Lecture 4 : Linear Quadratic Optimal Control II

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Outline

- Regulator Problem
- Solution of the Time-Varying Riccati Equation
- The Infinite-Time Regulator Problem

Solution of the Finite-time Regulator

Regulator problem

Consider the system

$$\dot{x} = A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0$$

The performance index is

$$V(x(t_0), u(\cdot), t_0) = \int_{t_0}^{T} (u^T Ru + x^T Qx)dt + x^T(T)Mx(T),$$

where $A(t), B(t)$ assumed continuous, the $Q(t) \geq 0$ and $R(t)$ have continuous entries, be symmetric, and be positive definite. The optimum performance index is $V^*(x(t_0), t_0)$.

To solve the problem can be done in two steps:

- show that if $V^*(x(t), t)$ exists, it must be of the form $x^T(t)P(t)x(t)$, where $P(t) \geq 0$.
- show that if $V^*(x(t), t)$ exists, it satisfies the so called Riccati Differential equations.

Solution of the Finite-time Regulator

Proof

Show that V^* must be of the quadratic form $x^T P x$.

- the function $V^*(x)$ is a quadratic form if and only if it is continuous in x and

$$V^*(\lambda x) = \lambda^2 V^*(x) \qquad \text{for all real } \alpha$$

$$V^*(x_1) + V^*(x_2) = \frac{1}{2}(V^*(x_1 + x_2) + V^*(x_1 - x_2)).$$

- Let u_x^* denotes the optimal control when the initial state is x. Since the plant model is linear and the performance index quadratic in x, we have

$$V(\lambda x, \lambda u_x^*, t) = \lambda^2 V^*(x, t) \qquad \text{and} \qquad V^*(\lambda x, t) = \lambda^2 V(x, \frac{1}{\lambda} u_{\lambda x}^*, t)$$

- Because the optimum is minimal, we also have

$$V^*(\lambda x, t) \leq V(\lambda x, \lambda u_x^*, t) \qquad \text{and} \qquad \lambda^2 V^*(x, t) \leq \lambda^2 V(x, \frac{1}{\lambda} u_{\lambda x}^*, t).$$

- Combining all together (to show both are equal.)

$$V^*(\lambda x, t) \leq \lambda^2 V^*(x, t) \qquad \text{and} \qquad \lambda^2 V^*(x, t) \leq V^*(\lambda, t).$$

Solution of the Finite-time Regulator

Proof

- The two inequalities imply that $V^*(x,t)$ satisfies the condition one.
- Similar reasoning gives the inequality

$$\begin{aligned} V^*(x_1,t)+V^*(x_2,t) &= \frac{1}{4}\left[V^*(2x_1,t)+V^*(2x_2,t)\right] \\ &\le \frac{1}{4}\left[V(2x_1,u^*_{x_1+x_2}+u^*_{x_1-x_2},t)+V(2x_2,u^*_{x_1+x_2}-u^*_{x_1-x_2},t)\right] \end{aligned}$$

by linearity

$$\begin{aligned} &= \frac{1}{2}\left[V(x_1+x_2,u^*_{x_1+x_2},t)+V(x_1-x_2,u^*_{x_1-x_2},t)\right] \\ &= \frac{1}{2}\left[V^*(x_1+x_2,t)+V^*(x_1-x_2,t)\right] \end{aligned}$$

- By using the controls $u^*_{x_1}$ and $u^*_{x_2}$, we establish the following inequality in a like manner:

$$\frac{1}{2}\left[V^*(x_1+x_2,t)+V^*(x_1-x_2,t)\right] \le V^*(x_1,t)+V^*(x_2,t)$$

Solution of the Finite-time Regulator
Proof

- This is trivial to show that $V^*(x(t), t)$ is continuous in $x(t)$. It is follows that it has the form

$$V^*(x(t), t) = x^T(t)P(t)x(t)$$

  for some matrix $P(t)$.
- The Positive semidefiniteness follows from the fact that $V^*(x(t), t)$ cannot be negative.
- $P(t)$ is symmetric. If it is not, one can replace it by the symmetric matrix $\frac{1}{2}(P + P^T)$ without altering the value of V^*.

Matrix Riccati Equation

The second step we show that the matrix $P(t)$ satisfies a particular matrix differential equation, the Riccati Differential Equation (RDE).

- the first form of the HJB equation

$$\frac{\partial V^*}{\partial t}(x(t), t) = -\min_{u(t)} \left(l(x, u, t) + \left[\frac{\partial V^*}{\partial x} \right]^T f(x, u, t) \right)$$

- for the linear regulator problem, we have

$$l(x, u, t) = x^T Q x + u^T R u \qquad \text{and} \qquad V^* = x^T P x \quad \text{for some } P(t).$$

- Thus, we have $f(x, u, t) = Ax + Bu$
- Since t and $x(t)$ are considered separately, we have

$$\frac{\partial V^*}{\partial x} = 2x^T(t)P(t) \quad \text{and} \quad \frac{\partial V^*}{\partial t} = x^T(t)\dot{P}(t)x(t)$$

Matrix Riccati Equation

- in the special case of the regulator problem

$$x^T \dot{P} x = -\min_u \left[x^T Q x + u^T R u + 2x^T PAx + 2x^T PBu \right]$$

- By completing the square:

$$\begin{aligned} u^T R u + x^T Q x + 2x^T PAx + 2x^T PBu &= (u + R^{-1}B^T Px)^T R(u + R^{-1}BPx) \\ &\quad + x^T(Q - PBR^{-1}B^T P + PA + A^T P)x \end{aligned}$$

- Because the matrix $R(t)$ is positive definite, the equation is minimized by setting

$$\begin{aligned} \bar{u}(t) &= -R^{-1}(t)B^T(t)P(t)x(t) \\ x^T \dot{P} x &= -x^T(PA + A^T P - PBR^{-1}B^T P + Q)x. \end{aligned}$$

- Since the equation holds for all x, we have the celebrated matrix Riccati Equation

$$-\dot{P} = PA + A^T P - PBR^{-1}B^T P + Q$$

Matrix Riccati Equation

A boundary condition for the Riccati Equation follows from the boundary condition for the HJB equation

$$V^*(x(T), T) = x^T(T)P(T)x(T) = x^T(T)Sx(T) \qquad \text{or} \qquad P(T) = S$$

where S is the penalty on the final state vector introduced previously.
In conclusion

- $\bar{u}$ is the optimal control input at time t, thus solving the Riccati equation and substituting the solution $P(t)$ back the optimal controller in the form of linear, time-varying state feedback

$$u^*(t) = F(t)x(t)$$
$$F(t) = -R^{-1}B^T P(t).$$

Solution of the Riccati Equation

Consider the Riccati differential equation

$$-\dot{P}(t) = P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t) + Q,$$

with boundary condition $P(T) = S$. Note that in the last lecture we consider only a constant matrix P .

First, we show that the Riccati equation can be solved by solving the linear system

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}, \qquad \begin{bmatrix} X(T) \\ Y(T) \end{bmatrix} = \begin{bmatrix} I \\ S \end{bmatrix},$$

where $X(t)$ and $Y(t)$ are square matrices of the same size as A. Having solved the above system for X and Y, one can compute the solution of the Riccati equation as

$$P(t) = Y(t)X^{-1}(t).$$

Solution of the Riccati Equation

This can be proved as

$$\frac{dP}{dt} = \frac{dYX^{-1}}{dt} = Y\frac{dX^{-1}}{dt} + \frac{dY}{dt}X^{-1}.$$

Differentiate both sides of $X(t)X^{-1}(t) = I$ we have

$$\frac{dX^{-1}}{dt} = -X^{-1}\frac{dX}{dt}X^{-1}$$

Substituting this result to get

$$\begin{aligned}\frac{dYX^{-1}}{dt} &= -YX^{-1}\frac{dX}{dt}X^{-1} + \frac{dY}{dt}X^{-1} \\ &= -YX^{-1}(AX - BR^{-1}B^TY)X^{-1} + (-QX - A^TY)X^{-1} \\ &= -YX^{-1}AXX^{-1} - A^TYX^{-1} + YX^{-1}BR^{-1}B^TYX^{-1} - QXX^{-1}\end{aligned}$$

Comparing with the Riccati equation, it is clear that $P(t) = Y(t)X^{-1}(t)$.

Solution of the Riccati Equation

Hamiltonian matrix

The matrix

$$H = \begin{bmatrix} A & -BR^{-1}B \\ -Q & -A^T \end{bmatrix}$$

is called the *Hamiltonian matrix* and plays an important role in the linear quadratic optimization. It has a property that if λ is an eigenvalue of H, then so is $-\lambda$. This can be proved as

$$H = -JH^TJ^{-1} \qquad \text{where} \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

If H has no eigenvalue on the imaginary axis there exists a nonsingular transformation U such that

$$U^{-1}HU = \begin{bmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{bmatrix},$$

where Λ_s is a matrix whose eigenvalues are the stable eigenvalues of H, and Λ_u is a matrix with the unstable eigenvalues of H.

Solution of the Riccati Equation

Hamiltonian matrix

Partitioning U as

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix},$$

the the columns of $\begin{bmatrix} U_{11} & U_{21} \end{bmatrix}^T$ span the eigenvalue space corresponding to the stable eigenvalues of H, whereas the columns of $\begin{bmatrix} U_{12} & U_{22} \end{bmatrix}^T$ span the eigenvalue space corresponding to the unstable eigenvalues. Applying the transformation

$$\begin{bmatrix} X \\ Y \end{bmatrix} = U \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix} \qquad \text{yields} \qquad \begin{bmatrix} \dot{\tilde{X}} \\ \dot{\tilde{Y}} \end{bmatrix} = \begin{bmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}$$

The solution at time T can be computed in terms of the solution at time t as

$$\tilde{X}(T) = e^{\Lambda_s (T-t)} \tilde{X}(t)$$
$$\tilde{Y}(T) = e^{\Lambda_u (T-t)} \tilde{Y}(t)$$

Solution of the Riccati Equation

Hamiltonian matrix

From $X(T) = I$ and $Y(T) = S$, we have

$$I = U_{11}\tilde{X}(T) + U_{12}\tilde{Y}(T)$$
$$S = U_{21}\tilde{X}(T) + U_{22}\tilde{Y}(T)$$

Defining $G = -(U_{22} - SU_{12})^{-1}(U_{21} - SU_{11})$ we obtain

$$\tilde{Y}(T) = G\tilde{X}(T).$$

Evaluating $X(t)$ and $Y(t)$, we have

$$X(t) = \left(U_{11} + U_{12}e^{-\Lambda_u(T-t)}Ge^{\Lambda_s(T-t)}\right)e^{-\Lambda_s(T-t)}\tilde{X}(T)$$
$$Y(t) = \left(U_{21} + U_{22}e^{-\Lambda_u(T-t)}Ge^{\Lambda_s(T-t)}\right)e^{-\Lambda_s(T-t)}\tilde{X}(T).$$

The Riccati differential equation is

$$\begin{aligned} P(t) &= Y(t)X^{-1}(t) \\ &= \left(U_{21} + U_{22}e^{-\Lambda_u(T-t)}Ge^{\Lambda_s(T-t)}\right)\left(U_{11} + U_{12}e^{-\Lambda_u(T-t)}Ge^{\Lambda_s(T-t)}\right)^{-1}. \end{aligned}$$

Solution of the Riccati Equation

Example

Consider a system $\dot{x} = u$ with a cost function $V(x(0), u(\cdot), 0) = \int_0^T (x^2 + u^2)dt$. We have the Hamilton matrix

$$H = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues are $\lambda_s = -1$ and $\lambda_u = 1$, and the transformation matrix is

$$U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

With $S = 0$, we have $G = 1$ and

$$P(t) = \frac{1 - e^{-2(T-t)}}{1 + e^{-2(T-t)}}.$$

Note that the time-varying state feedback gain is

$$K = -R^{-1}B^T P(t) \qquad \text{if } N = 0.$$

Solution of the Riccati Equation
Example

Consider the speed control problem, with a plant equation

$$\dot{\omega}(t) = -a\omega(t) + bu(t) \qquad \text{and cost function} \qquad \int_0^T (\omega^2(t) + \rho u^2(t))dt + s\omega^2(T).$$

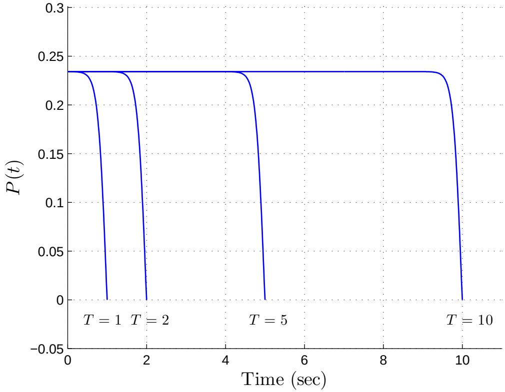
The numerical values are $a = 0.5$ sec^{-1}, $b = 150$ rad/(V sec^2), $\rho = 10^4$ and $s = 0$. Consider solutions of the Riccati equation $P(t)$ with different time horizons T, ranging from $T = 1$ to $T = 10$. The we have the Hamiltonian matrix as

$$H = \begin{bmatrix} 0.5 & -22.5 \\ -1 & -0.5 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 0.9737 & 0.9825 \\ 0.2280 & -0.1864 \end{bmatrix}$$

The solution or the Riccati equation is

$$P(t) = \frac{0.2280 - 0.2280e^{-9.5394(T-t)}}{0.9739 + 1.2017e^{-9.5394(T-t)}}.$$

Solution of the Riccati Equation
Example



It is clear that $P(t)$ is constant during most of the time interval and changes only when approaching the final time. If we select $T = \infty$, it will be no different between $P(t)$ and a constant P.

Existence of the Solution

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \qquad \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the performance index

$$V = \int_0^\infty (x_1^2 + x_2^2 + u^2) dt.$$

- from $\dot{x}_1 = x_1$, we have $x_1 = e^t$, which is independent of the control input. The cost function will contain the term e^{2t} no matter what control is chosen, and the performance index will be infinite.
- Why there exists no solution in this example?
- Firstly the unstable mode x_1 is uncontrollable.
- Secondly the unstable and uncontrollable mode is reflected in the performance index.
- We have two problems that must be addresses: a) does a solution to the problem exist, and b) is the solution stable.

Existence of the Solution

Existence

A solution P is the algebraic Riccati equation for the infinite time regulator problem exists if the system (A, B) is controllable. (It means $P < \infty$.)

- the controllability implies the existence of a state feedback gain F such that $(A - BF)$ is stable. Then the control input $u(t) = -Fx(t)$, which is not necessarily optimal, stabilizes the system, and the resulting performance index $V(x_0, u, t_0)$ will be finite. But it is clear that

$$V^*(x_0) \leq V(x_0, u) < \infty.$$

Since $V^*(x_0) = x_0^T P x_0$ (proof is in the references), we conclude that $x_0^T P x_0 < \infty$; and because this holds for any x_0, it is follows that P is bounded.

Stability

Controllability ensures the existence of bounded solution P to the algebraic Riccati equation and of a state feedback gain that minimizes the performance index. However it does not imply closed-loop stability. Consider a system

$$\dot{x} = x + u$$

and the performance index

$$V = \int_0^{\infty} u^2 dt.$$

The system is controllable, and the optimal control is $u^*(t) \equiv 0$, resulting in $V = 0$ and an unstable closed-loop system.

- The instability is due to the fact that the unstable mode is not reflected in the performance index. A bounded solution P guarantees stability if all modes are reflected in the performance index.

Stability

The stability of the closed-loop system is guaranteed if (A, C) is observable.

Stability

Proof

- The system $\dot{x} = \bar{A}x$ is stable if there exists a Lyapunov function $V = x^T P x$ such that $P > 0$ and $\dot{V} \leq 0$, and where $\dot{V} \equiv 0$ implies $x(t) \equiv 0$.
- Let $\bar{A}$ denote the optimal closed-loop state matrix

$$\bar{A} = A + BF^* = A - BR^{-1}B^T P.$$

- First show that (A, C) observable implies $P > 0$ by showing that $P \geq 0$ leads to a contradiction. Assume $P \geq 0$, then there exists a nonzero initial state $x_0 \neq 0$ such that

$$V = x_0^T P x_0 = \int_0^\infty \left(x^T C^T C x + u^T R u \right) dt = 0.$$

  But this can only be true if $Cx(t) = Ce^{At}x_0 \equiv 0$ for $0 \leq t < \infty$, and (A, C) observable then implies that $x_0 = 0$, which contradicts the assumption $x_0 \neq 0$.
- Next we prove that

$$\dot{V} = \frac{d}{dt}\left(x^T P x \right) \leq 0$$

  and that $\dot{V} \equiv 0$ implies $x(t) \equiv 0$.

Stability
Proof

- Observe that form the algebraic Riccati equation and from the definition of $\bar{A}$ we have

$$\bar{A}^T P + P\bar{A} = -PBR^{-1}B^T P - C^T C.$$

Substituting the right hand side in

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T \left(\bar{A}^T P + P\bar{A} \right) x$$

gives

$$\dot{V} = -x^T PBR^{-1}B^T P x - x^T C^T C x,$$

so clearly $\dot{V} \leq 0$, and $\dot{V} \equiv 0$ can only be true if $Cx(t) \equiv 0$, which by observability implies $x_0 = 0$ and thus $x(t) \equiv 0$. Invoking the Lyaponov stability result quoted above, this proves that observability of (A, C) guarantees stability of the optimal closed-loop system.

Closed-loop Eigenvalues

Using the algebraic Riccati equation, one can show that the optimal closed-loop eigenvalues are the stable eigenvalues of the Hamiltonian matrix: apply the similarity transformation

$$T^{-1}HT = \tilde{H}, \qquad T = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$$

to the Hamiltonian matrix. The result is

$$\tilde{H} = \begin{bmatrix} \bar{A} & -BR^{-1}B^T \\ 0 & -\bar{A}^T \end{bmatrix},$$

so the eigenvalues of H are the eigenvalues of $\bar{A}$ together with those of $-\bar{A}$.

The derivation of G

From $X(T) = I$ and $Y(T) = S$, we have

$$I = U_{11}\tilde{X}(T) + U_{12}\tilde{Y}(T) \tag{1}$$
$$S = U_{21}\tilde{X}(T) + U_{22}\tilde{Y}(T) \tag{2}$$

By multiplying (1) with S from the left we have

$$S = SU_{11}\tilde{X}(T) + SU_{12}\tilde{Y}(T)$$
$$S = U_{21}\tilde{X}(T) + U_{22}\tilde{Y}(T).$$

Then equal both equation we obtain

$$\tilde{Y}(T) = -(U_{22} - SU_{12})^{-1}(U_{21} - SU_{11})\tilde{X}(T) = G\tilde{X}(T),$$

where $G = -(U_{22} - SU_{12})^{-1}(U_{21} - SU_{11})$.

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