Lecture 1 Classical Feedback Control

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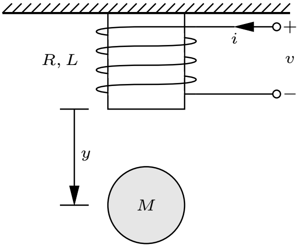
Magnetic-Ball Suspension System



The lifting force :

$$f(t) = K\frac{i^2(t)}{y(t)}$$

KVL :

$$Ri(t) + L\frac{di(t)}{dt} = v(t)$$

Newton's Law:

$$M\frac{d^2y(t)}{dt^2} = -K\frac{i^2(t)}{y(t)} + Mg$$

Magnetic-Ball Suspension System

The differential equation model

$$Ri(t) + L\frac{di(t)}{dt} = v(t)$$
$$M\frac{d^2y(t)}{dt^2} = -K\frac{i^2(t)}{y(t)} + Mg$$

Selecting $x_1(t) = i(t)$, $x_2(t) = y(t)$, $x_3(t) = \dot{y}(t)$ and $u(t) = v(t)$, the state-space model is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \frac{u(t)-Rx_1(t)}{L} \\ x_3(t) \\ -\frac{Kx_1^2(t)}{Mx_2(t)} + g \end{bmatrix}$$
$$y = x_2(t)$$

Linearize the system around an operating point with $y(t) = y_0$.

Magnetic-Ball Suspension System

Linearization

Definition

A triple of constant vectors $\begin{bmatrix} u_0 & x_0 & y_0 \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}$ is said to be an *operating point* of the system if

$$f(x_0, u_0) = 0$$
$$g(x_0, u_0) = y_0$$

$$\frac{u_0 - Rx_{10}}{L} = 0, \qquad x_{30} = 0$$
$$-\frac{Kx_{10}^2}{Mx_{20}} + g = 0, \qquad x_{20} = y_0$$

This gives

$$u_0 = R\sqrt{\frac{Mgy_0}{K}}, \qquad \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{Mgy_0}{K}} \\ y_0 \\ 0 \end{bmatrix}, \qquad y_0 = y_0$$

The physical meaning of an operating point is that if the system has initial condition x_0 and a constant input u_0 is applied, then the state and output will stay at constant values x_0 and y_0, respectively, for all time, i.e.,

$$u(t) = u_0, \quad x(0) = x_0 \Rightarrow x(t) = x_0, \quad y(t) = y_0.$$

Since f and g are sufficiently smooth, we can conclude that

$$u(t) - u_0,\ x(0) - x_0 \text{ are small } \Rightarrow x(t) - x_0,\ y(t) - y_0 \text{ are small.}$$

Denote $\tilde{u}(t) = u(t) - u_0$, $\tilde{x}(t) = x(t) - x_0$ and $\tilde{y}(t) = y(t) - y_0$.

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + b\tilde{u}(t), \qquad \tilde{y}(t) = c\tilde{x}(t) + d\tilde{u}(t)$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_0, u=u_0}, \ B = \left.\frac{\partial f}{\partial u}\right|_{x=x_0, u=u_0}$$

$$C = \left.\frac{\partial g}{\partial x}\right|_{x=x_0, u=u_0}, \ D = \left.\frac{\partial g}{\partial u}\right|_{x=x_0, u=u_0}$$

$$\tilde{u}(t) = u(t) - u_0 = u(t) - R\sqrt{\frac{Mgy_0}{K}}$$

$$\tilde{x}(t) = x(t) - x_0 = \begin{bmatrix} x_1(t) - \sqrt{\frac{Mgy_0}{K}} \\ x_2(t) - y_0 \\ x_3(t) \end{bmatrix}, \ \tilde{y}(t) = y(t) - y_0$$

The linearized model of the deviation variables is

$$\dot{\tilde{x}}(t) = \begin{bmatrix} -\frac{R}{L} & 0 & 0 \\ 0 & 0 & 1 \\ -2\sqrt{\frac{gK}{My_0}} & \frac{g}{y_0} & 0 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} \tilde{u}(t)$$

$$\tilde{y}(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tilde{x}(t) + [0]\tilde{u}(t)$$

Transfer Function

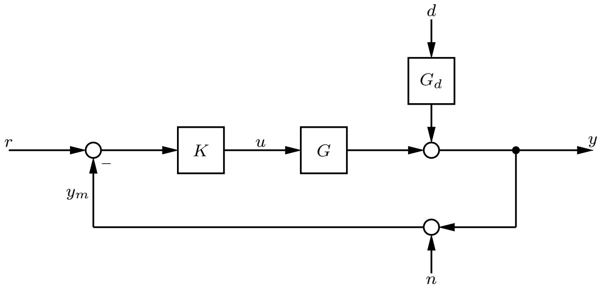
$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$G(s) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s + \frac{R}{L} & 0 & 0 \\ 0 & s & -1 \\ -2\sqrt{\frac{gK}{My_0}} & -\frac{g}{y_0} & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{-2\sqrt{\frac{gK}{My_0}}\frac{1}{L}}{\left(s + \frac{R}{L}\right)\left(s^2 - \frac{g}{y_0}\right)} = \frac{-2\sqrt{gy_0K}}{\sqrt{M}\,(Ls + R)\,(y_0 s^2 - g)}$$

Feedback Control

One degree-of-freedom controller

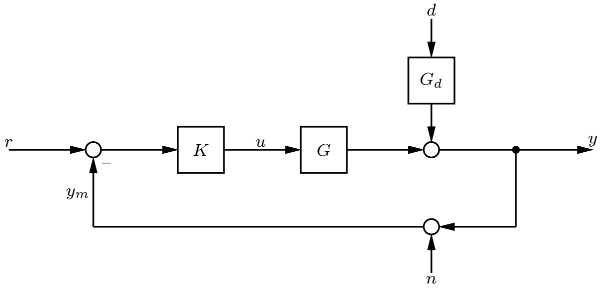


Objectives:

- Closed-loop stability
- Reference tracking
- Disturbance rejection
- Noise response

Feedback Control

One degree-of-freedom controller



The input to the plant is

$$u = K(s)(r - y - n)$$

The control error e is defined as

$$e = y - r$$

The plant model : $y = G(s)u + G_d(s)d$

For one-degree-of-freedom controller

$$y = GK(r - y - n) + G_d d$$

or

$$(I + GK)y = GKr + G_d d - GKn$$

hence

$$y = \underbrace{(I + GK)^{-1}GK}_{T}\, r + \underbrace{(I + GK)^{-1}}_{S}\, G_d d - \underbrace{(I + GK)^{-1}GK}_{T}\, n$$

$$e = y - r = -Sr + SG_d d - Tn$$

$$u = KSr - KSG_d d - KSn$$

Loop transfer function :

$$L(s) = G(s)K(s)$$

Sensitivity function :

$$S(s) = (I + GK)^{-1} = (I + L)^{-1}$$

Complementary sensitivity function :

$$T(s) = (I + GK)^{-1}GK = (I + L)^{-1}L$$

S is the closed-loop transfer function from the output disturbances to the outputs.

T is the closed-loop transfer function from the reference signals to the outputs.

$$S + T = (I + L)^{-1} + (I + L)^{-1}L = I$$

Sensitivity to plant gain changes

$$S(s) = \frac{\text{relative closed-loop response change}}{\text{relative open-loop response change}} = \frac{dT(s)/T(s)}{dG(s)/G(s)}$$

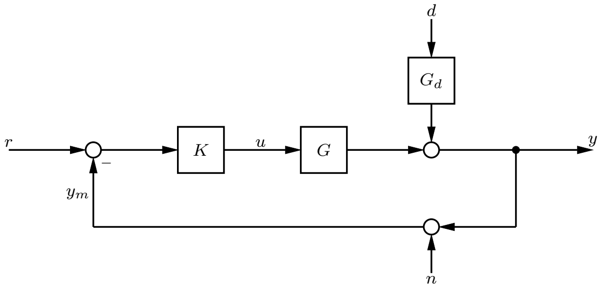
An change of α percent in the open-loop plant DC gain gives a change of $\alpha S(0)$ percent in the closed-loop DC gain.

Performance requirements

Reference tracking	$T(s) \approx 1$
Noise rejection	$T(s) \ll 1$
Disturbance rejection	$S(s)G_d(s) \ll 1$
Low plant sensitivity	$S(s) \ll 1$

Feedback Control

Why feedback?

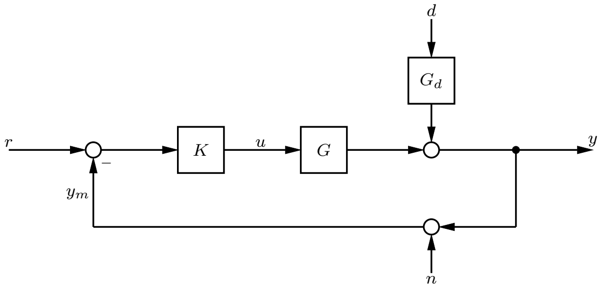


"Perfect" control can be obtained, even without feedback $(K = 0)$ by

$$K_r(s) = G^{-1}(s); \qquad K_d(s) = G^{-1}(s)G_d(s)$$

Let $u = K_r r - K_d d$ and we get

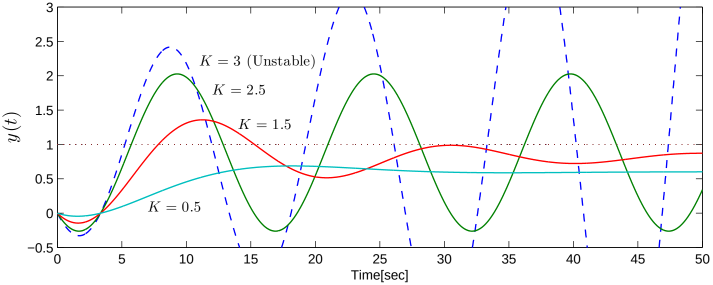
$$y = G(G^{-1}r - G^{-1}G_d d) + G_d d = r$$



The fundamental reasons for using feedback control are therefore the presence of

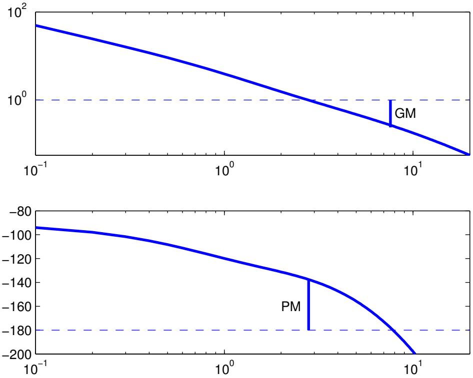
- Signal uncertainty – unknown disturbance (d)
- Model uncertainty (Δ)
- An unstable plant – unstable plants can only be stabilized by feedback.

Consider a system $G(s) = \dfrac{3(-2s+1)}{(5s+1)(10s+1)}$. If we close the feedback loop using a proportional gain K, the step responses of the closed-loop system $y(t)$ are shown below:



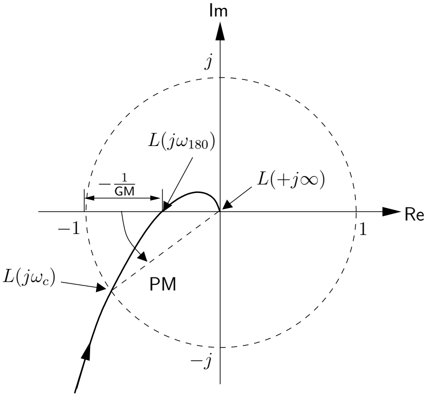
- Check the closed-loop poles That is the roots of $1 + L(s) = 1 + KG(s)$. The system is stable if and only if all roots are in left half plane (LHP).
- Use Bode' stability condition: Stability $\Leftrightarrow |L(j\omega_{180})| < 1$.
- Use Nyquist' stability criterion.

Bode plot
margins

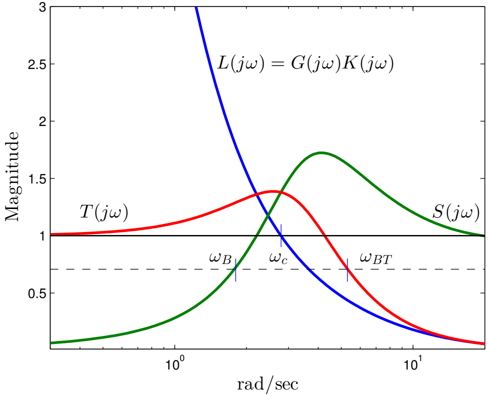


Nyquist plot

margins



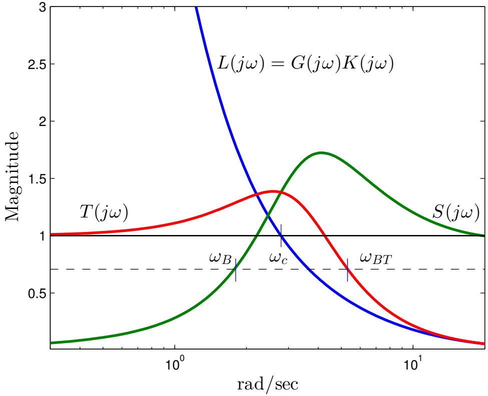
Closed-loop performance



$$G(s) = \frac{5e^{-0.1s}}{(s+1)(0.1s+1)} \quad K(s) = \frac{0.5s+1}{s}$$

Closed-loop performance

Bandwidth



- (Closed-loop) bandwidth ω_B : Frequency at which $|S(j\omega)| = -3\text{dB} = 1/\sqrt{2}$
- Crossover frequency ω_c : Frequency at which $|L(j\omega)| = 1$.
- if PM $< 90^\circ$ then $\omega_B < \omega_c < \omega_{BT}$

Loop shaping

- the classical loop-shaping approach is to shape the magnitude of the loop transfer function $L(s) = G(s)K(s)$ via the designed controller $K(s)$.
- Trade-offs in terms of L

$$e = y - r = -\underbrace{(I+L)^{-1}}_{S} r + \underbrace{(I+L)^{-1}}_{S} G_d d - \underbrace{(I+L)^{-1} L}_{T} n$$

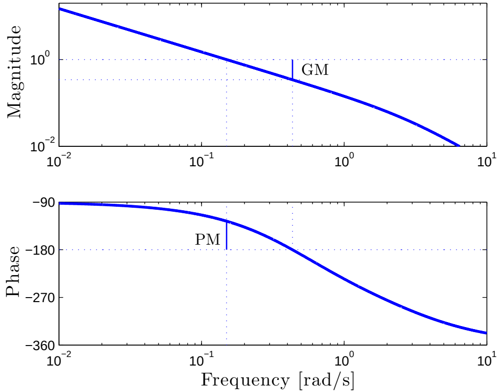
  - Performance, good disturbance rejection: needs large controller gains, i.e. L large.
  - Good command following, Stabilization of unstable plant: L large
  - Reduce measurement noise, Nominal stability, Robust stability: L small.

- We need a large loop gain $(|L| > 1)$ at low frequencies below crossover, and a small gain $(|L| < 1)$ at high frequencies above crossover.
  - Define he gain crossover frequency, ω_c, where $|L(j\omega_c)| = 1$.
  - We desire a slope of 20 dB/dec (or 1 in log-log scale) around crossover, and large roll-off at high frequencies. The desired slope at lower frequencies depends on the nature of the disturbance or reference signal.
  - Define the system type that is the number of pure integrators in $L(s)$.

Loop shaping

Example

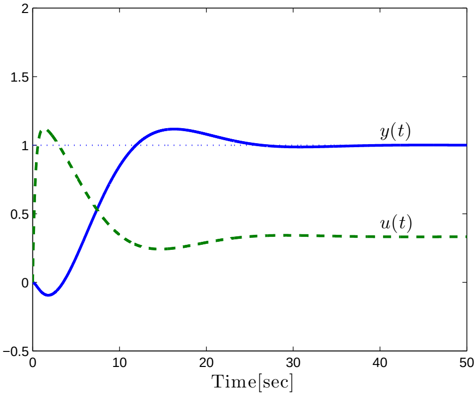
Consider $G(s) = \dfrac{3(-2s+1)}{(5s+1)(10s+1)}$ and $L(s) = 3K\dfrac{(-2s+1)}{s(2s+1)(0.33s+1)}$



The slope of $|L|$ is -1 up to 3 rad/s where it changes to -2.

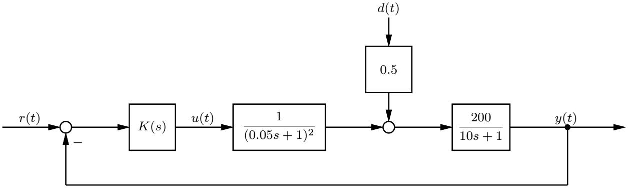
Loop shaping
Example

The controller corresponding to the loop-shape is $K(s) = 0.05\dfrac{(10s+1)(5s+1)}{s(2s+1)(0.33s+1)}$.

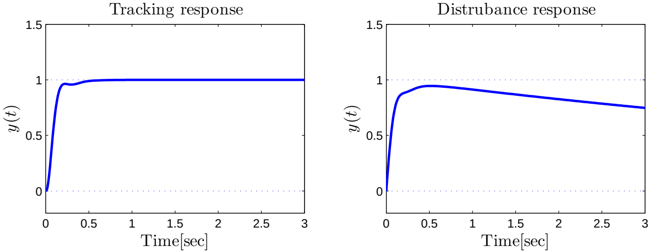


Response to step input of the loop-shaping design.

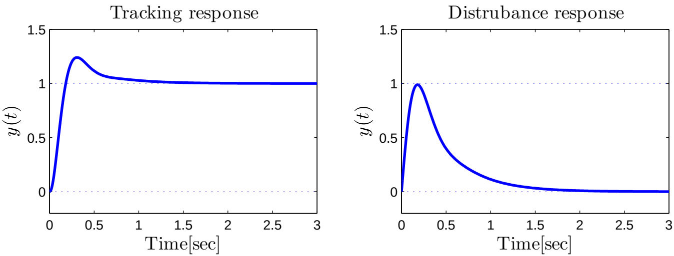
Disturbance rejection
Example



Using a controller $K_0(s) = \frac{1}{s}\frac{10s+1}{200}\frac{0.1s+1}{0.01s+1}$, which is for reference tracking objective we have

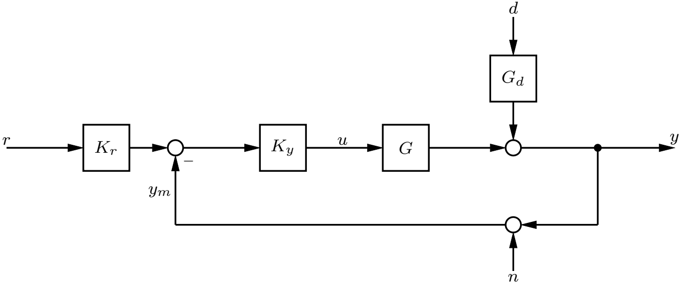


Same problem but this time we concentrate on disturbance rejection. The controller is then $K_3(s) = 0.5\dfrac{s+2}{s}\dfrac{0.05s+1}{0.005s+1}$. We have



- reference tracking is not good: large overshoot.
- disturbance rejection is good.
- to solve both objectives simultaneous, one can use a two-degree-of-freedom controller.

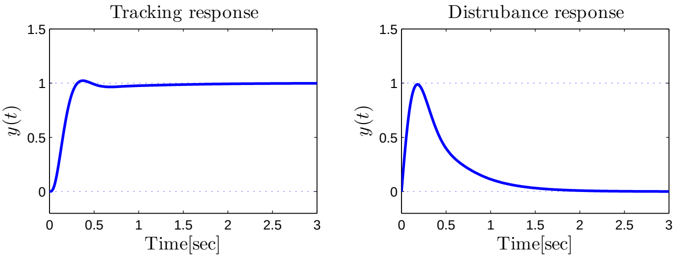
Two-degrees-of-freedom control



- We use the same feedback controller as in previous example.
- Here $K_r = \frac{0.5s+1}{0.65s+1}\frac{1}{0.03s+1}$.
- K_r can be used to improve the tracking reference separately from the disturbance rejection

$$T_{fb} = K_r \frac{GK_y}{1+GK_y}, \qquad T_d = G_d \frac{1}{1+GK_y}$$

Two-degrees-of-freedom control



- Finally, we are able to satisfy all requirement using a two-degree-of-freedom controller.

Reference

1. S. Skogestad and I. Postelthwaite, *Multivariable Feedback Control: Analysis and Design*, 2nd Edition, Wiley.
2. J. C. Doyle, B. A. Francis and A. R. Tannenbaum, *Feedback Control Theory*, McMillan, 1992.