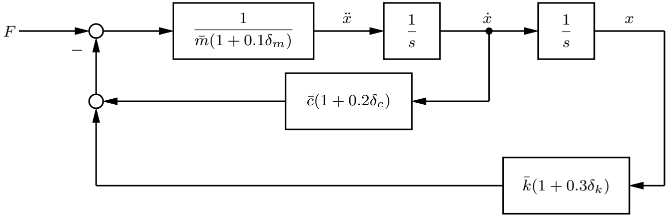
The dynamical equation of the system motion can be described by

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m}.$$

Suppose m, c, and k are not known exactly, but are believed to lie in known intervals as

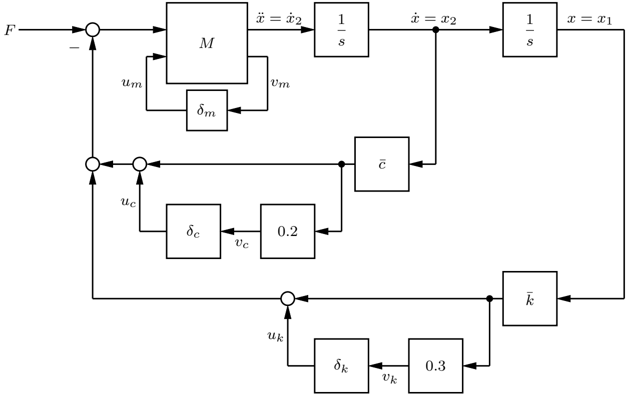
$$m = \bar{m} \pm 10\%, \quad c = \bar{c} \pm 20\%, \quad k = \bar{k} \pm 30\%$$

Introducing perturbations $\delta_m, \delta_c, \delta_k \in [-1, 1]$.



It is easy to check that $\frac{1}{m}$ can be represented as an LFT in δ_m:

$$\frac{1}{m} = \frac{1}{\bar{m}(1+0.1\delta_m)} = \frac{1}{\bar{m}} - \frac{0.1}{\bar{m}}\delta_m(1+0.1\delta_m)^{-1} = \mathcal{F}_l(M_1, \delta_m), \quad M_1 = \begin{bmatrix} \frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ 1 & -0.1 \end{bmatrix}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ v_k \\ v_c \\ v_m \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{c}}{\bar{m}} & \frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ 0.3\bar{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2\bar{c} & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & 1 & -1 & -1 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ F \\ u_k \\ u_c \\ u_m \end{bmatrix}, \quad \begin{bmatrix} u_k \\ u_c \\ u_m \end{bmatrix} = \Delta \begin{bmatrix} v_k \\ v_c \\ v_m \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mathcal{F}_l(M, \Delta) \begin{bmatrix} x_1 \\ x_2 \\ F \end{bmatrix}$$

where

$$M = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{c}}{\bar{m}} & \frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ \hline 0.3\bar{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2\bar{c} & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & 1 & -1 & -1 & -0.1 \end{array}\right], \quad \Delta = \begin{bmatrix} \delta_k & 0 & 0 \\ 0 & \delta_c & 0 \\ 0 & 0 & \delta_m \end{bmatrix}.$$

Consider an input/output relation

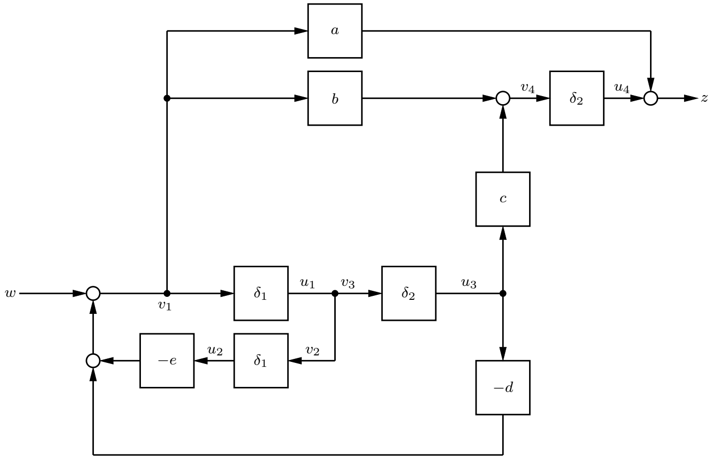
$$z = \frac{a + b\delta_2 + c\delta_1\delta_2^2}{1 + d\delta_1\delta_2 + e\delta_1^2} w := Gw$$

where $a, b, c, d,$ and e are given constants or transfer functions. we would like to write G as an LFT in terms of δ_1 and δ_2. We can do this in three steps:

1. Draw a block diagram for the input/output relation with each δ separated as shown in the next Figure.
2. Mark the inputs and outputs of the δ's as y's and u's, respectively. (This is essentially *pulling out the Δ's*
3. Write z and v's in terms of w and u's with all δ's taken out.

Linear Fractional Transformations

Basic Principle



$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ z \end{bmatrix} = M \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ w \end{bmatrix}$$

where

$$M = \left[\begin{array}{cccc|c} 0 & -e & -d & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -be & -bd+c & 0 & b \\ \hline 0 & -ae & -ad & 1 & a \end{array}\right], \text{then} \quad z \quad = \mathcal{F}_u(M, \Delta)w, \quad \Delta = \begin{bmatrix} \delta_1 I_2 & 0 \\ 0 & \delta_2 I_2 \end{bmatrix}$$

Reference

1. Herbert Werner "*Lecture Notes on Control Systems Theory and Design*", 2011
2. Mathwork "*Control System Toolbox: User's Guide*", 2014
3. Kemin Zhou and John Doyle "*Essentials of Robust Control*", Prentice Hall, 1998