Lecture 5 : Properties of the Optimal Regulator

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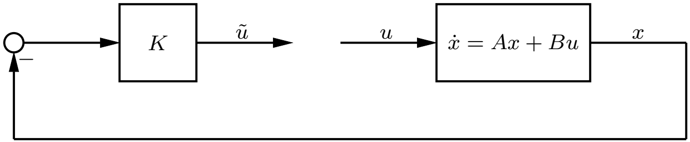
Kalman's Identity



The state-space model from u to $\tilde{u}$ is given by

$$\dot{x} = Ax + Bu, \qquad \tilde{u} = -Kx, \qquad \text{with} \qquad V = \int_0^\infty (x^T Q x + u^T R u)dt$$

which corresponds to the open-loop *negative-feedback* transfer matrix

$$L(s) = K(sI - A)^{-1}B.$$

The open-loop transfer matrix from the control signal u to the controlled output y is

$$T(s) = C(sI - A)^{-1}B + D.$$

Kalman's Identity

These transfer matrices are related by the so-called *Kalman's equality*:

Theorem (Kalman's equality)

For the LQR criterion, we have

$$(I + L^T(-s))R(I + L(s)) = R + D^T D + T(-s)T(s)$$

Should note that
$L^T(-s) = [K(-sI - A)^{-1}B]^T = [-K(sI + A)^{-1}B]^T = -B^T(sI + A^T)^{-1}K^T$.
By setting $s = j\omega$ and using the fact that for real-rational transfer matrices

$$L^T(-j\omega) = L^*(j\omega), \qquad T^T(-j\omega) = T^*(j\omega), \qquad D^T D + T^*(j\omega)T(j\omega) \geq 0.$$

Theorem (Kalman's inequality)

For the LQR criterion, we have

$$(I + L(j\omega)^* R(I + L(j\omega)) \geq R, \qquad \forall \omega \in \mathbb{R}$$

Kalman's Identity

Proof

Let $K = R^{-1}B^TP$ and $P \geq 0$ is a dolution of the algebraic Riccati equation

$$PA + A^TP + Q - PBR^{-1}B^TP = 0.$$

Using $K^TR = PB$ and adding and subtracting sP, we have

$$PA + A^TP + Q - K^TRK + sP - sP = 0$$

or

$$(sI + A^T)P - P(sI - A) + Q - K^TRK = 0.$$

Multiplying $-B^T(sI + A^T)^{-1}$ from the left and $(sI - A)^{-1}B$ from the right yields

$$\begin{aligned} -B^TP(sI - A)^{-1}B + B^T(sI + A^T)^{-1}PB - B^T(sI + A^T)^{-1}Q(sI - A)^{-1}B \\ + B^T(sI + A^T)^{-1}K^TRK(sI - A)^{-1}B = 0 \\ -RK(sI - A)^{-1}B + B^T(sI + A^T)^{-1}K^TR - B^T(sI + A^T)^{-1}Q(sI - A)^{-1}B \\ + B^T(sI + A^T)^{-1}K^TRK(sI - A)^{-1}B = 0 \end{aligned}$$

Kalman's Identity

Proof

$$-RL(s) - L^T(-s)R - B^T(sI + A^T)^{-1}Q(sI - A)^{-1}B - L^T(-s)RL(s) = 0$$

Since $Q = C^TC$, $T(s) = C(sI - A)^{-1}B + D$, and $T^T(-s) = -B^T(sI + A^T)^{-1}C^T + D^T$, we have

$$L^T(-s)RL(s) + RL(s) + L^T(-s)R = D^TD + T^T(-s)T(s)$$

Adding R to both side yields

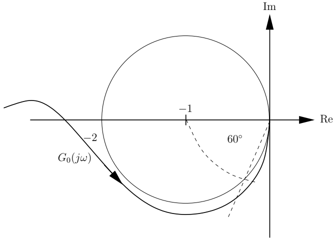
$$\begin{aligned}
R + L^T(-s)RL(s) + RL(s) + L^T(-s)R &= R + D^TD + T^T(-s)T(s) \\
R(I + L(s)) + L^T(-s)RL(s) + L^T(-s)R &= R + D^TD + T^T(-s)T(s) \\
R(I + L(s)) + L^T(-s)R(I + L(s) &= R + D^TD + T^T(-s)T(s) \\
(I + L^T(-s))R(I + L(s)) &= R + D^TD + T^T(-s)T(s).
\end{aligned}$$

Frequency Domain Properties: SISO case

For SISO system the Kalman's inequality simplifies to

$$|1 + L(j\omega)| \geq 1, \qquad \forall \omega \in \mathbb{R},$$

which refer to the fact that *the Nyquist plot* of $L(j\omega)$ does not enter a circle of radius 1 around the point -1 of the complex plane.



- the loop gain never enters a disc of radius 1 around the point -1.

Frequency Domain Properties: SISO case

- the number of encirclements of the critical point cannot change as long as $0.5 < k < \infty$.
- the optimal state feedback guarantees an infinite upper gain margin and a 0.5 lower gain margin.
- the phase margin is the amount of negative phase shift that must be introduced into the loop to make the Nyquist plot touch the critical point.
- From the Figure, it is clear that the smallest phase shift that could make any Nyquist plot touch the critical point is 60°.
- Unfortunately, the phase an gain margins are guaranteed only when the full state vector is available and used for feedback.

Sensitivity and Complementary Sensitivity

The sensitivity and the complementary sensitivity functions are given by

$$S(s) = \frac{1}{1 + L(s)}, \qquad T(s) = 1 - S(s) = \frac{L(s)}{1 + L(s)}.$$

Kalman's inequality guarantees that

$$|S(j\omega)| \leq 1, \quad |T(j\omega) - 1| \leq 1, \quad |T(j\omega)| \leq 2, \quad \Re(T(j\omega)) \geq 0, \quad \forall \omega \in \mathbb{R}.$$

Recall the following facts about the sensitivity function:

- A small sensitivity function is desirable for good disturbance rejection. Generally, this is very important at *low frequencies*.
- A complementary sensitivity function close to 1 is desirable for good reference tracking. Generally, this is very important at *low frequencies*.
- A small complementary sensitivity function is desirable for good noise rejection. Generally, this is especially important at *high frequencies*. It is clear that the noise rejection is rather poor.

Spectral Factorisation

Consider a SISO system

$$\dot{x} = Ax + bu, \qquad y = cx.$$

Here y is a fictitious output, corresponding to the choice $Q = c^T c$. The performance index is

$$V = \int_0^\infty (y^2 + \rho u^2) dt$$

If the transfer function is

$$\frac{b(s)}{a(s)} = c(sI - A)^{-1} b,$$

then the optimal closed-loop eigenvalues are the stable roots of the polynomial

$$p(s) = a(-s)a(s) + \frac{1}{\rho} b(s)b(-s)$$

We use the fact that for two matrices $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{n\times m}$, we have

$$\det(I_m - XY) = \det(I_n - YX)$$

(Proof see page 23.)
For the column vectors x and y this implies

$$\det(I_m - xy^T) = 1 - y^T x$$

Let $a_c(s)$ denotes the closed-loop characteristic polynomial., which can be factored as

$$\begin{aligned} a_c(s) &= \det(sI - A - bk) = \det((sI - A)(I - (sI - A)^{-1}bk)) \\ &= \det(sI - A)\det(I - (sI - A)^{-1}(bk)) \end{aligned}$$

Using the above determinant identity and the notation $\Phi(s) = (sI - A)^{-1}$, the last factor is

$$\det(I - (sI - A)^{-1}(bk)) = \det(I - k\Phi(s)b).$$

Spectral Factorisation

Thus, the closed-loop characteristic polynomial can be written as

$$a_c(s) = a(s)(1 - k\Phi(s)b)$$

and we have

$$1 - k\Phi(s)b = \frac{a_c(s)}{a(s)}.$$

Recall that the kalman's identity can be simplifies to

$$\rho|1 - k\Phi(s)b|^2 = \rho + b^T\Phi^T(-s)Q\Phi(s)b, \qquad \forall\omega.$$

Then

$$\rho(1 - b^T\Phi(-s)k^T)(1 - k\Phi(s)b) = \rho + b^T\Phi(-s)c^Tc\Phi(s)b$$

Spectral Factorisation

yields

$$\rho \frac{a_c(-s)a_c(s)}{a(-s)a(s)} = \rho + \frac{b(-s)b(s)}{a(-s)a(s)}$$

$$\rho a_c(-s)a_c(s) = \rho a(-s)a(s) + b(-s)b(s)$$

$$p(s) = a_c(-s)a_c(s) = a(-s)a(s) + \frac{1}{\rho} b(-s)b(s)$$

This result provides for single-input, single-output systems a simple way of computing the optimal state feedback gain: from the polynomial $p(s)$, find the stable roots, and solve a pole-assignment problem.

For MIMO system, see Ref 4.

In this section, we consider the question what conditions ca we be sure that the polynomial $p(s)$ has no roots on the imaginary axis? (This is same of the Hamiltonian matrix.)

Assume $p(s)$ has a purely imaginary eigenvalues, i.e. $p(j\omega_0) = 0$ for some ω_0. Then

$$p(j\omega_0) = |a(j\omega_0)|^2 + \frac{1}{\rho}|b(j\omega_0)|^2 = 0,$$

which can only be true if

$$a(j\omega_0) = b(j\omega_0) = 0.$$

But then both $a(s)$ and $b(s)$ can be factored as

$$a(s) = \tilde{a(s)}(s - j\omega_0) \quad \text{and} \quad b(s) = \tilde{b(s)}(s - j\omega_0).$$

Spectral Factorisation

Eigenvalues on the Imaginary Axis

Thus

$$\frac{a(s)}{b(s)} = \frac{\tilde{a}(s)(s - j\omega_0)}{\tilde{b}(s)(s - j\omega_0)}$$

- this transfer function must have a pole-zero cancellation.
- The sufficient condition to avoid this is that the state space system (A, b, c) is stabilizable and detectable.
- the stabilizability and detectability guarantee that $p(s)$ has no roots on the imaginary axis.
- It also guarantees that the Hamiltonian matrix has no eigenvalues on the imaginary axis.
- The spectral factorisation result proides a characterisation fo the optimal closed-loop poles in terms of the open-loop poles and zeros.

High Cost of Control

When the cost of control is infinite, i.e. $\rho \to \infty$, we have

$$p(s) = a_c(-s)a_c(s) = a(-s)a(s) + \frac{1}{\rho}b(s)b(-s).$$

In this case, it is clear that

$$a_c(-s)a_c(s) \quad \to \quad a(-s)a(s)$$

Because all roots of $a_c(s)$ must be in the left half plane, this means that

- Stable open-loop roots remain where they are. When the cost control is expensive, one should not spend control effort to move stable poles around.
- Unstable open-loop roots are reflected about the imaginary axis. From the spectral factorisation result, the optimal strategy is to move an unstable pole to its mirror image in the left half plane.

High Cost of Control
Example

Consider the scalar system $\dot{x} = ax + u, \quad x(0) = x_0$, $y = x$ and the performance index is

$$V = \int_0^\infty (y^2 + \rho u^2) dt.$$

Find the state feedback gain k such that $u = -kx$ minimizes V. The closed-loop system is $\dot{x} = (a - k)x$, and the closed-loop state trajectory is therefore

$$x(t) = x_0 e^{(a-k)t}.$$

Substituting this and the control law into the cost function gives

$$V = \int_0^\infty \left(x_0^2 e^{2(a-k)t} + \rho k^2 x_0^2 e^{2(a-k)t} \right) dt = \left(1 + \rho k^2\right) x_0^2 \int_0^\infty e^{2(a-k)t} dt$$

$$= \frac{1 + \rho k^2}{2(a-k)} x_0^2 e^{2(a-k)t} \Big|_0^\infty = \begin{cases} -\dfrac{1 + \rho k^2}{2(a-k)} x_0^2, & a - k < 0 \\ \infty & \text{otherwise} \end{cases}$$

Taking the limit as $\rho \to \infty$:

- this shows that if the open-loop system is stable $(a < 0)$, the choice $k = 0$ leads to a finite cost $V = -1/(2a)$. This is in fact the optimal solution for a stable plant.
- If the plant is unstable $(a > 0)$, we need $k > a$ to stabilize the closed-loop system; this excludes the choice $k = 0$.

To see how far the closed-loop pole must be shifted into the left half plane in order to minimize V, we take the derivative

$$\frac{dV}{dk} = \frac{d}{dk}\left(\frac{1 + \rho k^2}{2(a-k)}\right) = 0$$

This leads to

$$k^2 - 2ak - \frac{1}{\rho} = 0 \quad \Rightarrow \quad k_{1,2} = a \pm \sqrt{a^2 + \frac{1}{\rho}}.$$

When $\rho \to \infty$, we have $k = 0, 2a$. The stabilizing and optimal solution is $k = 2a$. The optimal closed-loop system is therefore $\dot{x} = (a - 2a)x = -ax$, and the optimal pole location is the mirror image of the unstable pole.

Low Cost of Control

Consider the optimal solution in the limiting case $\rho \to 0$. When the control is very cheap, it is intuitively clear that the closed-loop poles should be moved far into the left half plane in order to bring the states quickly to zero. In this case we have

$$a_c(-s)a_c(s) \quad \to \quad \frac{1}{\rho}b(-s)b(s)$$

Let $n = \deg a(s)$ and $m = \deg b(s)$, then the condition for a system to be physically realizable is $n > m$.

- Among the $2n$ roots of $a_c(-s)a_c(s)$ there must be the $2m$ roots of $b(-s)b(s)$. These are *finite roots*, i.e. roots at finite values of s and the $2(n-m)$ roots at infinity.
- From $p(s) = a(-s)a(s) + \frac{1}{\rho}b(s)b(-s)$ at large values of s, the polynomials are dominated by the highest power of s and we can ignore the lower order terms.
- With $a(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$ and $b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0$, we can thus use the approximation

$$a(-s)a(s) = (-1)^n s^{2n}, \qquad b(-s)b(s) = (-1)^m b_m^2 s^{2m}.$$

Low Cost of Control

For large values of s, we can write the equation as

$$(-1)^n s^{2n} + \frac{1}{\rho}(-1)^m b_m^2 s^{2m} = 0$$

or

$$s^{2(n-m)} = -(-1)^{m-n}\frac{1}{\rho}b_m^2 = (-1)^{m-1-n}\frac{1}{\rho}b_m^2.$$

The magnitude of the solution is

$$|s| = \left(\frac{b_m^2}{\rho}\right)^{\frac{1}{2(n-m)}},$$

where the right hand side is the radius of a circle on which the roots are located. This pole configuration is known in network theory and filter design as *Butterworth configuration*.

Root locus

Consider a plant described by the following state equation:

$$\dot{x} = \begin{bmatrix} 1 & -1 \\ 3 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

The transfer function of this plant with the output Cx is

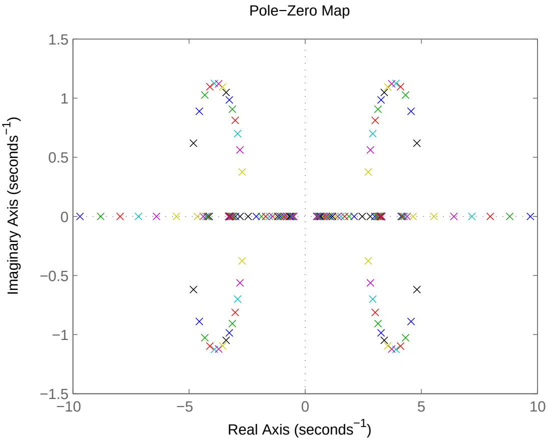
$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 1 \\ -3 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{s+4}{(s+3.3)(s-0.3)}.$$

The optimal control input is desired to minimize the cost function:

$$V = \int_0^\infty x^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \rho u^2 dt,$$

where ρ is a variable.

Root locus



- the poles approach the open-loop pole location at -3.3 and the negative of the open-loop pole location at 0.3 as the control weighting becomes large.

Reference

1. Herbert Werner "*Lecture Notes on Control Systems Theory and Design*", 2011
2. João P. Hespanha "*Linear Systems Theory*", Princeton University Press, 2009
3. Jeffrey B. Burl "*Linear Optimal Control: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Methods*", 1999

Proof of $\det(I_m - XY) = \det(I_n - YX)$

By using the fact that

$$\det\left(\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}\right) = \det(AD).$$

Then, we can show that $\det(I_m - XY) = \det(I_n - YX)$, where $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{n\times m}$, as follow:

$$\begin{aligned}
\det(I_n - YX) &= \det\left(\begin{bmatrix} I_m & 0 \\ Y & I_n - YX \end{bmatrix}\right) = \det\left(\begin{bmatrix} I_m & X \\ Y & I_n \end{bmatrix}\begin{bmatrix} I_m & -X \\ 0 & I_n \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} I_m & X \\ Y & I_n \end{bmatrix}\right)\det\left(\begin{bmatrix} I_m & -X \\ 0 & I_n \end{bmatrix}\right) = \det\left(\begin{bmatrix} I_m & X \\ Y & I_n \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} I_m & X \\ Y & I_n \end{bmatrix}\begin{bmatrix} I_m & 0 \\ -Y & I_n \end{bmatrix}\right) = \det\left(\begin{bmatrix} I_m - XY & X \\ 0 & I_n \end{bmatrix}\right) \\
&= \det(I_m - XY).
\end{aligned}$$

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