Lecture 3 : Linear Quadratic Optimal Control

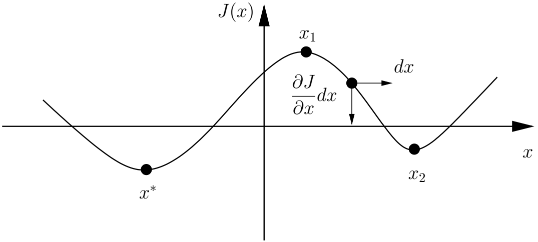
Dr.-Ing. Sudchai Boonto
Assistant Professor

Department of Control System and Instrumentation Engineering
King Mongkuts Unniversity of Technology Thonburi
Thailand

- Optimization is the problem of determining a set of parameters that minimize or minimize a given function.
- Finding the minimum of a smooth function $J : \mathbb{R}^n \to \mathbb{R}$. We wish to find a point $x^* \in \mathbb{R}^n$ such that $J(x^*) \leq J(x)$ for all $x \in \mathbb{R}^n$. A necessary condition for x^* to be a minimum is that the gradient of the function be zero at x^*:

$$\frac{\partial J}{\partial x}(x^*) = 0.$$

- In control system, $J(x)$ is often called a cost function and x^* is the optimal value for x.



- The function $J(x)$ has a local minimum at x^* if and only if

$$J(x^* + \delta x) \geq J(x^*), \qquad \forall \delta x \text{ sufficiently small.}$$

- It is equivalent to

$$\Delta J(x^*, \delta x) = J(x^* + \delta x) - J(x^*) \geq 0, \qquad \forall \delta x \text{ sufficiently small.}$$

- Using a Taylor series arond the point x^*, the optimality condition can be written

$$\begin{aligned}
\Delta J(x^*, \delta x) &= J(x^* + \delta x) - J(x^*) \\
&= J(x^*) + \frac{\partial J}{\partial x}(x^*)\delta x + \frac{\partial^2 J}{\partial x^2}(x^*)\delta x^2 + \text{ H.O.T } - J(x^*) \geq 0 \\
&= \frac{\partial J}{\partial x}(x^*)\delta x + \frac{\partial^2 J}{\partial x^2}(x^*)\delta x^2 + \text{ H.O.T } \geq 0.
\end{aligned}$$

Here δx is called the variation of x, and the term in the increment that is linear in δx is called the variation of J and is denoted $\delta J(x^*, \delta x)$.

- A necessary condition for x^* to be a local minimum is as follows:

$$\frac{\partial J}{\partial x}(x^*) = 0 \qquad \forall \delta x.$$

- Consider $J(x) = x^2 + 6x + 8$, we have

$$\begin{aligned} \Delta J(x, \delta x) &= (x + \delta x)^2 + 6(x + \delta x) + 8 - (x^2 + 6x + 8) \\ &= (2x + 6)\delta x + \delta x^2. \end{aligned}$$

The necessary condition for x to be minimum is that the variation of $J(x)$ equals zero:

$$\begin{aligned} &\frac{\partial J}{\partial x}(x^*) = (2x^* + 6)\delta x = 0 \\ &2x^* + 6 = 0 \qquad \Rightarrow x^* = -3. \end{aligned}$$

- the problem is more complicated if constraints are present.
- Let $G_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, k$ be a set of smooth functions with $G_i(x) = 0$ representing the constrains.

$$\min_{x \in \mathbb{R}^n} \quad J(x)$$
$$\text{S.T.} \quad G_i(x) = 0, \qquad i = 1, \ldots, k$$

- This situation can be visualized as constraining the point to a surface (defined by the constraints) and searching for the minimum of the cost function along this surface.
- The necessary condition for being at a minimum is that there are no directions tangent to the constraints that also decrease the cost.
- The tangent directions to the surface can be computed by considering small perturbations of the constraint that remain on the surface:

$$G_i(x + \delta x) \approx G_i(x) + \frac{\partial G_i}{\partial x}(x)\delta x = 0 \qquad \Rightarrow \qquad \frac{\partial G_i}{\partial x}(x)\delta x = 0,$$

where $\delta x \in \mathbb{R}^n$ is a vanishingly small perturbation.

Optimization

Lagrange Multipliers

- It follows that the normal directions to the surface are spanned by $\partial G_i/\partial x$, since these are precisely the vectors that annihilate an admissible tangent vector δx.
- Using this characterization of the tangent and normal vectors to the constraint, a necessary condition for optimization is that the gradient of J is spanned by vectors that are nomal to the constraints, so that the only directions that increase the cost violate the constraints. We thus require that there exist scalars $\lambda_i, \ i = 1, \ldots, k$ such that

$$\frac{\partial J}{\partial x}(x^*) + \sum_{i=1}^{k} \lambda_i \frac{\partial G_i}{\partial x}(x^*) = 0.$$

or in matrix form

$$\frac{\partial J}{\partial x}(x^*) + \lambda^T \frac{\partial G}{\partial x}(x^*) = 0$$

- Defining $\tilde{J}(x, \lambda) = J(x) + \lambda^T G(x)$, the necessary condition becomes

$$\frac{\partial \tilde{J}}{\partial x}(x^*) = 0$$

The cost function and the constraint are

$$J(x,y) = x^2 + y^2; \qquad G(x,y) = 2x + y + 4 = 0.$$

The augmented cost function is

$$\begin{aligned}
\tilde{J}(x,y,\lambda) &= x^2 + y^2 + \lambda(2x + y + 4) \\
\frac{\partial \tilde{J}}{\partial x} &= 2x^* + 2\lambda^* = 0 \\
\frac{\partial \tilde{J}}{\partial y} &= 2y^* + \lambda^* = 0 \\
\frac{\partial \tilde{J}}{\partial \lambda} &= 2x^* + y^* + 4 = 0
\end{aligned}$$

Solving for the values of x^*, y^* and λ, yields

$$\begin{bmatrix} x^* \\ y^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -1.6 \\ -0.8 \\ 1.6 \end{bmatrix}.$$

Suppose the plant is described by the nonlinear time-varying dynamical equation

$$\dot{x}(t) = f(x, u, t),$$

which state $x(t) \in \mathbb{R}^n$ and control input $u(t) \in \mathbb{R}^m$. The performance index is

$$J(x, u, t) = \int_{t_0}^{T} L(x(t), u(t), t)dt + V(x(T), T),$$

where $[t_0, T]$ is the time interval of interest.

optimal control problem

The optimal control problem is to find the input $u^*(t)$ on the time interval $[t_0, T]$ that drives the plant along with the trajectory $x^*(t)$ such that the cost function is minimized, and such that

$$\psi(x(T), T) = 0$$

for a given function $\psi \in \mathbb{R}^p$.

Using Lagrange multipliers to adjoint the constraints to performance the cost function.

- $\lambda(t) \in \mathbb{R}^n$ for time function constraint.
- $\nu \in \mathbb{R}^p$ for the final value constraint.

The augmented performance index is thus

$$\begin{aligned}\tilde{J}(x(\cdot), u(\cdot), \lambda(\cdot), \nu(\cdot)) = V(x(T), T) + \nu^T \psi(x(T), T) \\ + \int_0^T \left[L(x, u, t) + \lambda^T (f(x, u, t) - \dot{x}) \right] dt\end{aligned}$$

Define the *Hamiltonian function* as

$$H(x, u, t) = L(x, u, t) + \lambda^T f(x, u, t),$$

Then

$$\tilde{J} = V(x(T), T) + \nu^T \psi(x(T), T) + \int_0^T \left[H(x, u, t) - \lambda^T \dot{x} \right] dt.$$

Linearize the cost function around the optimal solution $x(t) = x^*(t) + \delta x(t)$, $u(t) = u^*(t) + \delta u(t)$, $\lambda(t) = \lambda^*(t) + \delta\lambda(t)$ and $\nu = \nu^* + \delta\nu$, then the incremental cost can be written as

$$\begin{aligned}\delta\tilde{J} &= \tilde{J}(x^* + \delta x, u^* + \delta u, \lambda^* + \delta\lambda, \nu^* + \delta\nu) - \tilde{J}(x^*, u^*, \lambda^*, \nu^*) \\ &\approx \int_0^T \left(\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u - \lambda^T \delta\dot{x} + \left(\frac{\partial H}{\partial \lambda} - \dot{x}^T\right)\delta\lambda \right) dt \\ &\quad + \frac{\partial V}{\partial x}\delta x(T) + \nu^T \frac{\partial \psi}{\partial x}\delta x(T) + \delta\nu^T \psi(x(T), T) + H.O.T.\end{aligned}$$

where (t) is omitted and all derivatives are evaluated along the optimal solution. We can eliminate the dependence on $\delta\dot{x}$ using integration by parts:

$$-\int_0^T \lambda^T \delta\dot{x} dt = -\lambda^T(T)\delta x(T) + \lambda^T(0)\delta x(0) + \int_0^T \dot{\lambda}^T \delta x dt.$$

Since $x(0) = x_0$, the second term vanishes and substituting this into $\delta\tilde{J}$ yields

$$\begin{aligned}\Delta\tilde{J} \approx \int_0^T \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u + \left(\frac{\partial H}{\partial \lambda} - \dot{x}^T\right)\delta\lambda\right] dt \\ + \left(\frac{\partial V}{\partial x} + \nu^T \frac{\partial \psi}{\partial x} - \lambda^T(T)\right)\delta x(T) + \delta\nu^T \psi(x(T), T).\end{aligned}$$

To be optimal, we need $\Delta\tilde{J} = 0$ for all δx, δu, $\delta\lambda$ and $\delta\nu$, and we obtain the local conditions in the next theorem.
For the general case, consider a nonlinear system

$$\dot{x} = f(x, u, t), \qquad x(0) = x_0$$

We wish to minimize a cost function J with terminal constraints:

$$J = \int_0^T L(x, u, t)dt + V(x(T)), \qquad \psi(x(T)) = 0.$$

The *Hamiltonian* is then

$$H(x,u,t) = L(x,u,t) + \lambda^T f(x,u,t)$$

A set of necessary conditions for a solution to be optimal was derived by Pontryagin.

Theorem (Maximum Principle)

If (x^*, u^*) *is optimal, then there exists* $\lambda^*(t) \in \mathbb{R}^n$ *and* $\nu^* \in \mathbb{R}^q$ *such that*

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial \lambda_i}, \qquad -\dot{\lambda}_i = \frac{\partial H}{\partial x_i}, \qquad x(0) = x_0, \qquad \psi(x(T)) = 0 \\ \lambda(T) &= \frac{\partial V}{\partial x}(x(T)) + \nu^T \frac{\partial \psi}{\partial x} \\ H(x^*(t), u^*(t), \lambda^*(t)) &\le H(x^*(t), u(t), \lambda^*(t)) \quad \forall \quad u \in \Omega \end{aligned}$$

The necessary condition for the optimal (control) input is

$$\frac{\partial H}{\partial u} = 0.$$

A system $\dot{x} = ax + bu$, where $x \in \mathbb{R}$ is a state , $u \in \mathbb{R}$ is an input, the initial state $x(t_0) = x_0$ and $a, b \in \mathbb{R}$ are positive constants. We wish to find a trajectory $(x(t), u(t))$ that minimizes the cost function

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)dt + \frac{1}{2}cx^2(t_f),$$

where the terminal time t_f is given and $c > 0$ is a constant. Here

$$L = \frac{1}{2}u^2(t) \qquad V = \frac{1}{2}cx^2(t_f)$$

The Hamiltonian is

$$H = L + \lambda f = \frac{1}{2}u^2 + \lambda(ax + bu),$$
$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -a\lambda, \qquad \lambda(t_f) = \frac{\partial V}{\partial x} = cx(t_f).$$

This is a final value problem for a linear differential equation in λ and the solution is

$$\lambda(t) = cx(t_f)e^{a(t_f - t)}.$$

The optimal control is given by

$$\frac{\partial H}{\partial u} = u + b\lambda = 0 \qquad \Rightarrow \qquad u^*(t) = -b\lambda(t) = -bcx(t_f)e^{a(t_f - t)}.$$

Substituting this control input into the dynamics system yields a first-order ODE in x:

$$\dot{x} = ax - b^2 cx(t_f)e^{a(t_f - t)}.$$

$$x^*(t) = x(t_0)e^{a(t - t_0)} + \frac{b^2 c}{2a} x^*(t_f)\left[e^{a(t_f - t)} - e^{a(t + t_f - 2t_0)}\right].$$

Setting $t = t_f$ and solving for $x(t_f)$ gives

$$x^*(t_f) = \frac{2ae^{a(t_f - t_0)}x(t_0)}{2a - b^2 c\left(1 - e^{2a(t_f - t_0)}\right)}$$

and finally we can write

$$u^*(t) = -\frac{2abce^{a(2t_f - t_0 - t)}x(t_0)}{2a - b^2c\left(1 - e^{2a(t_f - t_0)}\right)}$$

$$x^*(t) = x(t_0)e^{a(t-t_0)} + \frac{b^2ce^{a(t_f - t_0)}x(t_0)}{2a - b^2c\left(1 - e^{2a(t_f - t_0)}\right)}\left[e^{a(t_f - t)} - e^{a(t + t_f - 2t_0)}\right].$$

Setting $t = t_f$ and taking the limit we find that

$$\lim_{c \to \infty} x^*(t_f) = 0.$$

It is desired to heat a room using the least possible energy. If $\theta(t)$ is the temperature in the room, $\theta_a = 60^\circ$ is the ambient air temperature, and $u(t)$ is the rate of heat supply to the room, then the dynamics equation is

$$\dot{\theta} = -a(\theta - \theta_a) + bu.$$

By defining the state as $x(t) \triangleq \theta(t) - \theta_a$ and the state equation is

$$\dot{x} = -ax + bu.$$

To control the temperature on the fixed time interval $[0, T]$ with the least possible supplied energy, define the performance index as

$$J = \frac{1}{2} \int_0^T u^2(t)dt.$$

The Hamiltonian is

$$H = \frac{u^2}{2} + \lambda(-ax + bu).$$

We have

$$\dot{x} = \frac{\partial H}{\partial \lambda} = -ax + bu,$$
$$\dot{\lambda} = -\frac{\partial H}{\partial x} = a\lambda,$$
$$0 = \frac{\partial H}{\partial u} = u + b\lambda.$$

Then the optimal control is given by $u(t) = -b\lambda(t)$. To determine $u^*(t)$ we need only find the optimal costate $\lambda^*(t)$. Substituting the control input back to the state-costate equations

$$\dot{x} = -ax - b^2\lambda, \qquad \dot{\lambda} = a\lambda(t)$$

The solution of $\lambda(t)$ is $\lambda(t) = e^{-a(T-t)}\lambda(T)$. Using the solution yields

$$\dot{x} = -ax - b^2\lambda(T)e^{-a(T-t)}.$$

Using Laplace transform to solve this gives

$$\begin{aligned} sX(s) - x(0) &= -aX(s) - \frac{b^2\lambda(T)e^{-aT}}{(s-a)} \\ X(s) &= \frac{x(0)}{s+a} - \frac{b^2\lambda(T)e^{-aT}}{(s+a)(s-a)} \\ &= \frac{x(0)}{s+a} - \frac{b^2}{a}\lambda(T)e^{-aT}\left(\frac{-1/2}{s+a} + \frac{1/2}{s-a}\right) \end{aligned}$$

so that

$$x(t) = x(0)e^{-at} - \frac{b^2}{a}\lambda(T)e^{-aT}\sinh(at).$$

Both $x(t)$ and $\lambda(t)$ give the optimal costate $\lambda^*(t)$ and state $x^*(t)$ in terms of the unknown final costate $\lambda(T)$. Suppose $x(0) = 0^\circ$ and the control objective is to drive the final temperature $\theta(T)$ exactly to 70° at the final time of T seconds. The final state is required to take on the fixed value of $x(T) = 10^\circ$.

To find $\lambda(T)$, we have

$$x(T) = x(0)e^{-aT} - \frac{b^2}{2a}\lambda(T)(1 - e^{-2aT}).$$

Since $x(0) = 0^\circ$ and $x(T) = 10^\circ$, the final costate is

$$\lambda(T) = \frac{-20a}{b^2(1 - e^{-2aT})}, \qquad \text{and} \qquad \lambda^*(t) = -\frac{10ae^{at}}{b^2 \sinh(aT)}.$$

Finally, the optimal rate of heat supply to the room is given by

$$u^*(t) = -b\lambda^*(t) = \frac{10ae^{at}}{b \sinh(aT)}, \qquad 0 \le t \le T.$$

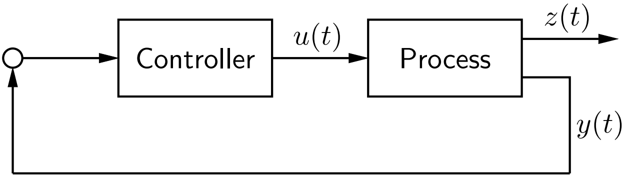
Solving for the state trajectory yields

$$x^*(t) = 10\frac{\sinh(at)}{\sinh(aT)} \quad \Rightarrow \quad x^*(T) = 10^\circ$$

Deterministic Linear Quadratic Regulation (LQR)

The plant to be controlled in state space model is

$$\dot{x} = Ax + Bu$$
$$y = Cx(t), \qquad z = Gx + Hu$$



- $y(t)$ is the measured output
- $z(t)$ is the controlled output. Sometimes $z(t) = y(t)$ but mostly not, e.g. $z(t) = \begin{bmatrix} y(t) & \dot{y}(t) \end{bmatrix}^T$.

Optimal Regulation

- The LQR problem is to find the control input $u(t), t \in [0, \infty)$ that makes the following criterion as small as possible:

$$J_{\mathsf{LQR}_1} := \int_0^\infty \|z(t)\|^2 + \rho \|u(t)\|^2 dt,$$

  where ρ is a positive constant.

- The term

$$\int_0^\infty \|z(t)\|^2 dt$$

  corresponds to the *energy of the controlled output*, and the term

$$\int_0^\infty \|u(t)\|^2 dt$$

  corresponds to the *energy of the control signal*.

- Here ρ is a penalty factor to the control input $u(t)$.

Optimal Regulation

- the more general cost function is

$$J_{\mathsf{LQR}_2} := \int_0^\infty z^T(t)\bar{Q}z(t) + \rho u^T(t)\bar{R}u(t)dt,$$

  where $\bar{Q}$ and $\bar{R}$ are symmetric positive-definite matrices. and ρ is a positive constant.
- the most general form for a quadratic cost function is

$$J_{\mathsf{LQR}} := \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t) + 2x^T(t)Nu(t)dt$$

- Since $z = Gx + Hu$, it is not hard to see that for J_{LQR_1}

$$Q = G^TG, \qquad R = H^TH + \rho I, \qquad N = G^TH$$

  and for J_{LQR_2}

$$Q = G^T\bar{Q}G, \qquad R = H^T\bar{Q}H + \rho\bar{R}, \qquad N = G^T\bar{Q}H.$$

Feedback Invariants

Given a continuous-time LTI system (AB-CLTI)

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k,$$

we say that a function

$$H(x(\cdot), u(\cdot))$$

is *feedback invariant* for the system if its value depends only on the initial condition $x(0)$ and not the specific input signal $u(\cdot)$.

Feedback invariant

For every symmetric matrix P, the functional

$$H(x(\cdot), u(\cdot)) := -\int_0^\infty (Ax(t) + Bu(t))^T Px(t) + x^T(t)P(Ax(t) + Bu(t))\, dt$$

is a feedback invariant for the system (AB-CLTI), as long as $\lim_{t\to\infty} x(t) = 0$.

Feedback Invariants

- **Proof:** We can rewrite H as

$$\begin{aligned} H(x(\cdot), u(\cdot)) &= -\int_0^\infty \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t)dt \\ &= -\int_0^\infty \frac{d(x^T(t)Px(t))}{dt}dt = x^T(0)Px(0) - \lim_{t\to\infty} x^T(t)Px(t) \\ &= x^T(0)Px(0), \end{aligned}$$

  as long as $\lim_{t\to\infty} x(t) = 0$.
- $\lim_{t\to\infty} x(t) = 0$ simply mean the system is stable.

Feedback Invariants in Optimal Control

- a cost function J is minimized by an appropriate choice of the input $u(\cdot)$ in the form

$$J = H(x(\cdot), u(\cdot)) + \int_0^\infty L(x(t), u(t))dt,$$

where H is a feedback invariant and the function $L(x, u)$ has the property that for every $x \in \mathbb{R}^n$

$$\min_{u \in \mathbb{R}^k} L(x, u) = 0$$

- In control system, the optimal control input

$$u^*(t) = \arg \min_{u \in \mathbb{R}^k} L(x, u),$$

minimizes the criterion J, and the optimal value of J is equal to the feedback invariant

$$J = H(x(\cdot), u^*(\cdot)).$$

Optimal State Feedback

The LQR criterion

$$J_{\mathsf{LQR}} := \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t) + 2x^T(t)Nu(t)dt$$

By adding and subtracting the feedback invariant to the LQR criterion, we have

$$\begin{aligned}
J_{\mathsf{LQR}} &:= \int_0^\infty x^TQx + u^TRu + 2x^TNudt \\
&= H(x(\cdot),u(\cdot)) + \int_0^\infty x^TQx + u^TRu + 2x^TNu + (Ax+Bu)^TPx + x^TP(Ax+Bu)dt \\
&= H(x(\cdot),u(\cdot)) + \int_0^\infty x^T(A^TP + PA + Q)x + u^TRu + 2u^T(B^TP + N^T)xdt
\end{aligned}$$

By completing the square, we have

$$(u+Kx)^TR(u+Kx) = u^TRu + x^T(PB+N)R^{-1}(PB+N)^Tx + 2u^T(PB+N)^Tx,$$

where $K := R^{-1}(PB+N)^T$ and P is a symmetric matrix.

Optimal State Feedback

We conclude that

$$J_{\mathsf{LQR}} = H(x(\cdot), u(\cdot)) + \int_0^\infty x^T(A^TP + PA + Q - (PB+N)R^{-1}(PB+N)^T)x \\ + (u+Kx)^T R(u+Kx)dt.$$

- If we are able to select the matrix P so that, the Riccati equation,

  $$A^TP + PA + Q - (PB+N)R^{-1}(PB+N)^T = 0,$$

  we have

  $$L(x,u) := (u+Kx)^T R(u+Kx),$$

  which has a minimum equal to zero for

  $$u = -Kx, \qquad K := R^{-1}(PB+N)^T,$$

Optimal State Feedback

- leading to the closed-loop system

$$\dot{x} = \left(A - BR^{-1}(PB+N)^T\right)x.$$

Theorem

Assume that there exists a symmetric solution P to the algebraic Riccati equation for which $A - BR^{-1}(PB+N)^T$ is a stability matrix. Then the feedback law

$$u(t) := -Kx(t), \qquad \forall t \geq 0, \qquad K := R^{-1}(PB+N)^T$$

minimizes the LQR criterion and leads to

$$J_{LQR} := \int_0^\infty x^T Q x + u^T R u + 2x^T N u \; dt = x^T(0)Px(0).$$

Optimal State Feedback

Example: a double integrator

Consider a double integrator system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

with quadratic cost given by

$$Q = \begin{bmatrix} \frac{1}{q^2} & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = 1.$$

Let P be a symmetric positive definite matrix of the form

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then the Riccati equation becomes

$$\begin{bmatrix} -b^2 + q^2 & a - bd \\ a - bd & 2b - d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow P = \begin{bmatrix} \sqrt{\frac{2}{q^3}} & \frac{1}{q} \\ \frac{1}{q} & \sqrt{\frac{2}{q}} \end{bmatrix}$$

The controller is given by

$$K = R^{-1}B^TP = \begin{bmatrix} \frac{1}{q} & \sqrt{\frac{2}{q}} \end{bmatrix}.$$

The feedback law minimizing the given cost function is then $u = -Kx$.
The closed-loop matrix is then

$$A_{cl} = A - BK = \begin{bmatrix} 0 & 1 \\ -1/q & -\sqrt{2/q} \end{bmatrix}.$$

The characteristic polynomial of the matrix is

$$\lambda^2 + \sqrt{\frac{2}{q}}\lambda + \frac{1}{q} = 0.$$

Comparing this to $\lambda^2 + 2\zeta\omega_0\lambda + \omega_0^2$, we have a good tradeoff between rise time and overshoot:

$$\omega_0 = \sqrt{1/q}, \qquad \zeta = \frac{1}{\sqrt{2}} = 0.707.$$

To select the cost function weights Q and R, we must use the knowledge of the system we are controlling. The simplest method is to use diagonal weights

$$Q = \begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{bmatrix}, \qquad R = \begin{bmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_n \end{bmatrix}.$$

The individual weights for the (diagonal) elements of the Q and R matrix can be selected as follow:

1. choose q_i and ρ_j as the inverse of the square of the maximum value for the corresponding x_i or u_j.
2. modify the elements to obtain a compromise among response time, damping and control effort.

The command `[K,P,E] = lqr(A,B,Q,R,N)` solves the algebraic Riccati equation

$$A^T P + PA + Q - (PB + N)R^{-1}(B^T P + N^T) = 0$$

and computes the optimal state feedback matrix gain

$$K = R^{-1}(B^T P + N^T)$$

that minimizes the LQR criteria

$$J = \int_0^\infty x^T Qx + u^T Ru + 2x^T Nudt$$

for the continuous-time process

$$\dot{x} = Ax + Bu.$$

This command also returns the poses `E` of the closed-loop system

$$\dot{x} = (A - BK)x.$$

Optimal State Feedback with Reference inputs

Simple method

Suppose we have an equilibrium point (x_d, u_d) . The normal state spae control law to include the nominal input:

$$u = u_d - K(x - x_d).$$

This allows the required input $u = u_d$ to be applied when $x = x_d$ to achieved the equilibrium point.

- to adjust the equilibrium point based on the reference r. We need to find x_d and u_d as a function of r as

$$\begin{aligned} 0 &= Ax_d + Bu_d \\ r &= Cx_d + Du_d \end{aligned}$$

Since it is a linear equation, then we have

$$\begin{bmatrix} x_d \\ u_d \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} r = \begin{bmatrix} N_x \\ N_u \end{bmatrix} r$$

- Then $u = -K(x - N_x r) + N_u r = -Kx + Nr$, where $N = N_u + KN_x$.

Optimal State Feedback with Reference inputs

Simple method

- The scalar N represents a *feedforward* gain. The closed loop system is

$$\dot{x} = (A - BK)x + BNr$$
$$y = (C - DK)x + DNr$$

  where r represents the reference input and y is the output for the closed loop plant.
- This approach is not very robust and if the system model is not corrent then this approach will generate steady state errors.
- One way to see this is that the resulting control law does not compare the actual output to the reference output, and hence it cannot correct for errors in the output.

An alternative way to achieving zero steady state error is to add integral action to a state-space feedback controller. By augmenting the plant:

$$\dot{x} = Ax + Bu$$
$$\dot{x}_i = e = r - Cx$$
$$y = Cx.$$

This can be written in terms of the augmented state $\begin{bmatrix} x & x_i \end{bmatrix}^T$:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} r.$$

We design a control law for the case where $r = 0$ to get state feedback and

$$u = -\begin{bmatrix} K & K_i \end{bmatrix} \begin{bmatrix} x - x_d \\ x_i \end{bmatrix} + u_d$$

This method is more robust than the previous method. In fact, the controller will automatically adjust the integrator state to provide sufficient input to hold the output at the reference value.

LQR with Matlab

Matlab command to solve the algebraic Riccati equation

[K,P,E] = lqr(A, B, Q, N)

The optimal state feedback matrix gain is

$$K = R^{-1}(PB + N)^T$$

that minimizes the LQR criteria

$$J := \int_0^\infty x^T Q x + u^T R u + 2x^T N u \; dt$$

for the continuous-time process

$$\dot{x} = Ax + Bu.$$

This command also returns the poles E of the closed-loop system

$$\dot{x} = (A - BK)x.$$

LQR Design Example

An Aricraft roll dynamics equation is

$$\dot{\theta} = \omega, \qquad \dot{\omega} = -0.875\omega - 20\tau, \qquad \dot{\tau} = -50\tau + 50u$$

Defining $x := \begin{bmatrix} \theta & \omega & \tau \end{bmatrix}^T$, we write the state-space model as

$$\dot{x} = Ax + Bu, \text{ where}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.875 & -20 \\ 0 & 0 & -50 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}.$$

The controlled output was chosen to be $z = \begin{bmatrix} \theta & \gamma\dot{\theta} \end{bmatrix}^T$, which corresponds to

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Find a state-feedback gain K that the closed-loop system can be tracked a step input.

Reference

1. Herbert Werner "*Lecture Notes on Control Systems Theory and Design*", 2011
2. Jeffrey B. Burl "*Linear Optimal Control: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Methods*", 1999
3. Brian D. O. Anderson and John B. Moore "*Linear Optimal Control*", Prentice-Hall, Inc., 1989
4. João P. Hespanha "*Linear Systems Theory*", Princeton University Press, 2009
5. Mathwork "*Control System Toolbox: User's Guide*", 2014
6. Richard M. Murray "Optimization-Based Control", CDS, 2010