Lecture 6 : State-Estimate Feedback

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State Estimate Feedback

- The state feedback LQR is suffered from the requirement of the whole state x of the process has to be measured. This is sometime impossible in real-life.
- To overcome this problem one can construct an estimate $\hat{x}$ of the state of the process based on the past values of the measured ouput $y(t)$ and control signal $u(t)$, and use

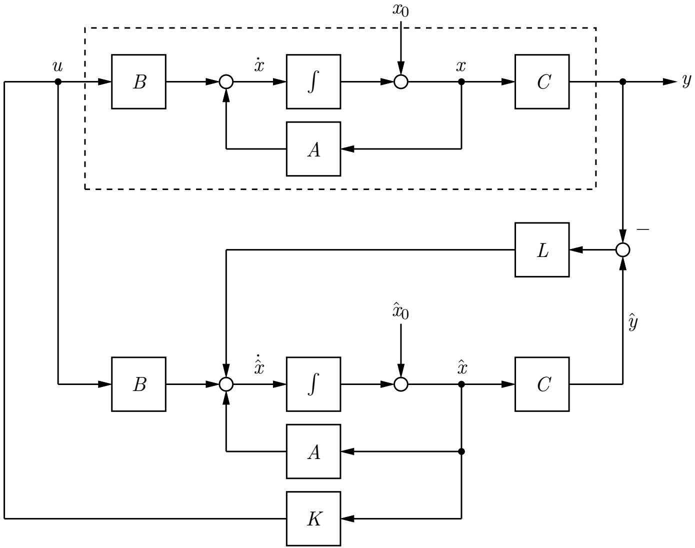
$$u(t) = -K\hat{x}(t)$$

  instead of using the real state.
- This approach is usually known as the state estimate feedback. See next page.
- Subtracting the estimator equation

$$\dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x}) \quad \text{from} \quad \dot{x} = Ax + Bu$$

  gives $\dot{\epsilon} = (A - LC)\epsilon$.
- The state feedback gain F and the estimator gain L can be designed to place the closed-loop plant eigenvalues. The estimator eigenvalues should be faster than the closed-loop plant eigenvalues, but the upper limit should be trade-off to avoid high frequency noise.

State Estimate Feedback



Deterministic Minimum-Energy Estimation (MEE)

Consider a continuous-time LTI system of the form

$$\dot{x} = Ax + Bu, \qquad y = Cx, \qquad x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}, y \in \mathbb{R}^{n_y}$$

- Estimating the state x at time t can be viewed as solving for the unknown $x(t)$, for given $u(\tau)$, $y(\tau)$, and $\tau \leq t$.
- $x(t)$ can be reconstructed exactly using the observability Gramian matrix

$$x(t) = W_o(t)^{-1} \left(\int_0^t e^{A^T(\tau - t)} C^T y(\tau) d\tau + \int_0^t \int_\tau^t e^{A^T(\tau - t)} C^T C e^{A(\tau - s)} Bu(s) ds d\tau \right),$$

where

$$W_o(t) = \int_0^t e^{A^T(\tau - t)} C^T C e^{A(\tau - t)} d\tau$$

Stochastic Model

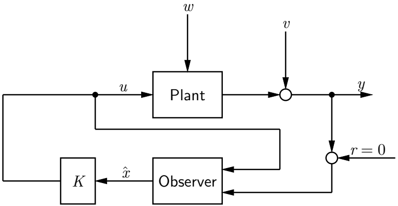
In practice, the model (CLTI) is never exact, and the measured output y is include the effect of stochastic disturbances as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + w(t)\\ y(t) &= Cx(t) + v(t),\end{aligned}$$

where $w(t)$ is *process noise* and $v(t)$ is *measurement noise*. Both noise processes are assumed to be white, zero mean, Gaussian and uncorrelated, and satisfy

$$\mathsf{E}\left[w(t)w^T(t+\tau)\right] = Q_e\delta(\tau), \qquad \mathsf{E}\left[v(t)v^T(t+\tau)\right] = R_e\delta(\tau),$$

where $\mathsf{E}\left[\cdot\right]$ is an expectation operator .



Optimal Estimation Problem

- Let $\varepsilon(t) = x(t) - \hat{x}(t)$ denote the estimation error, and let q be a weighting vector such that $q^T\varepsilon$ is a linear combination of the errors .
- The estimation problem is as follow: Given the estimator structure as a previous figure, find the estimator gain L that minimizes the stochastic cost function

$$V_e = \lim_{t\to\infty} \mathsf{E}\left[\varepsilon^T q q^T \varepsilon\right].$$

- Here the limit is taken as $t \to \infty$ because we are interested in the steady state.
- Subtracting the estimator equation

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \qquad \Rightarrow \qquad \dot{\varepsilon} = (A - LC)\varepsilon + \xi,$$

where ξ is the white, zero mean, Gaussian noise process and

$$\xi = w - Lv, \qquad \mathsf{E}\left[\xi(t)\xi^T(t+\tau)\right] = \left(Q_e + LR_eL^T\right)\delta(\tau).$$

Stochastic-Deterministic Dualism

- The estimation problem above is a stochastic optimization problem. It is easier to solve this problem by using deterministic method since they have the same structure.
- The deterministic problem is known as the linear regulator problem. When a stochastic problem has the same structure and can be solved by the same methods as an equivalent deterministic problem, we call a stochastic-deterministic dualism.
- To establish this dualism, we first consider a simple deterministic and a simple stochastic problem.

Stochastic-Deterministic Dualism

Deterministic Problem

Given the autonomous plant model

$$\dot{x} = Ax, \qquad x(t) = e^{At}x_0, \qquad x(0) = x_0,$$

assume we are interested in the value of the cost function

$$V = \int_0^\infty x^T Q x dt, \qquad Q = Q^T > 0.$$

Substituting for $x(t)$ in the cost function, we get

$$V = \int_0^\infty (x_0^T e^{A^T t} Q e^{At} x_0)\, dt = x_0^T \int_0^\infty (e^{A^T t} Q e^{At} dt) x_0 = x_0^T P x_0,$$

where $P = \int_0^\infty (e^{A^T t} Q e^{At} dt)$ is the positive definite matrix satisfying

$$PA + A^T P + Q = 0$$

Stochastic-Deterministic Dualism
Stochastic Problem

Consider the process

$$\dot{x} = Ax + \xi,$$

where ξ is a white, zero mean, Gaussian noise process that satisfies

$$\mathsf{E}\left[\xi(t)\xi^T(t+\tau)\right] = Q_s\delta(\tau),$$

and assume we want to find the value of the stochastic cost function

$$V_s = \lim_{t\to\infty} \mathsf{E}\left[x^T q q^T x\right],$$

where q is a weighting vector as before. Here we should note that $x(t_0)$ is a gaussian random variable, of mean m and is independent of $\xi(t)$, that is

$$\mathsf{E}\left[x(t_0)\xi^T(t)\right] = 0, \quad \forall t$$

Let $x(t_0) = x_0$, the solution to the state equation above is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}\xi(\tau)\,d\tau.$$

Using the properties of ξ, we have

$$\begin{aligned}
\mathsf{E}\left[xx^T\right] &= \mathsf{E}\left[e^{At}x_0x_0^Te^{A^Tt} + \int_0^t\int_0^t e^{A(t-\tau_1)}\xi(\tau_1)\xi^T(\tau_2)e^{A^T(t-\tau_2)}\,d\tau_1\,d\tau_2\right] \\
&= e^{At}\mathsf{E}\left[x_0x_0^T\right]e^{A^Tt} + \int_0^t\int_0^t e^{A(t-\tau_1)}Q_s\delta(\tau_1-\tau_2)e^{A^T(t-\tau_2)}\,d\tau_1\,d\tau_2 \\
&= e^{At}\mathsf{E}\left[x_0x_0^T\right]e^{A^Tt} + \int_0^t e^{A(t-\tau)}Q_se^{A^T(t-\tau)}\,d\tau, \text{ by sampling property} \\
&= e^{At}\mathsf{E}\left[x_0x_0^T\right]e^{A^Tt} + \int_0^t e^{A\tau}Q_se^{A^T\tau}\,d\tau
\end{aligned}$$

Assuming that A is stable and taking the limit as $t \to \infty$

$$\lim_{t\to\infty} \mathsf{E}\left[xx^T\right] = 0 + \int_0^\infty e^{A\tau} Q_s e^{A^T\tau}\, d\tau,$$

where in the last term the variable of integration has been changed. Multiplying the above equation from left and right by q^T and q respectively yields

$$\lim_{t\to\infty} \mathsf{E}\left[q^T x x^T q\right] = \lim_{t\to\infty} \mathsf{E}\left[x^T q q^T x\right] = q^T \int_0^\infty e^{A\tau} Q_s e^{A^T\tau}\, d\tau\, q,$$

because $q^T x x^T q$ is scalar. The left hand side is the stochastic cost function V_s, and the above equation can be written as

$$V_s = q^T P_s q$$

when the positive definite matrix P_s is defined as

$$P_s = \int_0^\infty e^{A\tau} Q_s e^{A^T\tau}\, d\tau$$

We have shown that the value of the stochastic cost is given by $V_s = q^T P_s q$. Next we have to show that P_s is the solution of ARE

$$P_s A^T + A P_s + Q_s = 0.$$

Notice that this problem as its solution have the same structure as the deterministic proble: A is replaced by A^T, x_0 by q, Q by Q_s and P by P_s. Substituting P_s in ARE yields

$$\int_0^\infty \left(e^{A\tau} Q_s e^{A^T\tau} A^T + A e^{A\tau} Q_s e^{A^T\tau} \right) d\tau + Q_s = 0$$

$$\int_0^\infty \frac{d}{d\tau} \left(e^{A\tau} Q_s e^{A^T\tau} \right) d\tau + Q_s = 0$$

$$e^{A\tau} Q_s e^{A^T\tau} \Big|_0^\infty + Q_s = 0 - Q_s + Q_s = 0 \text{ stable system },$$

which proves that P_s satisfies ARE.

The optimal estimation problem is

$$\min_L \lim_{t\to\infty} \mathsf{E}\left[\varepsilon^T q q^T \varepsilon\right],$$

where the estimation error is governed by

$$\dot{\varepsilon} = (A - LC)\varepsilon + \xi, \qquad \mathsf{E}\left[\xi(t)\xi^T(t+\tau)\right] = (Q_e + LR_eL^T)\delta(\tau).$$

Applying the above result with the replacements

$$A \to A - LC, \qquad Q_s \to Q_e + LR_eL^T, \text{ with } V_e = q^T P_e q,$$

where P_e is the positive definite solution to

$$P_e(A - LC)^T + (A - LC)P_e + Q_e + LR_eL^T = 0.$$

Linear Regulator Problem

The linear regulator problem: Given the closed-loop system

$$\dot{x} = (A - BK)x, \qquad x(0) = x_0,$$

Find the state feedback gain K that minimizes

$$V = \int_0^\infty (x^T Q x + u^T R u)\, dt = \int_0^\infty x^T (Q + K^T R K) x dt$$

The optimal solution was shown to be $K = R^{-1} B^T P$, where P is the positive semidefinite solution to the ARE

$$PA + A^T P - PBR^{-1} B^T P + Q = 0, \text{ and } V = x_0^T P x_0.$$

With $PB = K^T R$, it is straightforward to show that the ARE can also be written as

$$P(A - BK) + (A - BK)^T P + Q + K^T R K = 0.$$

Linear Regulator Problem

Comparing this with the optimal estimation problem, we find that both problems are equivalent with the replacement

$$Q \to Q_e, \qquad R \to R_e, \qquad A \to A^T, \qquad B \to C^T,$$
$$x_0 \to q, \qquad K \to L^T, \qquad P \to P_e.$$

We can construct the following result by duality: the optimal estimator gain is

$$L = P_e C^T R_e^{-1},$$

where P_e is the positive definite solution to

$$P_e A^T + A P_e - P_e C^T R_e^{-1} C P_e + Q_e = 0.$$

This equation is know as the *filter algebraic Riccati equation (FARE)*.

Consider the first order system $\dot{x} = ax + w$, $y = x + v$, where w and v are white, zero mean, Gaussian noise processes that satisfy

$$\mathsf{E}\left[w(t)w(t+\tau)\right] = q_e\delta(\tau), \qquad \mathsf{E}\left[v(t)v(t+\tau)\right] = r_e\delta(\tau).$$

For the estimator $\dot{\hat{x}} = a\hat{x} + L(y - \hat{x}) = (a - L)\hat{x} - Ly$, find the optimal estimator gain L. The FARE is

$$2ap_e - \frac{1}{r_e}p_e^2 + q_e = 0 \quad \Rightarrow \quad p_e^2 - 2ar_ep_e - r_eq_e = 0$$

with positive solution $p_e = ar_e + \sqrt{a^2r_e^2 + r_eq_e}$. The optimal estimator gain is therefore

$$L = \frac{p_e}{r_e} = a + \sqrt{a^2 + \frac{q_e}{r_e}}.$$

Substituting the optimal gain in the estimator equation yields

$$\dot{\hat{x}} = \left(-\sqrt{a^2 + \frac{q_e}{r_e}}\right)\hat{x} + Ly.$$

Linear Regulator Problem

Example

The solution depends only on the ratio q_e/r_e of the intensities of process and measurement noise. The two limiting cases are $q_e/r_e \to 0$ and $q_e/r_e \to \infty$.

- When we have very large measurement noise, $q_e/r_e \to 0$. In this case the estimator equation becomes

$$\dot{\hat{x}} = -|a|\hat{x} - Ly$$

  Notice that

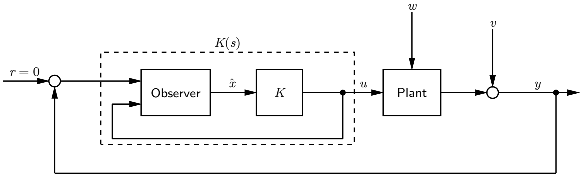
$$L = \begin{cases} 0, & a < 0 \\ 2a, & a \geq 0 \end{cases}$$

- When $q_e/r_e \to \infty$, it is corresponds to a situation with no measurement noise. In this case, we have $L = \infty$, i.e. the optimal solution is to make the estimator dynamics infinitely fast.

Linear Quadratic Gaussian

- When the full state is not available for feedback and state estimate feedback must be used. The problem is changed. The closed-loop system is redrawn as a figure below. We let $K(s)$ denote the transfer function from the state error $r - y$ to u, and define the optimal control problem as the problem of finding the controller $K(s)$ that minimizes the stochastic cost function

$$V_{\mathsf{LQG}} = \lim_{t\to\infty} \mathsf{E}\left[x^T Q x + u^T R u\right].$$



- This problem is known as the *Linear Quadratic Gaussian (LQG)* control problem.
- The cost function has the structure of the cost for the linear regulator problem with two terms penalizing state error and control energy respectively: state estimate feedback must be used and the presence of process and measurement noise is taken into account.

The Separation Principle

- The solution to the LQG problem is based on the *separation principle*: the controller $K(s)$ that minimizes the LQG cost function is obtained by combining the optimal state feedback gain K and the optimal estimator gain L.
- This is the sketch proof is based on two properties:
  - State estimate and estimation error are uncorrelated, i.e. $\mathsf{E}\left[\hat{x}\varepsilon^T\right] = 0$
  - the output error $y - C\hat{x}$ is white and zero mean, its covariance is $\mathsf{E}\left[(y - C\hat{x})(y - C\hat{x})^T\right] = R_e$ (this can be proved by using Kalman's identity).
- Because $x = \hat{x} + \varepsilon$, we can use the first property and rewrite the LQG cost as

$$V_{\mathsf{LQG}} = \lim_{t\to\infty} \mathsf{E}\left[\hat{x}^T Q\hat{x} + u^T Ru + \varepsilon^T Q\varepsilon\right].$$

The LQG cost can be split into two terms that can be minimized independently:

  - a term that represents the cost of estimation $V_e = \lim_{t\to\infty} \mathsf{E}\left[\varepsilon^T Q\varepsilon\right]$,
  - a term that represents the cost of control $V_c = \lim_{t\to\infty} \mathsf{E}\left[\hat{x}^T Q\hat{x} + u^T Ru\right]$.

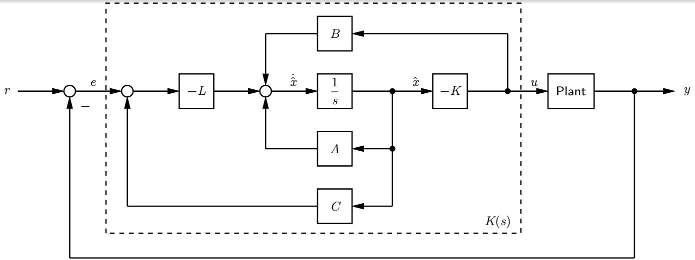
The Separation Principle

- The cost V_e is independent of the state feedback gain K, it depens only on the estimator gain L.
- The estimator gain that minimizes V_e is the solution to the optimal estimation problem $L = P_e C^T R^{-1}$, where P_e is the solution to the FARE.
- The cost V_c is independent of the estimator gain and depends only on the state feedback gain K; here the optimal estimator we know that the state estimate $\hat{x}$ is the state x perturbed by an additive white, zero mean noise process.
- The solution to the problem of minimizing V_c is the same as that to minimizing the deterministic cost function for the linear regulator problem: it is $K = R^{-1} B^T P$, where P is the solution to ARE.
- When a stochastic problem can be solved by replacing the stochastic variables with deterministic ones, we call that the *certainty equivalence principle* holds.

Finally, the controller that minimizes the LQG cost is obtained by solving two Riccati equations: the ARE to find the optimal state feedback gain, and the FARE to find the optimal estimator gain.

LQG Control

Output feedback



The LQG controller is (using `K = lqr(A,B,Q,R)` and `L = lqr(A',C',Qe,Re)`)

$$K(s) = \left[\begin{array}{c|c} A - BK - LC & -L \\ \hline -K & 0 \end{array}\right].$$

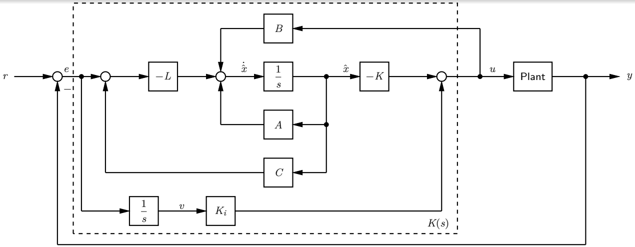
The closed-loop system is

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 0 \\ -L \end{bmatrix} r(t)$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

LQG Control

Output feedback with integral action



The LQG controller is (using `K = lqr(A,B,Q,R)` and `L = lqr(A',C',Qe,Re)`)

$$K(s) = \left[\begin{array}{cc|c} A - BK - LC & 0 & -L \\ 0 & 0 & I \\ \hline -K & K_i & 0 \end{array}\right].$$

The closed-loop system is

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A & -BK & BK_i \\ LC & A - BK - LC & BK_i \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -L \\ I \end{bmatrix} r(t)$$

$$y = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ e \end{bmatrix}$$

LQG Example

Consider a system with

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Do it by youself!

Reference

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