Lecture 2 : Linear Systems Reviews

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Descriptions of Linear Dynamical Systems

Let a finite dimensional linear time invariant (FDLTI) dynamical system be described as follow:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0$$
$$y(t) = Cx(t) + Du(t),$$

where

- $u(t) \in \mathbb{R}^n$ is called the system state
- $x(t_0)$ is called the *initial condition* of the system
- $y(t) \in \mathbb{R}^p$ is the system *output*
- The A, B, C, and D are appropriately dimensioned real constant matrices

Descriptions of Linear Dynamical Systems

- A dynamical system with single-input $(m = 1)$ and single-output $(p = 1)$ is called a SISO (single-input and single-output) system;
- otherwise it is called a MIMO (multiple-input and multiple-output) system.
- The corresponding transfer matrix from u to y is defined as

$$Y(s) = G(s)U(s),$$

  where $U(s)$ and $Y(s)$ are the Laplace transforms of $u(t)$ and $y(t)$ with zero initial condition $(x(0) = 0)$.
- We have $G(s) = G(sI - A)^{-1}B + D$.

Descriptions of Linear Dynamical Systems

The system can be written in a more compact matrix form:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

to expedite calculations involving transfer matrices, we shall use the following notation:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI - A)^{-1}B + D$$

MATLAB command:

```
G = ss(A,B,C,D)              % Construct state-space model
[A,B,C,D]  = ssdata(G)       % access to state-space data
```

Descriptions of Linear Dynamical Systems

- Now given the initial condition $x(t_0) = x_0$ and the input $u(t)$:

$$x(t) = e^{A(t-t_0)}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$
$$y(t) = Cx(t) + Du(t)$$

- If $u(t) = 0, \forall t \geq 0$,

$$x(t) = e^{A(t-t_1)}x(t_1)$$

- The matrix $e^{A(t-t_1)}$ acts as a transformation from one state to another, and thus $e^{A(t-t_1)} = \Phi(t, t_1)$ is usually called the *state transition matrix*.
- Proof see page 64.

- For LTI system, the state-transition matrix and the output are

$$\Phi(t) = e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots$$
$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- For MIMO system, if the inputs are impulses then, the outputs are impulse response $g_{ij}(t)$:

$$g_{ij}(t) = y_i(t), \text{ where } u_k(t) = \begin{cases} \delta(t) & k = j \\ 0 & k \neq j \end{cases}, \; x(0) = 0$$

$$G(t) = \begin{bmatrix} g_{11}(t) & \cdots & g_{1n_u}(t) \\ \vdots & \ddots & \vdots \\ g_{n_y1}(t) & \cdots & g_{n_yn_u}(t) \end{bmatrix} = \begin{cases} Ce^{At}B + D\delta(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

```
G = ss(A,B,C,D)      % create a constant system matrix
[y,x,t] = step(G(Yu,Iu))  % step response of input
                          % Iu to output Yu
[y,x,t] = initial(G,x_0)  % initial response with
                          % initial conditions x_0
[y,x,t] = impulse(G(Yu,Iu)) % Impulse response of input
                          % Iu to output Yu
[y,x] = lsim(G,U,T)       % simulate time response of
                          % the system
```

Consider a system described by

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-1 & 2\\ 0 & -3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}$$

$$\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} \text{ with } \begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} = \begin{bmatrix}5\delta(t)\\ 2\mathbb{1}(t)\end{bmatrix},$$

where $\mathbb{1}(t)$ is a unit step function. The state-transition matrix is

$$e^{At} = \begin{bmatrix}1 + (-t) + \frac{(-t)^2}{2!} + \dots & 1 - 1 + (-t) - (-3t) + \frac{(-t)^2}{2!} - \frac{(-3t)^2}{2!} + \dots\\ 0 & 1 + (-3t) + \frac{(-3t)^2}{2!} + \dots\end{bmatrix}$$
$$= \begin{bmatrix}e^{-t} & e^{-t} - e^{-3t}\\ 0 & e^{-3t}\end{bmatrix}$$

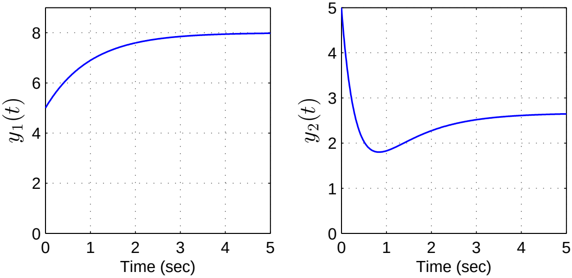
The impulse response is

$$g(t) = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^{-t} & e^{-t} - e^{-3t}\\ 0 & e^{-3t}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix} = \begin{bmatrix}e^{-t} & 4e^{-t}\\ e^{-t} & 4e^{-t} - 8e^{-3t}\end{bmatrix}$$

The output (zero-state response) is $g(t) * u(t)$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \int_0^t \begin{bmatrix} e^{-(t-\tau)} & 4e^{-(t-\tau)} \\ e^{-(t-\tau)} & 4e^{-(t-\tau)} - 8e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} 5\delta(\tau) \\ 2\mathbb{1}(\tau) \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 8 - 3e^{-t} \\ \frac{8}{3} - 3e^{-t} + \frac{16}{3} e^{-3t} \end{bmatrix}$$



Transfer function

- Taking the Laplace transform of the state and output equations and assuming that all initial conditions are zero yields

$$
\begin{aligned}
sx(s) &= Ax(s) + Bu(s) \\
y(s) &= Cx(s) + Du(s)
\end{aligned}
$$

- We have

$$y(s) = \left[C(sI - A)^{-1}B + D\right]u(s) = G(s)u(s)$$

- Taking inverse Laplace transform

$$y(t) = \mathcal{L}^{-1}\left\{C(sI - A)^{-1}B + D\right\} * u(t)$$

- Since

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} \;\Rightarrow\; g(t) = \mathcal{L}^{-1}\{G(s)\}$$

Transfer function
Example

Given the system described by the state model,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$
$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

the transfer function matrix is

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} s+1 & -2 \\ 0 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{4}{s+1} \\ \dfrac{1}{s+1} & \dfrac{-4(s-1)}{(s+1)(s+3)} \end{bmatrix}.$$

Then

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \begin{bmatrix} e^{-t} & 4e^{-t} \\ e^{-t} & 4e^{-t} - 8e^{-3t} \end{bmatrix}.$$

- the transfer function of a SISO system is a ratio of polynomials in s-domain, where all initial conditions are zero:

$$G(s) = C(sI - A)^{-1}B + D = \frac{C\,\mathrm{adj}(sI - A)B + D\det(sI - A)}{\det(sI - A)}$$

- the roots of the numerator $\mathrm{num}(s) = 0$ are called the system zeros. Or the value of z_i such that

$$G(z_i) = 0$$

- the roots of the denominator $\mathrm{den}(s) = 0$ are called the system poles. Or the value of p_i such that

$$|G(p_i)| = \infty.$$

Poles and Zeros

- the transfer function of a MIMO system is a matrix of SISO transfer functions.
- the system poles are defined as the union of the poles of each of the SISO transfer functions.
- for state-space realization of the system, the poles are the eigenvalues of the state matrix.
- the zoers of a MIMO system are the values of s such that the transfer function matrix has less than full rank:

$$\text{rank}[G(s)] < \min\{n_y, n_u\}$$

- for some system the output is zero for some nonzero input:

$$y(s) = 0 = G(s)u(s).$$

- then for the special case of square transfer functions, the zeros are the values of s such that

$$\det\left[G(s)\right] = 0$$

The Laplace transform of the state model is

$$sx(s) = Ax(s) + Bu(s)$$
$$y(s) = 0 = Cx(s) + Du(s)$$

Rewriting

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = 0$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-1 & 2\\ 0 & -3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}$$
$$\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix},$$

the transfer function is

$$G(s) = \begin{bmatrix}\dfrac{-s}{s+1} & \dfrac{4}{s+1}\\ \dfrac{1}{s+1} & \dfrac{-s^2-8s+1}{(s+1)(s+3)}\end{bmatrix}$$

The poles are the solution of

$$\det(sI - A) = \det\begin{bmatrix}s+1 & -2\\ 0 & s+3\end{bmatrix} = (s+1)(s+3) = 0,$$
$$p_1 = -1 \text{ and } p_2 = -3.$$

Poles and Zeros
Example

The zeros are the solution of

$$\det \begin{bmatrix} \frac{-s}{s+1} & \frac{4}{s+1} \\ \frac{1}{s+1} & \frac{-s^2-8s+1}{(s+1)(s+3)} \end{bmatrix} = \frac{s^2+7s-12}{(s+1)(s+3)} = 0,$$

which are $z_1 = -8.42$ and $z_2 = 1.42$. The better method that can be avoided the poles-zeros cancellation is

$$\det \begin{bmatrix} s+1 & -2 & -1 & 0 \\ 0 & s+3 & 0 & -4 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix} = s^2+7s-12 = 0.$$

Matlab code:

```
sys = ss(A,B,C,D);
tzero(sys)
```

Stability

Definition (BIBO Stability)

A system is bounded input/bounded output-stable if for every bounded input,

$$|u_i(t)| < M_1 \text{ for all } t \text{ and all } i,$$

the output is bounded:

$$|y_j(t)| < M_2 \text{ for all } t \text{ and all } j,$$

provided that the initial conditions are zero.

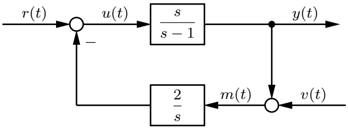
Definition (stability)

A causal, linear, time-invariant system is stable if and only if all of its poles have negative real parts.

Internal Stability

a motivated example

Consider a system



$v(t)$ is the measurement error. The transfer function $T_{yr}(s)$ is

$$\frac{y(s)}{u(s)} = \frac{\frac{s}{(s-1)}}{1+\frac{2}{s}\frac{s}{s-1}} = \frac{s}{s+1}$$

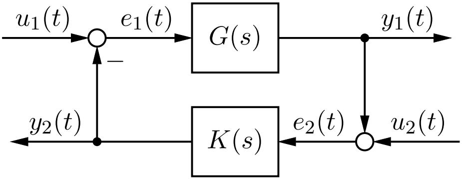
The transfer function $T_{vu}(s)$ is

$$\frac{u(s)}{v(s)} = \frac{\frac{-2}{s}}{1+\frac{2}{s}\frac{s}{s-1}} = \frac{-s(s-1)}{s(s+1)}$$

which is stable. Actually, the control input of this system is unbounded if the measurement error contains a constant bias.

Internal Stability

- A linear feedback system is internally stable if all internal signals and all possible outputs remain bounded given that all possible inputs are bounded.
- Internal stability is evaluated by considering all of the possible transfer functions associated with the feedback system.



The inputs $u_1(t)$ and $u_2(t)$ are applied to four possible outputs $y_1(t)$, $y_2(t)$, $e_1(t)$, and $e_2(t)$.

Internal Stability

The eight possible transfer functions are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (I+GK)^{-1}G & -(I+GK)^{-1}GK \\ (I+KG)^{-1}KG & (I+KG)^{-1}K \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} G_{y_1u_1} & G_{y_1u_2} \\ G_{y_2u_1} & G_{y_2u_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix};$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} (I+KG)^{-1} & -(I+KG)^{-1}K \\ (I+KG)^{-1}G & (I+GK)^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} G_{e_1u_1} & G_{e_1u_2} \\ G_{e_2u_1} & G_{e_2u_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

- the inputs can represent input disturbances, ooutput disturbances, reference inputs, and measurement noise.
- the outputs represent the plant output, the controller output, the plant input, and the controller input.

Internal Stability

Definition (Internal Stability)

The feedback system consisting of the plant $G(s)$ and the controller $K(s)$ is internally stable if each of the eight transfer functions are stable.

- Internal stability is a stronger condition than stability and will be required when designing feedback systems.
- To check the internal stability, considering:

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \left(\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right)$$

Internal Stability

The transfer functions from u to y are then related to the transfer functions from u to e:

$$\begin{bmatrix} G_{y_1u_1} & G_{y_1u_2} \\ G_{y_2u_1} & G_{y_2u_2} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \left(\begin{bmatrix} G_{e_1u_1} & G_{e_1u_2} \\ G_{e_2u_1} & G_{e_2u_2} \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right).$$

Since all of the transformation matrices relating $G_{e_iu_j}$ to $G_{y_iu_j}$ are constants and therefore stable, the following result is obtained:

Internal Stability

The feedback system consisting of the plant $G(s)$ and the controller $K(s)$ is internally stable if and only if each of the four transfer functions $G_{e_1u_1}$, $G_{e_1u_2}$, $G_{e_2u_1}$, and $G_{e_2u_2}$ are stable.

Similarity Transformations

- The state equations of a system are not unique. There are infinite number of state representations of a given physical system.
- *Similarity transformation* can be used to generate special state models that have nice algebraic and numerical properties. The poles and zeros of the system are invariant under the operation.

Consider a system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t).$$

A new state vector can be defined:

$$\tilde{x}(t) = T^{-1}x(t), \quad \text{and} \quad x(t) = T\tilde{x}(t)$$

where T is a constant, invertible transformation matrix.

Similarity Transformations

- Since $\dot{x}(t) = T\dot{\tilde{x}}(t)$,

$$T\dot{\tilde{x}}(t) = AT\tilde{x}(t) + Bu(t)$$
$$y(t) = CT\tilde{x}(t) + Du(t)$$

- Multiplying both sides by T^{-1} yields a state model in term of the new state $\tilde{x}(t)$:

$$\dot{\tilde{x}}(t) = (T^{-1}AT)\tilde{x}(t) + (T^{-1}B)u(t)$$
$$y(t) = (CT)\tilde{x}(t) + Du(t).$$

- the new state model is generated by

$$A \Rightarrow T^{-1}AT;\ B \Rightarrow T^{-1}B;\ C \Rightarrow CT;\ D \Rightarrow D.$$

Given

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 3u(t).$$

A new model has states that are the sum and difference of the original states:

$$\begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The transformation matrices are then

$$T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \; T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Performing the similarity transformation yields the new model:

$$\begin{bmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} + 3u(t).$$

- The new state model has no coupling between states.
- The state matrix is diagonal.
- The eigenvalues (poles) of the system are then simply the diagonal elements of the state matrix.

Controllability

Consider a system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0.$$

Definition (Controllability)

The dynamical system described above or the pair (A, B) is said to be *controllable* if, for any initial state $x(0) = x_0$, $\quad t_1 > 0$ and final state x_1, there exists a (piecewise continuous) input $u(\cdot)$ such that the solution of the system satisfies $x(t_1) = x_1$. Otherwise, the system or the pair (A, B) is said to be *uncontrollable*.

Consider

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \qquad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad x(t_f) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

From the solution of the state equation

$$x_1(t) = x_2(t) = \int_0^t e^{\lambda(t-\tau)} u(\tau) d\tau$$

It is clear that there are no such input $u(t)$ that will bring the system to the final state $x(t_f)$.

Controllability

Theorem

The following are equivalent:

(i) (A, B) *is controllable.*

(ii) *The matrix*

$$W_c(t) := \int_0^t e^{A\tau} B B^* e^{A^*\tau} d\tau$$

is positive definite for any $t > 0$*.*

(iii) *The controllability matrix*

$$\mathcal{C}(A, B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

has full-row rank.

(iv) *The matrix* $[A - \lambda I, B]$ *has full-row rank for all* λ *in* $\mathbb{C}$*.*

Controllability

Theorem

(v) *Let λ and x be any eigenvalue and any corresponding left eigenvector of A (i.e., $x^*A = x^*\lambda$); then $x^*B \neq 0$.*

(vi) *The eigenvalues of $A + BF$ can be freely assigned (with the restriction that complex eigenvalues are in conjugate pairs) by a suitable choice of F.*

- Proof See page 65

Stabilizability

Definition

An unforced dynamical system $\dot{x} = Ax$ is said to be *stable* if all the eigenvalues of A are in the open left half plane; that is , $\mathrm{Re}\,\lambda(A) < 0$. A matrix A with such a property is said to be stable or Hurwitz.

Definition

The dynamical system, or the pair (A, B), is said to be *stabilizable* if there exists a state feedback $u = Fx$ such that the system is stable (i.e., $A + BF$ is stable).

It is more appropriate to call this stabilizability the *state feedback stabilizability* to differentiate it from the *output feedback stabilizability*.

Stabilizability

Theorem

The following are equivalent:

(i) (A, B) *is stabilizable.*

(ii) *The matrix* $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ *has full row rank for all* $\operatorname{Re}\lambda \geq 0$*.*

(iii) *For all* λ *and* x *such that* $x^* A = x^* \lambda$ *and* $\operatorname{Re}\lambda \geq 0$, $x^* B \neq 0$*.*

(iv) *There exists a matrix* F *such that* $A + BF$ *is Hurwitz.*

Observability

Consider a system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = 0$$
$$y(t) = Cx(t) + Du(t).$$

Definition

The dynamical system described above or by the pair (C, A) is said to be *observable* if, for any $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input $u(t)$ and the output $y(t)$ in the interval of $\left[0, t_1\right]$. Otherwise, the system, or (C, A), is said to be *unobservable*.

Observability

Theorem

The following are equivalent:

(i) (C, A) *is observable.*

(ii) *The matrix*

$$W_o(t) := \int_0^t e^{A^*\tau} C^* C e^{A\tau} d\tau$$

is positive definite for any $t > 0$.

(iii) *The observability matrix*

$$\mathcal{O} = \begin{bmatrix} C^* & (CA)^* & (CA^2)^* & \cdots & (CA^{n-1})^* \end{bmatrix}^*$$

has full column rank.

Observability

Theorem

(iv) *The matrix* $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ *has full column rank for all* λ *in* $\mathbb{C}$.

(v) *Let* λ *and* y *be any eigenvalue and any corresponding right eigenvector of* A, *i.e.,* $Ay = \lambda y$, *then* $Cy \neq 0$.

(vi) *The eigenvalues of* $A + LC$ *can be freely assigned by a suitable choice of* L.

(vii) (A^*, C^*) *is controllable.*

- Proof See page 71

Detectability

Definition

The system, or the pair (C, A), is *detectable* if $A + LC$ is stable for some L.

Theorem

The following are equivalent:

(i) (C, A) *is detectable.*

(ii) *The matrix* $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ *has full column rank for all* $\operatorname{Re}\lambda \geq 0$.

(iii) *For all* λ *and* x *such that* $Ax = \lambda x$ *and* $\operatorname{Re}\lambda \geq 0$, $Cx \neq 0$.

(iv) *There exists a matrix* L *such that* $A + LC$ *is Hurwitz.*

(v) (A^*, C^*) *is stabilizable.*

Controllability and Observability

Matlab Code

```
Cc = ctrb(A,B);    % Compute the controllability matrix
Oc = obsv(A,C);    % Compute the observability matrix
Wc = gram(SYS,'c');    % Controllability gramian
Wc = gram(SYS,'o');    % Observability gramian
```

Controllability and Observability Example

We are given the system described by the following state model:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

The controllability test matrix is

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & -1 & 8 \\ 0 & 4 & 0 & -12 \end{bmatrix},$$

which has full rank (a rank of 2). The system is therefore controllable. The observability test matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 5 \end{bmatrix},$$

which has full rank (a rank of 2). The system is observable.

State Feedback

Consider a system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

- A simple controller is $u(t) = -Kx(t)$, where K is a vector of state feedback gains.
- The closed-loop state equation is

$$\dot{x}(t) = (A - BK)x(t)$$

- Closed-loop poles placement is one method to satisfy the requirement.

$$\det(sI - A + BK) = (s - p_1)(s - p_2)\cdots(s - p_{n_x})$$

An ac motor is described by the state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

where the two states are motor shaft angle and angle rate, and the control input is the applied voltage. We want to have closed-loop poles at $p_{1,2} = -2 \pm 2j$.

$$\det\left[sI - A + BK)\right] = (s + 2 - 2j)(s + 2 + 2j) = s^2 + 4s + 8$$

$$\det \begin{bmatrix} s & -1 \\ k_1 & s + 1 + k_2 \end{bmatrix} = s^2 + (1 + k_2)s + k_1$$

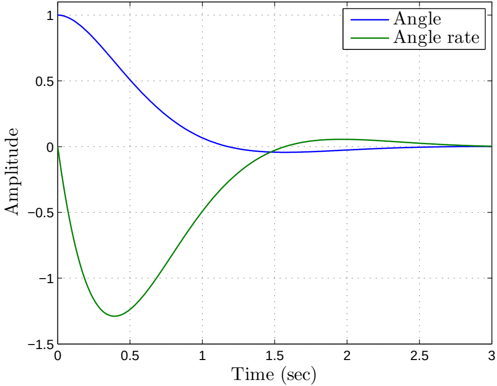
Then $k_1 = 8$ and $k_2 = 3$, the control input is

$$u(t) = -\begin{bmatrix} 8 & 3 \end{bmatrix} x(t).$$

This can be done using a Matlab command `K = -place(A,B,[-2+2j,-2-2j])`.

State Feedback

Example



State feedback control of an ac motor. This method is useful only the case that all states can be measured.

Observers and Observer-Based Controllers

Consider a system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t).$$

Theorem

An observer exists iff (C, A) is detectable. Further, if (C, A) is detectable, then a full-order Luenberger observer is give by

$$\dot{q} = Aq + Bu + L(Cq + Du - y)$$
$$\hat{x} = q,$$

where L is any matrix such that $A + LC$ is stable.

Observers and Observer-Based Controllers

- If (A, B) is controllable and state x is available for feedback, then there is a state feedback $u = Fx$ such that the closed-loop poles of the system can be arbitrarily assigned.
- Similarly, if (C, A) is observable, then the system observe poles can be arbitrarily placed so that the state estimator $\hat{x}$ can be made to approach x arbitrarily fast.
- If the system states are not available for feedback so that the estimated state has to be used. Hence, the controller has the following dynamics:

$$\dot{\hat{x}} = (A + LC)\hat{x} + Bu + LDu - Ly$$
$$u = F\hat{x}.$$

Observers and Observer-Based Controllers

- Then the total system state equations are give by

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}.$$

- Let $e := x - \hat{x}$ then the system equation becomes

$$\begin{bmatrix} \dot{e} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A + LC & 0 \\ -LC & A + BF \end{bmatrix} \begin{bmatrix} e \\ \hat{x} \end{bmatrix}$$

- The closed-loop poles consist of two parts: the poles resulting from state feedback $\lambda_i(A + BF)$ and the poles resulting from the state estimation $\lambda_j(A + LC)$.

Observers and Observer-Based Controllers

- If (A, B) is controllable and (C, A) is observable, then there exist F and L such that the eigenvalues of $A + BF$ and $A + LC$ can be arbitrarily assigned.
- The controller given above is called an observer-based controller and is denoted as

$$u = K(s)y$$

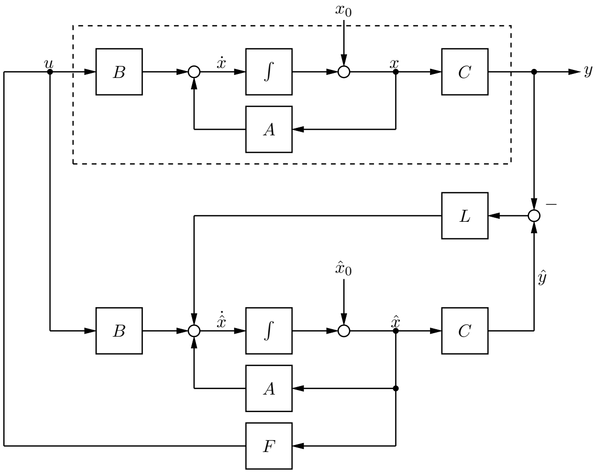
and

$$K(s) = \left[\begin{array}{c|c} A + BF + LC + LDF & -L \\ \hline F & 0 \end{array}\right].$$

Observers and Observer-Based Controllers

- In general, if a system is stabilizable through feeding back the output y, then it is said to be *output feedback stabilizable*.
- It is clear that a system is output feedback stabilizable if and only if (A, B) is stabilizable and (C, A) is detectable.

Observers and Observer-Based Controllers



Assume $D = 0$.

- Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. We shall design a state feedback $u = Fx$ such that the closed-loop poles are at $\{-2, -3\}$. This can be done by choosing $F = \begin{bmatrix} -6 & -8 \end{bmatrix}$ using

```
F = -place(A,B,[-2,-3]).
```

- If the states are not available for feedback and we want to construct an observer so that the observer poles are at $\{-10, -10\}$. The $L = \begin{bmatrix} -21 \\ -51 \end{bmatrix}$ can be obtained by using

```
L = -acker(A',C',[-10,-10])'
```

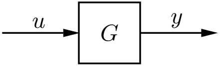
and the observer-based controller is given by

$$A_K = A + BF + LC + LDF, \quad B_K = -L, \quad C_K = F, \quad D_K = 0$$

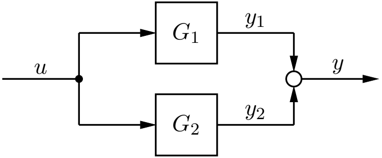
and

$$K(s) = \frac{-534(s + 0.6966)}{(s + 34.6564)(s - 8.6564)}.$$

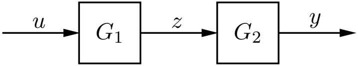
Operations on Systems



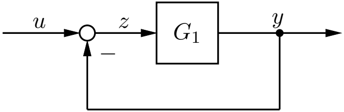
(a) single block



(b) parallel



(c) cascade



(d) negative feedback

Operations on Systems

Consider the two LTI systems

$$G_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right], \quad G_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right].$$

In parallel case: $u = u_1 = u_2$ and $y = y_1 + y_2$ we then have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u,$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (D_1 + D_2) u$$

$$G_1 + G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

In cascade case: $u = u_1$, $y = y_2$, and $z = y_1 = u_2$ we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2D_1u$$

$$G_1G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ \hline D_2C_1 & C_2 & D_2D_2 \end{array}\right]$$

In negative feedback case: $u_1 = z = u - y_1$, $y = y_1$ we have

$$\dot{x}_1 = (A_1 - B_1(I + D_1)^{-1}C_1)x_1 + B_1(I - (I + D)^{-1}D_1)u$$

$$y = (I + D_1)^{-1}C_1 x_1 + (I + D_1)^{-1}D_1 u$$

$$G = \left[\begin{array}{c|c} (A_1 - B_1(I + D_1)^{-1}C_1 & B_1(I - (I + D)^{-1}D_1 \\ \hline (I + D_1)^{-1}C_1 & (I + D_1)^{-1}D_1 \end{array}\right]$$

Sometimes feedback interconnections are *ill-posed*. In this example, this would happen if the matrix $I + D_1$ was singular.

Cascade: To create a system `sys` from the cascade connection of the system `sys1` whose output is connected to the input of `sys2`, we use a command

`sys = series(sys1,sys2)` or `sys = sys2*sys1`

Parallel: To create a system `sys` from the parallel connection of the systems `sys1` and `sys2`, we use a command

`sys = parallel(sys1,sys2)` or `sys = sys1 + sys2`

Feedback: The command

```
sys = feedback(sys1,sys2)
```

creates a system `sys` from the negative feedback interconnection of the system `sys1` in the forward loop, with the system `sys2` in the backward loop.

A positive feedback interconnection can be obtained using

```
sys = feedback(sys1, sys2, 1)
```

State-Space Realizations of Transfer Matrices

Assume that $G(s)$ is a real rational transfer matrix that is *proper*. Then we call a state-space model (A, B, C, D) such that

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

a *realization* of $G(s)$.

Definition

A state-space realization (A, B, C, D) of $G(s)$ is said to be a minimal realization of $G(s)$ if A has the smallest possible dimension.

Definition

A state-space realization (A, B, C, D) of $G(s)$ is minimal if and only if (A, B) is controllable and (C, A) is observable.

State-Space Realizations of Transfer Matrices

Controllable canonical form

Consider a SISO system

$$G(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

Rewritten as

$$G(s) = \frac{b(s)}{a(s)} \quad \text{and} \quad y(s) = b(s)\frac{1}{a(s)}u(s) = b(s)v(s)$$

$$u(s) = (s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n)v(s)$$

$$y(s) = (\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n)v(s)$$

and

$$x_1 = v^{(n-1)}(t),\ x_2 = v^{(n-2)}(t),\ \ldots,\ x_{n-1} = \dot{v}(t),\ x_n = v(t)$$

$$y(t) = \beta_1 x_1(t) + \beta_2 x_2(t) + \ldots + \beta_{n-1} x_{n-1}(t) + \beta_n x_n$$

Then a *controllable canonical form* or *controller canonical form* of the system is

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

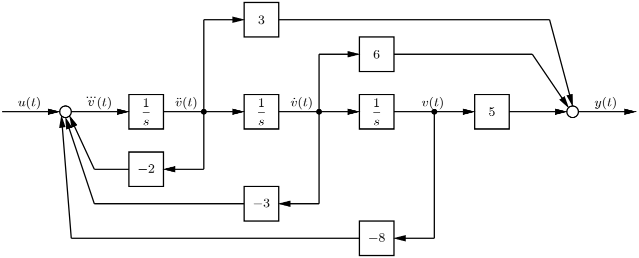
$$C_c = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \end{bmatrix}$$

A system $G(s) = \dfrac{3s^2+6s+5}{s^3+2s^2+3s+8}$, $u(s) = (s^3+2s^2+3s)v(s)$, and $y(s) = (3s^2+6s+5)v(s)$. Then $x_3 = v(t)$, $x_2 = \dot{v}(t)$ $x_1 = v(t)$ and

$$\dot{x}(t) = \begin{bmatrix} -2 & -3 & -8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 6 & 5 \end{bmatrix} x(t)$$

with a block diagram.



Then an *observable canonical form* or *observer canonical form* of the system is

$$A_o = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

State-Space Realizations of Transfer Matrices

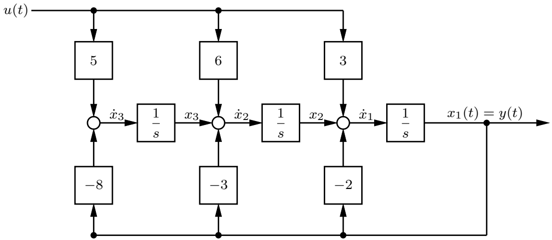
Observable canonical form example

A system $G(s) = \dfrac{3s^2 + 6s + 5}{s^3 + 2s^2 + 3s + 8}$, we have $x_1(t) = y(t)$ and
$s\left[s\left[sy(s) + (2y(s) - 3u(s))\right] + (3y(s) - 6u(s))\right] + (8y(s) - 5u(s)) = 0$. Then
$\dot{x}_3(t) = -8x_1(t) + 5u(t)$, $\dot{x}_2(t) = x_3(t) - 3x_1(t) + 6u(t)$, $\dot{x}_1(t) = x_2 - 2x_1(t) + 3u(t)$ and

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -8 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t)$$

with a block diagram



State-Space Realizations of Transfer Matrices

Matlab example

A system $G(s) = \dfrac{3s^2 + 6s + 5}{s^3 + 2s^2 + 3s + 8}$

Matlab Code

```
num = [3 6 5]; den = [1 2 3 8];

% controllable canonical form
[Ac,Bc,Cc,Dc] = tf2ss(num,den);

% observable canonical form
Ab = Ac'; Bb = Cc'; Cb = Bc'; Db = Dc;
```

Reference

1. Kemin Zhou and John Doyle "*Essentials of Robust Control*", Prentice Hall, 1998
2. Chi-Tsong Chen "*Linear System Theory and Design*", Oxford University Press, 1999
3. Herbert Werner "*Lecture Notes on Control Systems Theory and Design*", 2011
4. Jeffrey B. Burl "*Linear Optimal Control: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Methods*", 1999

Solutions of State Equations

- Consider a SISO

$$\dot{x}(t) = ax(t) + bu(t),\ x(0) = x_0$$

- The solution to the problem

$$\frac{d}{dt}\left(e^{at}a\right) = e^{-at}\left(\dot{x}(t) - ax(t)\right) = e^{-at}bu$$

Integration from 0 to t and multiplication by e^{at} yields

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

- MIMO version

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- Note that $\frac{d}{dt}(e^{At}) = Ae^{At}$. Back

- $(ii) \leftrightarrow (i)$ Suppose $W_c(t_1) > 0$ for some $t_1 > 0$, and let the input be defined as

$$u(\tau) = -B^* e^{A^*(t_1-\tau)} W_c^{-1}(t_1)(e^{At_1}x_0 - x_1).$$

It is easy to verify that $x(t_1) = x_1$ as follow:

$$x(t_1) = e^{At_1}x_0 - W_c(t_1)W_c^{-1}(t_1)(e^{At_1}x_0 - x_1)$$

Therefore, the system (A, B) is controllable if the $W_c(t)$ is invertible. Thus it has full rank and positive definite for any $t > 0$.
To show that the controllability of (A, B) implies that $W_c(t) > 0$ for any $t > 0$, assume that (A, B) is controllable but $W_c(t)$ is singular for some $t_1 > 0$.

- Since $e^{At}BB^*e^{A^*t} \geq 0$ for all t, there exists a real vector $0 \neq v \in \mathbb{R}^n$ such that

$$v^* e^{At} B = 0, \quad t \in [0, t_1].$$

- Back

Controllability
Proof

- Let $x(t_1) = x_1 = 0$, and then from the solution of the system, we have

$$0 = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau.$$

Pre-multiply the above equation by v^* to get

$$0 = v^* e^{At_1} x_0.$$

If we chose the initial state $x_0 = e^{-At_1}v$, then $v = 0$, and this is contradiction. Hence, $W_c(t)$ cannot be singular for any $t > 0$.

- $(ii) \leftrightarrow (iii)$ First assume that $W_c(t) > 0$ for all $t > 0$ but the controllability matrix $\mathcal{C}(A, B)$ does not have full row rank. Then there exists a $v \in \mathbb{R}^n$ such that

$$v^* A^i B = 0, \quad i = 0, 1, \ldots, n-1$$

Hence $v^* e^{At} B = 0$ for all t or, equivalently, $v^* W_c(t) = 0$ for all t; this is a contradiction, and hence, the controllability matrix $\mathcal{C}(A, B)$ must be full row rank.

- Back

Controllability

Proof

- Conversely, assume suppose $\mathcal{C}(A, B)$ has full row rank but $W_c(t_1)$ is singular for some t_1. Then there exists a vector $v \neq 0 \in \mathbb{R}^n$ such that $v^* e^{At} B = 0$ for all $t \in [0, t_1]$. Therefore, set $t = 0$, and we have

$$v^* B = 0$$

- Next, evaluate the i-th derivative of $v^* e^{At} B = 0$ at $t = 0$ to get

$$v^* A^i B = 0, \quad i > 0.$$

  Hence, we have

$$v^* \begin{bmatrix} B & AB & A^2 B & \cdots & A^{n-1} B \end{bmatrix} = 0$$

  or, in other words, the controllability matrix $\mathcal{C}$ does not have full row rank. This is again a contradiction.

- Back

Controllability

Proof

- $(iii) \to (iv)$: Suppose, on the contrary, that the matrix

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix}$$

does not have full row rank for some $\lambda \in \mathbb{C}$.

- Then there exists a vector $x \in \mathbb{C}^n$ such that

$$x^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0$$

i.e., $x^* A = \lambda x^*$ and $x^* B = 0$. However, this will result in

$$x^* \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} x^* B & \lambda x^* B & \cdots & \lambda^{n-1} x^* B \end{bmatrix} = 0$$

i.e., the controllability matrix $\mathcal{C}(A, B)$ does not have full row rank, and this is a contradiction.

- $(iv) \to (v)$ This is obvious from the proof of $(iii) \to (iv)$.
- Back

- $(v) \to (iii)$ Assume that (v) holds but rank $\mathcal{C}(A, B) = k < n$. By using a transformation T such that

$$TAT^{-1} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix}, \quad TB = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

  with $\bar{A}_{\bar{c}} \in \mathbb{R}^{(n-k)\times(n-k)}$. Let λ_1 and $x_{\bar{c}}$ be any eigenvalue and any corresponding left eigenvector of $\bar{A}_{\bar{c}}$, i.e., $x_{\bar{c}}^* \bar{A}_{\bar{c}} = \lambda_1 x_{\bar{c}}^*$. Then $x^*(TB) = 0$ and

$$x = \begin{bmatrix} 0 & x_{\bar{c}} \end{bmatrix}^*$$

  is an eigenvector of TAT^{-1} corresponding to the eigenvalue λ_1, which implies that (TAT^{-1}, TB) is not controllable. This is a contradiction since similarity transformation does not change controllability.
- Back

Controllability
Proof

- $(vi) \rightarrow (i)$ This follows the same arguments as in the proof of $(v) \rightarrow (iii)$ assume that (vi) holds but (A, B) is uncontrollable. Then, there is a decomposition so that some subsystems are not affected by the control, but this contradicts the condition (vi).
- $(i) \rightarrow (vi)$ We can construct a matrix F so that the eigenvalues of $A + BF$ are in the desired locations.
- Back

Observability

Proof

Proof. First, we show the equivalence between condition (i) and (iii). Once this is done, the rest will follow by the duality or condition (vii).

- $(i) \leftarrow (iii)$ Note that given the input $u(t)$ and the initial condition x_0, the output in the time interval $[0, t_1]$ is given by

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

Since $y(t)$ and $u(t)$ are known, assuming $u(t) = 0, \forall t$. Hence, $y(t) = Ce^{At}x_0, \quad t \in [0, t_1]$. From this equation, we have

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0$$

The observability matrix $\mathcal{O}$ has full column rank, there is a unique solution x_0.

- Back

Observability

Proof

- $(i) \rightarrow (iii)$ Assume that (C, A) is observable but that the observability matrix does not have full column rank, i.e., there is a vector x_0 such that $\mathcal{O}x_0 = 0$ or equivalently $CA^i x_0 = 0, \forall i \geq 0$ by the Cayley-Hamilton Theorem. Now suppose the initial state $x(0) = x_0$, then $y(t) = Ce^{At}x_0 = 0$. This implies that the system is not observable since x_0 cannot be determined from $y(t) = 0$.
- Back