Lecture 12: Robust Stability and Robust Performance Analysis and Synthesis

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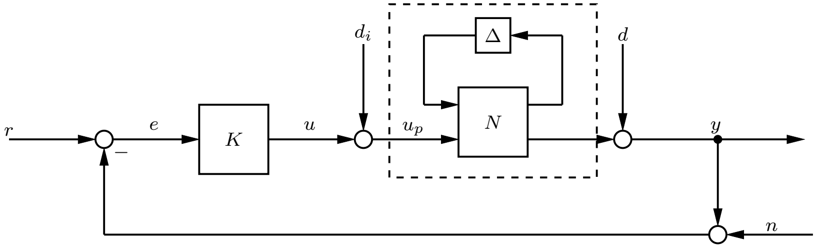
Feedback System with Uncertainty



$$P = \mathcal{F}_u(N, \Delta), \qquad \|\Delta\|_\infty \leq 1$$

where

- N is a nominal plant
- Δ is possibly a diagonal matrix with real and dynamic uncertainties.

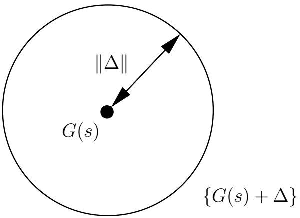
- **Nominal stability (NS)**: Feedback system is internally stable when $\Delta = 0$.
- **Robust stability (RS)**: Feedback system is internally stable for any norm-bounded Δ.
- **Nominal performance (NP)**: Feedback system is stable and satisfies certain performance for $\Delta = 0$.
- **Robust performance (RP)**: Feedback system is stable and satisfies certain performance for any norm-bounded Δ.

Model sets:

$$G_p(s) \in \{G(s) + \Delta | \quad \|\Delta\| \leq \gamma\}$$

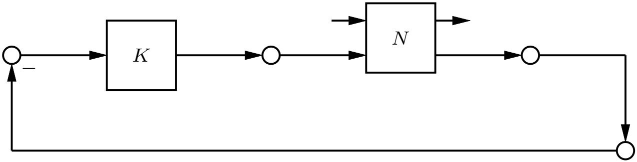
$G(s) =$ Nominal plant

$\Delta =$ unknown, but bounded perturbation (i/o operator)



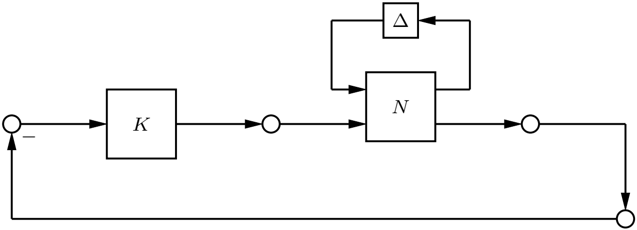
Typically, Δ is stable, causal and satisfies, $\|\Delta\|_\infty \leq \gamma$.

Norminal Stability



- **Analysis:** Given a controller K, check if the feedback (FB) system above is internally stable.
- **Synthesis:** Design K such that the feedback system is internally stable.

Robust Stability



- **Analysis:** Given nominally internally stabilizing controller K, check if the feedback system above is internally stable for all stable *structured* Δ with $\|\Delta\|_\infty \leq 1$
- **Synthesis:** Design K such that the feedback system is robustly internally stable.

Structure of an Uncertainty

An uncertainty is called **structured** if it has a fixed structure, e.g.,

- Some components are zero
- Some components are real, or dynamic uncertainty
- Some components are the same uncertainty

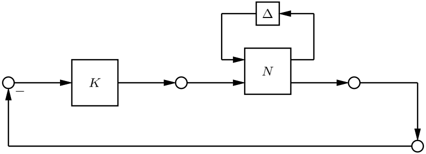
$$\Delta = \begin{bmatrix} \delta_1 & & & & \\ & \delta_2 & & & \\ & & \delta_3 I_2 & & \\ & & & \Delta_4(s) & \\ & & & & \Delta_5(s) \end{bmatrix}$$

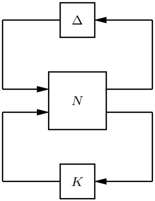
where each δ_i and $\Delta_j(s)$ represents a specific source of uncertainty

- δ_1, δ_2, $\delta_3 \in \mathbb{R}$,
- $\Delta_4(s)$, $\Delta_5(s) \in \mathcal{H}_\infty$ are set of stable functions

LFT Representation

For analysis and synthesis purpose, we use an LFT representation by extracting K:

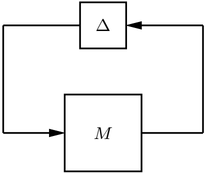




$-K$ is redefined as K

Robust Stability Condition

- Assume that fixed $M(s)$ and a structured uncertain $\Delta(s)$ are stable
- FB system below is internally stable for any structured Δ with $\|\Delta\|_\infty < 1$



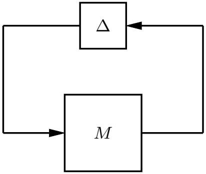
if and only if

$$\det(I - M(j\omega)\Delta(j\omega)) \neq 0, \qquad \forall \omega$$
$$\forall \Delta : \text{ structured } , \|\Delta\|_\infty \leq 1$$

- This condition is impractical to check because it involves uncertain Δ.

Robust Stability Condition
Special case

- Assume that a fixed $M(s)$ and an unstructured uncertain $\Delta(s)$ are stable.
- FB system below is internally stable for any unstructured Δ with $\|\Delta\|_\infty \le 1$ if and only if



$$\|M\|_\infty < 1$$

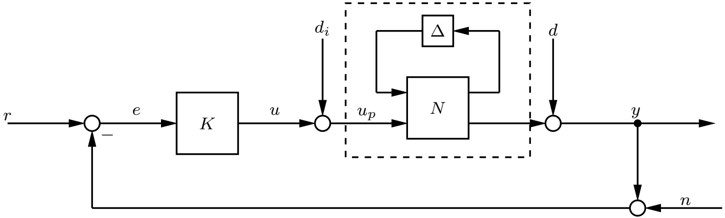
- This condition is practical because the condition is without Δ.

Remarks on Robust Stability

- For unstructured uncertainty,
  - Analysis is a computation of $\mathcal{H}_\infty$ norm of a system.
    - We study the Bounded Real Lemma
    - In MATLAB, use `norm(sys,inf)`
  - Robust stabilization is by $\mathcal{H}_\infty$ controller design
    - In MATLAB, use `hinfsyn(sys)`
- For structured uncertainty
  - Analysis is by μ-analysis
  - Robust stabilization is by μ-synthesis.

Robust Performance

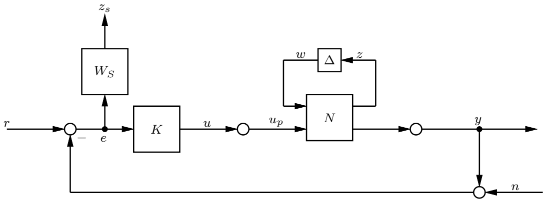
- **Analysis**: Given robustly internally stabilizing K, check if the feedback system below satisfies performance for all stable *structured* Δ with $\|\Delta\|_\infty \leq 1$

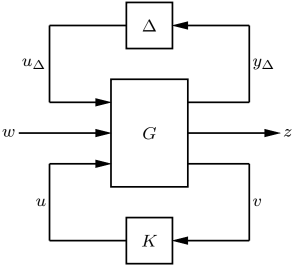


- $\|W_S S\|_\infty < 1$
- **Synthesis**: Design K such that the feedback system satisfies robust performance.

LFT Representation

- For analysis and synthesis purpose, we use an LFT representation by attracting K:





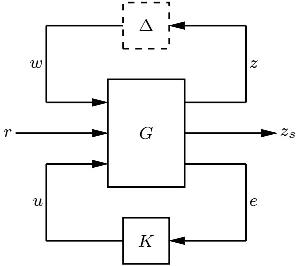
Nominal Performance Condition

- **Analysis**: Given a nominally stabilizing K, check if

$$\|T_{z_s r}\|_\infty < 1$$

- **Synthesis**: Design a nominally stabilizing K such that

$$\|T_{z_s r}\|_\infty < 1$$

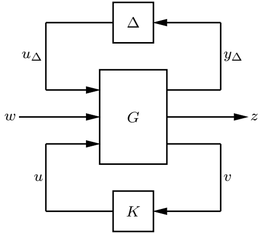


Robust Performance Condition

- **Analysis**: Given a robustly stabilizing K, check if

$$\|T_{z_s r}\|_\infty < 1, \forall\Delta$$
$$\Delta : \text{ Structured }, \|\Delta\|_\infty \leq 1$$

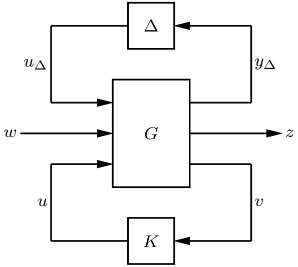
- **Synthesis**: Design a robustly stabilizing K such that the above condition is satisfied.



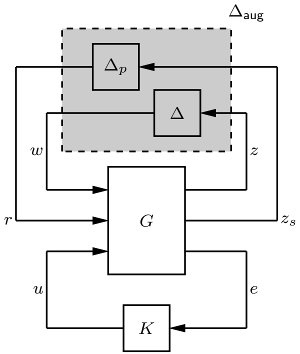
Robust Performance Condition

Reducing robust performance to robust stability

- Robust performance problems are equivalent to robust stability problems with augmented uncertainty



$$\|T_{z_s r}\|_\infty < 1$$



Remarks on NP and RP

- For nomianl performance
  - Analysis is computation for $\mathcal{H}_\infty$ norm of a system
  - Controller design for nominal performance is by $\mathcal{H}_\infty$ controller design
- For robust performance
  - Analysis is by μ-analysis
  - Robust stabilization by μ-synthesis
  - Same difficulty as the difficulty for robust stability analysis and robust stabilization for structured uncertainty.

Uncertain system

Example 1

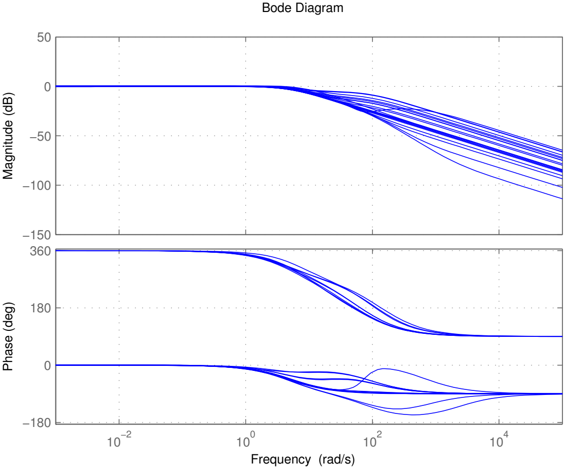
$$G(s) = \frac{1}{\frac{1}{b_w}s + 1}(1 + W(s)\Delta(s)) \qquad b_w = 5(1 + 0.1\delta), \qquad \delta \in [-1, 1]$$

$$W(s) = \frac{s + 9(0.05)}{\frac{s}{10} + 9}, \qquad \|\Delta\|_\infty \le 1$$

Uncertain system

```
clc; clear all;

bw = ureal('bw',5,'Percentage',10);
Gnom = tf(1,[1/bw 1]);

W = makeweight(0.05,9,10);
Delta = ultidyn('Delta',[1 1]);
G = Gnom*(1+W*Delta);
```

Uncertain system
Example 1



Uncertain system

Example 1

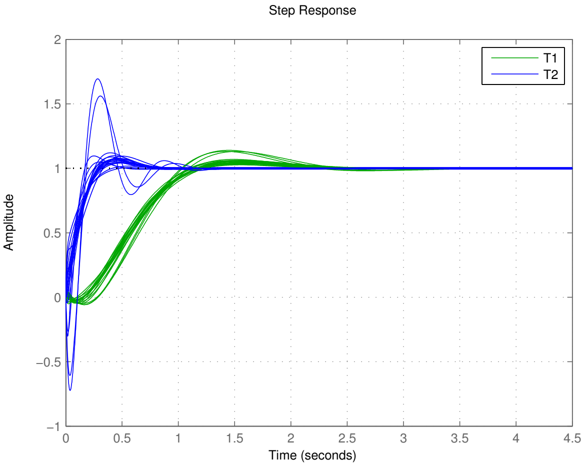
- PI controllers

```
xi = 0.707;
wn = 3;
K1 = tf([(2*xi*wn/5-1) wn*wn/5],[1 0]);
wn = 7.5;
K2 = tf([(2*xi*wn/5-1) wn*wn/5],[1 0]);
```

- Complementary sensitivity functions

```
T1 = feedback(G*K1,1);
T2 = feedback(G*K2,1);
```

Uncertain system
Example 1



Uncertain system
Example 1

- Robust stability analysis

```
[stabmarg1,destabu1,report1] = robuststab(T1)
```

```
stabmarg1 =     LowerBound: 4.0323
                UpperBound: 4.0323
    DestabilizingFrequency: 4.0938
report1 = Uncertain system is robustly stable to
    modeled uncertainty.
 -- It can tolerate up to 403% of the modeled
    uncertainty.
 -- A destabilizing combination of 403% of the
    modeled uncertainty was found.
 -- This combination causes an instability at 4.09
    rad/seconds.
```

Uncertain system

Example 1

- Robust stability analysis

```
[stabmarg2,destabu2,report1] = robuststab(T2)
```

```
stabmarg2 =      LowerBound: 1.2616
                 UpperBound: 1.2616
    DestabilizingFrequency: 9.8187
report1 = Uncertain system is robustly stable to
    modeled uncertainty.
 -- It can tolerate up to 126% of the modeled
    uncertainty.
 -- A destabilizing combination of 126% of the
    modeled uncertainty was found.
 -- This combination causes an instability at 9.82
    rad/seconds.
```

Uncertain system

Example 1

- Sensitivity peak analysis

```
S1 = feedback(1,G*K1);
S2 = feedback(1,G*K2);
[maxgain1,wcu1] = wcgain(S1);
[maxgain2,wcu2] = wcgain(S2);
```

Uncertain system

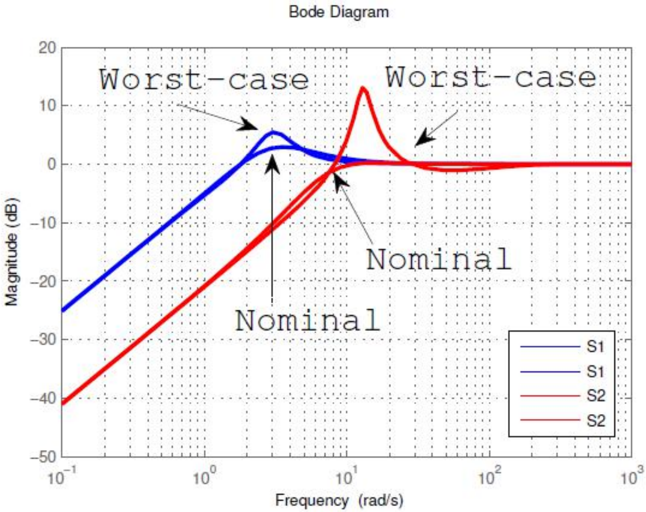
Example 1

```
>> maxgain1
maxgain1 =
           LowerBound: 1.8778
           UpperBound: 1.8779
    CriticalFrequency: 3.0583
>> maxgain2
maxgain2 =
           LowerBound: 4.5400
           UpperBound: 4.5402
    CriticalFrequency: 13.1431
```

```
bodemag(S1.NominalValue,'b',usubs(S1,wcu1),'b');
hold on, grid on
bodemag(S2.NominalValue,'r',usubs(S2,wcu2),'r');
hold off
```

Uncertain system

Example 1



Uncertain system
Example 2

$$G(s) = \left[\begin{array}{cc|cc} 0 & p & 1 & 0 \\ -p & 0 & 0 & 1 \\ \hline 1 & p & 0 & 0 \\ -p & 1 & 0 & 0 \end{array}\right] \times \left(1 + \begin{bmatrix} W_1(s)\Delta_1(s) & 0 \\ 0 & W_2(s)\Delta_2(s) \end{bmatrix}\right)$$

$$\begin{aligned} p &= 10(1 + 0.1\delta), & \delta &\in [-1, 1] \\ W_1(s) &= \frac{s + 20 \cdot 0.1}{\frac{s}{50} + 20}, & \|\Delta_1\|_\infty &\leq 1 \\ W_2(s) &= \frac{s + 45 \cdot 0.2}{\frac{s}{50} + 45}, & \|\Delta_2\|_\infty &\leq 1 \end{aligned}$$

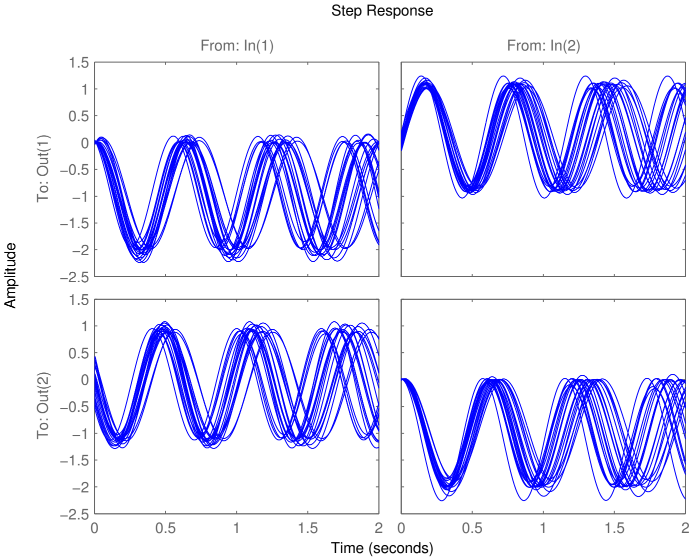
Uncertain system

Example 2

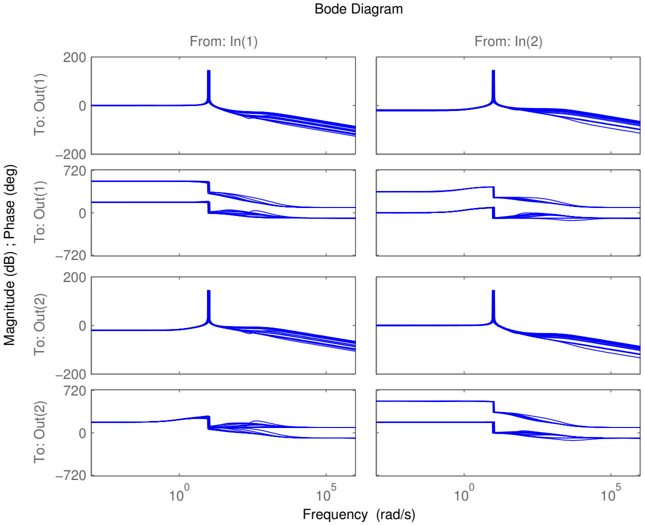
```
p = ureal('p',10,'Percentage',10);
A = [0 p; -p 0]; B = eye(2);
C = [1 p; -p 1];
H = ss(A,B,C,[0 0; 0 0]);


W1 = makeweight(0.1,20,50);
W2 = makeweight(0.2,45,50);
Delta1 = ultidyn('Delta1',[1 1]);
Delta2 = ultidyn('Delta2',[1 1]);

G = H*blkdiag(1+W1*Delta1, 1+W2*Delta2);
```

Uncertain system

Example 2

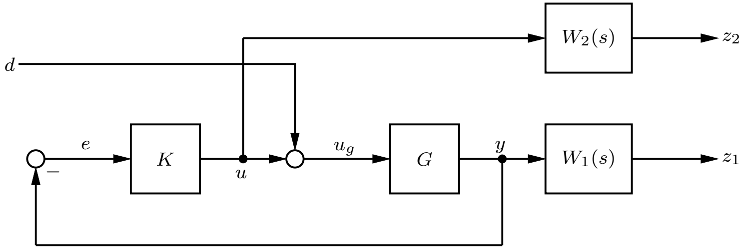


Uncertain system
Example 2



Uncertain system

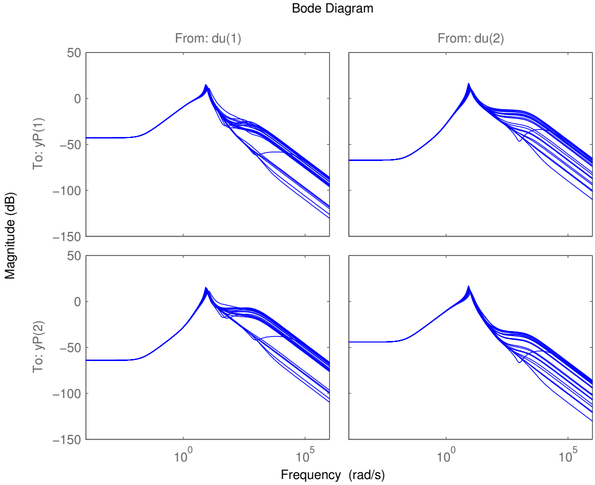
Closed-loop robust analysis



$$L_i = KP, \qquad S_i = (1 + L_i)^{-1}, \qquad T_i = I - S_i$$
$$L_o = KP, \qquad S_o = (1 + L_o)^{-1}, \qquad T_o = I - S_o$$

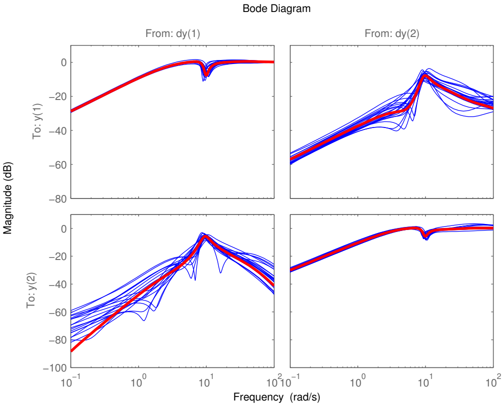
```
>> load mimoKexample
>> F = loopsense(G,K)
```

The transmission of disturbances at the plant input to the plant output



Bode magnitude of the nominal output sensitivity function.

```
bodemag(F.So,'b',F.So.NominalValue,'r',{1e-1 100})
```



- Nominal peak gain (largest singular value)

```
PeakNom =
    1.1317
freq =
    7.0483
```

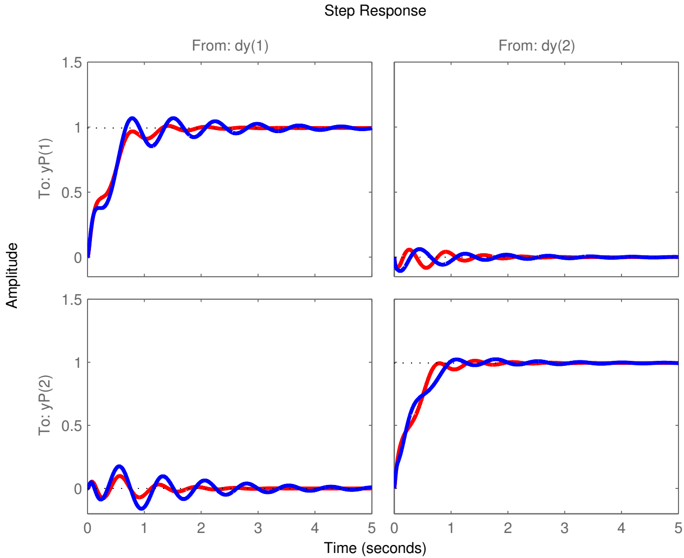
- Worst-case gain

```
[maxgain,wcu] = wcgain(F.So)
maxgain =
           LowerBound: 2.1459
           UpperBound: 2.1466
    CriticalFrequency: 8.4435
```

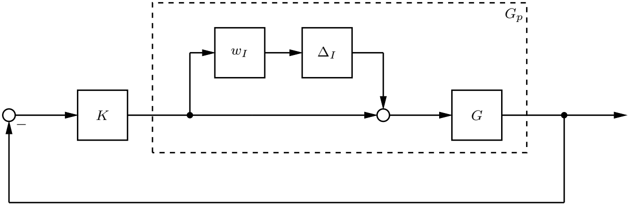
- The analysis indicates that the worst-case gain is somewhere between 2.1 and 2.2. The frequency where the peak is achieved is about 8.5.
- We can replace the values of `Delta1`, `Delta2` and `p` that achieve the gain of 2.1, using `usubs`

```
step(F.To.NominalValue,'r',usubs(F.To,wcu),'b',5)
```

- The perturbed response, which is the worst combination of uncertain values in terms of output sensitivity amplification, does not show significant degradation of the command response.
- The setting time is increased by about 50%, from 2 to 4, and the off-diagonal coupling is increased by about a factor of about 2, but is still quite small.



Step Response
From: dy(1)
From: dy(2)
Amplitude
To: yP(1)
To: yP(2)
1.5
1
0.5
0
0
1
2
3
4
5
Time (seconds)

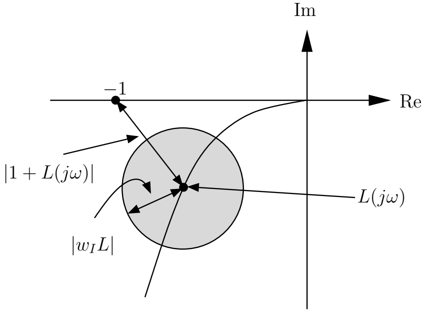


The loop transfer function is

$$L_p = G_P K = GK(1 + w_I \Delta_I) = L + w_I L \Delta_I, \qquad |\Delta_I(j\omega)| \leq 1, \forall \omega$$

- the system is NP and L_p is stable

$$\begin{aligned} \text{RS} \;&\Leftrightarrow\; \text{System stable } \forall L_p \\ &\Leftrightarrow\; L_p \text{ should not encircle the point } -1, \quad \forall L_p \end{aligned}$$



- $|-1-L| = |1+L|$ is the distance from the point -1 to the center of the disc representing L_p, and $|w_I L|$ is the radius of the disc.

$$\text{RS} \Leftrightarrow \quad |w_I L| < |1+L|, \quad \forall\omega \quad \Leftrightarrow \quad \left|\frac{w_I L}{1+L}\right| < 1, \quad \forall\omega$$

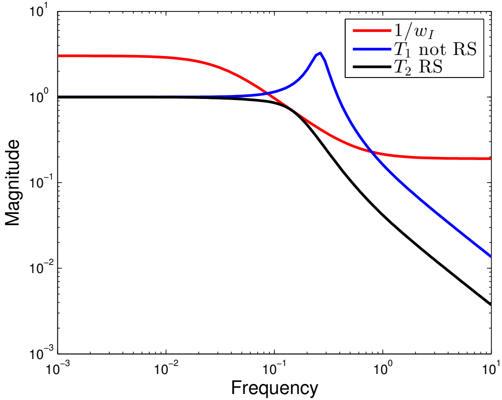
$$\Leftrightarrow \quad |w_I T| < 1, \quad \forall\omega \quad \Leftrightarrow \quad \|w_I T\|_\infty < 1$$

SISO Robust Stability
Example

Consider the following nomianl plant and PI-controller

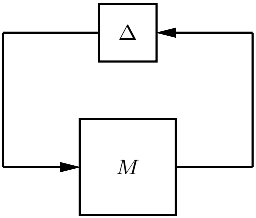
$$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)},\ K(s) = K_c\frac{12.7s+1}{12.7s},\ w_I(s) = \frac{10s+0.33}{(10/5.25)s+1},$$

$$K_{c1} = 1.13, K_{c2} = 0.31$$



Consider a transfer function of the Δ output to Δ input of the feedback system with multiplicative uncertainty. We have

$$w_I K(1 + GK)^{-1} G = w_I T = M$$

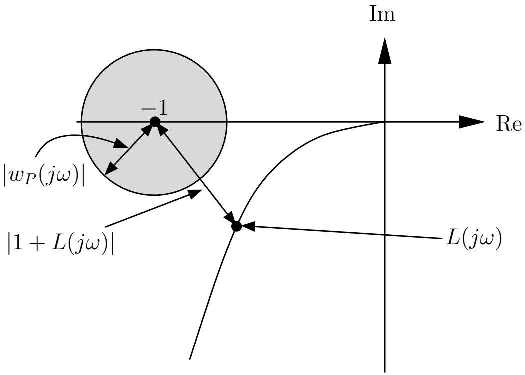


- The Nyquist stability condition then determines RS if and only if the "loop transfer function" $M\Delta$ does not encircle -1 for all Δ.

$$\text{RS} \Leftrightarrow |1 + M\Delta| > 0, \quad \forall\omega, \forall|\Delta| \leq 1$$

The condition is most easily violated (the worst case) when Δ is selected at each frequency such that $|\Delta| = 1$ and the terms $M\Delta$ and 1 have opposite signs (point to the opposite direction). We therefore get

$$\begin{aligned} \text{RS} &\Leftrightarrow \quad 1 - |M(j\omega)| > 0, \quad \forall\omega \\ &\Leftrightarrow \quad |M(j\omega)| < 1, \quad \forall\omega \quad = \|\omega_I T\| < 1 \end{aligned}$$



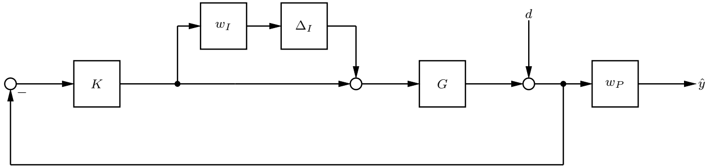
$$\text{NP} \quad \Leftrightarrow \quad |w_P S| < 1 \quad \forall\omega \quad \Leftrightarrow \quad |w_P| < |1 + L| \quad \forall\omega$$

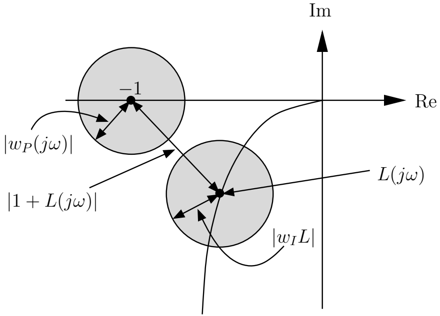
For robust performance we need the previous condition to be satisfied for all possible plants, that is, including the worst-case uncertainty.

$$\begin{aligned} \text{RP} \quad &\Leftrightarrow \quad |w_P S_p| < 1 \quad \forall S_p, \forall \omega \\ &\Leftrightarrow \quad |w_P| < |1 + L_p| \quad \forall L_p, \forall \omega \end{aligned}$$

This corresponds to requiring $|\hat{y}/d| < 1 \; \forall \Delta_I$, where we consider multiplicative uncertainty, and the set of possible loop transfer functions is

$$L_p = G_p K = L(1 + w_I \Delta_I) = L + w_I L \Delta_I$$





For RP we must require that all possible $L_p(j\omega)$ stay outside a disc of radius $|w_P(j\omega)|$ centered on -1. Since L_p at each frequency stays within a disc of radius $w_I L$ centered on L, we see that the condition for RP is that the two discs, with radii $|w_P|$ and $|w_I L|$, do not overlap.

Since their centers are located a distance $|1+L|$ apart, the RP-condition becomes

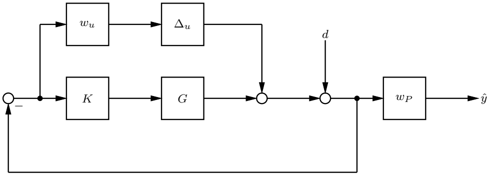
$$\begin{aligned} \text{RP} \quad &\Leftrightarrow \quad |w_P| + |w_I L| < |1+L|, \quad \forall\omega \\ &\Leftrightarrow \quad |w_P(1+L)^{-1}| + |w_I L(1+L)^{-1}| < 1, \quad \forall\omega \end{aligned}$$

or in other words

$$\text{RP} \quad \Leftrightarrow \quad \max_{\omega} \left(|w_P S| + |w_I T|\right) < 1$$

Consider robust performance of the SISO system in Figure, for which we have

$$\text{RP} \quad \Leftrightarrow \quad \left|\frac{\hat{y}}{d}\right| < 1, \quad \forall\omega; \quad w_P(s) = 0.25 + \frac{0.1}{s}; \quad w_u(s) = r_u\frac{s}{s+1}$$



- Derive a condition for robust performance (RP).
- For what values of r_u is it impossible to satisfy the robust performance condition?
- Let $r_u = 0.5$, consider two cases for the nominal loop transfer function: 1) $GK_1(s) = 0.5/s$ and 2) $GK_2(s) = \frac{0.5}{s}\frac{1-s}{1+s}$. For each system, sketch the magnitudes of S and its performance bound as a function of frequency. Does each system satisfy robust performance?

a) the requirement for RP is $|w_P S_p| < 1, \forall S_p, \forall \omega$, where the possible sensitivity are given by

$$S_p = \frac{1}{1 + GK + w_u \Delta_u} = \frac{S}{1 + w_u \Delta_u S}$$

The condition for RP then becomes

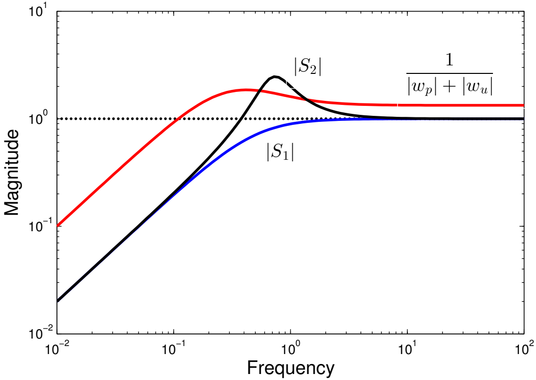
$$\text{RP} \quad \Leftrightarrow \quad \left| \frac{w_P S}{1 + w_u \Delta_u S} \right| < 1, \quad \forall \Delta_u, \forall \omega$$

A simple analysis shows that the worst case corresponds to selecting Δ_u with magnitude 1 such that the term $w_u \Delta_u S$ is purely real and negative, and hence we have

$$\begin{aligned} \text{RP} \quad &\Leftrightarrow \quad |w_P S| < 1 - |w_u S|, \quad \forall \omega \\ &\Leftrightarrow \quad |w_P S| + |w_u S| < 1, \quad \forall \omega \\ &\Leftrightarrow \quad |S(j\omega)| < \frac{1}{|w_P(j\omega)| + |w_u(j\omega)|}, \quad \forall \omega \end{aligned}$$

SISO Robust Performance
Example

b) Since any real system is strictly proper we have $|S| = 1$ at high frequencies and therefore we must require $|w_u(j\omega)| + |w_P(j\omega)| < 1$ as $\omega \to \infty$. With the weight given, this is equivalent to $r_u + 0.25 < 1$. Therefore, we must at least require $r_u < 0.75$ for RP, so RP cannot be satisfied if $r_u \geq 0.75$.



SISO Robust Performance
Example

c) Design S_1 yields RP, while S_2 does not. This is seen by checking the RP-condition graphically as shown in Figure above; $|S_1|$ has a peak of 1 while $|S_2|$ has a peak of about 2.45.

General Control Configuration with Uncertainty

The uncertain perturbations in a block diagonal matrix,

$$\Delta = \operatorname{diag}\{\delta_i, \Delta_j\} = \begin{bmatrix} \delta_1 I & & & \\ & \ddots & & \\ & & \Delta_j & \\ & & & \ddots \end{bmatrix}$$

where each δ_i, Δ_j represents a specific source of uncertainty

$$\begin{aligned} \Delta_j &= \text{ input uncertainty} \\ \delta_i &= \text{ parametric uncertainty where } \delta_i \text{ is real.} \end{aligned}$$

General Control Configuration with Uncertainty

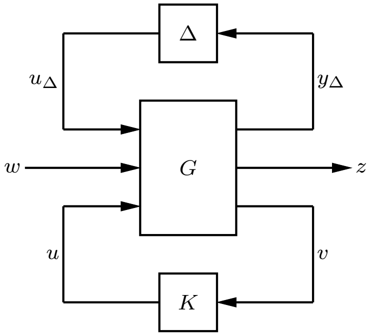


Figure: General control configuration

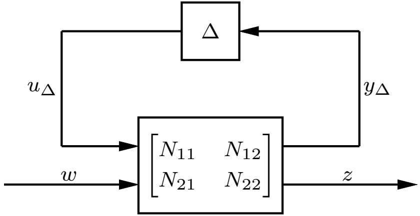


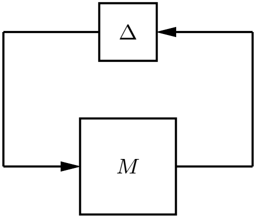
Figure: $N\Delta$-structure for robust performance analysis

$$N = \mathcal{F}_l(P, K) \triangleq P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

$$F = \mathcal{F}_u(N, \Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

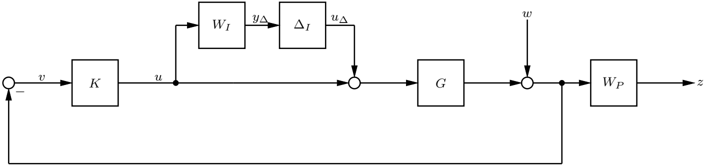
General Control Configuration with Uncertainty

$M\Delta$-structure for robust stability analysis



To analyze robust stability of M, we can rearange the system into the $M\Delta$-structure where $M = N_{11}$ is the transfer function from the output to the input of the perturbations.

Obtaining P, N and M



The inputs are $\begin{bmatrix} u_\Delta & w & u \end{bmatrix}^T$ and outputs $\begin{bmatrix} y_\Delta & z & v \end{bmatrix}^T$. By writing down the equations we get

$$P = \begin{bmatrix} 0 & 0 & W_I \\ W_P G & W_P & W_P G \\ -G & -I & -G \end{bmatrix}, \quad P_{11} = \begin{bmatrix} 0 & 0 \\ W_P G & W_P \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} -G & -I \end{bmatrix}, \quad P_{22} = -G.$$

Obtaining P, N and M

Find N from $N = \mathcal{F}_l(P, K)$ or directly from the system we get

$$N = \begin{bmatrix} -W_I KG(I+KG)^{-1} & -W_I K(I+GK)^{-1} \\ W_P G(I+KG)^{-1} & W_P(I+GK)^{-1} \end{bmatrix}$$

The upper left block, N_{11} is the transfer function from u_Δ to y_Δ. This is the transfer function M for $M\Delta$-structure for evaluating robust stability. Thus, we have

$$M = -W_I KG(I+KG)^{-1} = -W_I T_I$$

Robust Stability of the $M\Delta$-Structure

Consider the uncertain $N\Delta$-system for which the transfer function from w to z is given by

$$\mathcal{F}_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

- Suppose the system is nominally stable (with $\Delta = 0$), that is, N is stable (which means that the whole of N, and not only N_{22} must be stable).
- The only possible source of instability is the feedback term $(I - N_{11}\Delta)^{-1}$.
- The nominal stability (NS), the stability of the system is equivalent to the stability of the $M\Delta$-structure where $M = N_{11}$.

Robust Stability of the $M\Delta$-Structure

Theorem (Determinant stability condition)

For a fixed stable $M(s)$, the $M\Delta$-structure system is internally stable for any structured Δ with $\|\Delta\|_\infty \leq 1$ if and only if

$$\text{Nyquist plot of } \det(I - M\Delta(s)) \text{ does not encircle the origin } \forall\Delta \tag{1}$$

$$\Leftrightarrow \det(I - M\Delta(j\omega)) \neq 0, \quad \forall\Delta \tag{2}$$

$$\Leftrightarrow \lambda_i(M\Delta) \neq 1, \quad \forall i, \forall\omega, \forall\Delta \tag{3}$$

Proof:

- The first condition is simply the generalized Nyquist Theorem applied to a positive feedback system with a stable loop transfer function $M\Delta$.

Robust Stability of the $M\Delta$-Structure

- (1) $\Rightarrow$ (2): This is obvious sine by "encirclement of the origin" we also include the origin itself.
- (2) $\Leftarrow$ is proved by proving not(1) $\Rightarrow$ not(2): First note that with $\Delta = 0$, $\det I - M\Delta = 1$ at all frequencies. Assume there exists a perturbation Δ' such that the image of $\det(I - M\Delta'(s))$ encircles the origin as s traverses the Nyquist $\mathcal{D}$-contour. Because the Nyquist contour and its map is closed, there then exists another perturbation in the set, $\Delta'' = \epsilon\Delta'$ with $\epsilon \in [0, 1]$, and an ω' such that $\det(I - M\Delta''(j\omega')) = 0$.
- (3) is equivalent to (2) since $\det(I - A) = \prod_i \lambda_i(I - A)$ and $\lambda_i(I - A)$ and $\lambda_i(I - A) = 1 - \lambda_i(A)$.

Robust Stability of the $M\Delta$-Structure

Theorem (Spectral radius condition for complex perturbations)

Assume that the nominal system $M(s)$ and the perturbations $\Delta(s)$ are stable. Consider the class of perturbations, Δ, such that if Δ' is an allowed perturbation then so is $c\Delta'$ where c is any complex scalar such that $|c| \leq 1$. Then the $M\Delta$-system is stable for all allowed perturbations if and only if

$$\rho(M\Delta(j\omega)) < 1, \quad \forall\omega, \forall\Delta \tag{4}$$

or equivalently

$$\textit{RS} \quad \Leftrightarrow \quad \max_{\Delta} \rho(M\Delta(j\omega)) < 1, \quad \forall\omega$$

RS for Complex Unstructured Uncertainty

Theorem (RS for Unstructured Perturbations)

Assume that the nominal system $M(s)$ is stable (NS) and that the perturbations $\Delta(s)$ are stable. Then the $M\Delta$-system is stable for all perturbations Δ satisfying $\|\Delta\|_\infty \leq 1$ if and only if

$$\bar{\sigma}(M(j\omega)) < 1, \quad \forall\omega \quad \Leftrightarrow \quad \|M\|_\infty < 1$$

Proof: We can show that

$$\det(I - M\Delta) \neq 0, \quad \forall\omega, \forall\Delta \quad \Leftrightarrow \quad \lambda_i(M\Delta) < 1, \quad \forall i, \forall\omega, \forall\Delta$$

For Δ that $\bar{\Delta} \leq 1$, we have

$$\max_{\Delta} \rho(M\Delta) = \max_{\Delta} \bar{\sigma}(M\Delta) = \max_{\Delta} \bar{\sigma}(M)\bar{\sigma}(\Delta) = \bar{\sigma}(M)$$

Then RS $\quad \Leftrightarrow \quad \bar{\sigma}(M(j\omega)) < 1, \quad \forall\omega.$

RS with Structured Uncertainty

- Consider the presence of structured uncertainty, where $\Delta = \mathrm{diag}\{\Delta_i\}$ is block diagonal. The test for robust stability is changed to

$$\text{RS} \quad \text{if} \quad \bar{\sigma}(M(j\omega)) < 1, \quad \forall \omega$$

  Here we write "if" rather than "if and only if" since this condition is only sufficient for RS when Δ has "no structure".

- To take the advantage of the fact that $\Delta = \mathrm{diag}\{\Delta_i\}$ is structured to obtain an RS-condition which is tighter than the unstructured one. We can use the block-diagonal scaling matrix

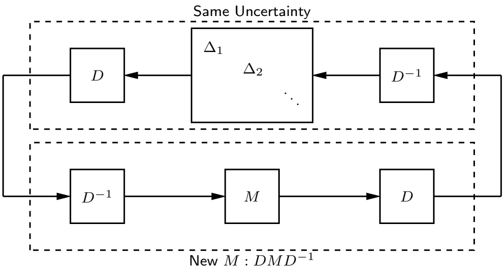
$$D = \mathrm{diag}\{d_i I_i\}$$

  where d_i is a scalar and I_i is an identity matrix of the same dimension as the Δ_i.

RS with Structured Uncertainty

- Moreover we have $\Delta D = D\Delta$. This means the RS condition must also apply if we replace M by DMD^{-1} and we have

$$\text{RS} \quad \text{if} \quad \bar{\sigma}(DMD^{-1}) < 1, \quad \forall\omega$$



Structured Singular Value μ

The structured singular value (μ) is a function which provides a generalization of the singular value, $\bar{\sigma}$, and the spectral radius, ρ. μ can be used to get necessary and sufficient conditions for RS and RP.

Definition (Structured Singular Value)

Let M be a given complex matrix and let $\Delta = \mathrm{diag}\{\Delta_i\}$ denote a set of complex matrices matrices with $\bar{\sigma}(\Delta) \leq 1$ and with a given block-diagonal structure. The real non-negative function $\mu(M)$, called the structured singular value, is defined by

$$\mu(M) \triangleq \left(\min_{\Delta} \{k_m | \det(I - k_m M\Delta) = 0, \quad \bar{\sigma}(\Delta) \leq 1\} \right)^{-1}$$

If no such structured Δ exists then $\mu(M) = 0$.

RS and RP with Structured Uncertainty

Theorem (RS for block-diagonal perturbations)

Assume that the nominal system M and the perturbations Δ are stable. Then the $M\Delta$-system is stable for all allowed perturbations with $\bar{\sigma}(\Delta) \leq 1, \quad \forall\omega$, if and only if

$$\mu(M(j\omega)) < 1, \qquad \forall\omega$$

Theorem (RP for block-diagonal perturbations)

Rearrange the uncertain system into the $N\Delta$-structure. Assume nominal stability such that N is stable. Then

$$\textit{RS} \quad \Leftrightarrow \quad \mu_{\hat{\Delta}}(N(j\omega)) < 1, \quad \forall\omega.$$

μ-Synthesis

- At present there is no direct method to synthesize a μ-optimal controller. However, for complex perturbations a method known as DK-iteration is available.
- The method combines $\mathcal{H}_\infty$-synthesis and μ-analysis, and often yields good results.
- The idea is to find the controller that minimizes the peak value over frequency of this upper bound, namely

$$\min_{K} \min_{D \in \mathcal{D}} \|DND^{-1}\|_\infty$$

by alternating between minimizing $\|DN(K)D^{-1}\|_\infty$ with respect to either K or D (while holding the other fixed).

DK-iteration

The DK-iteration proceeds as follows:

1. K-step: Synthesize and $\mathcal{H}_\infty$ controller for the scaled problem,

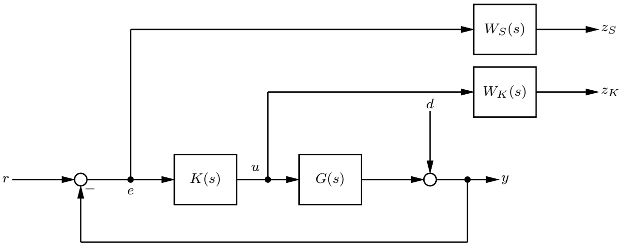
$$\min_K \|DN(K)D^{-1}\|_\infty \text{ with fixed } D(s)$$

2. D-step: Find $D(j\omega)$ to minimize at each frequency $\bar{\sigma}(DND^{-1}(j\omega))$ with fixed N.
3. Fit the magnitude of each element of $D(j\omega)$ to a stable and minimum phase transfer function $D(s)$ and go to Step 1.

Consider a two-input, two-output system with transfer function matrix

$$G(s) = \begin{bmatrix} \frac{k_1}{T_1 s+1} & -\frac{0.05}{0.1s+1} \\ \frac{0.1}{0.3s+1} & \frac{k_2}{T_2 s-1} \end{bmatrix}$$

where the coefficients k_1 and k_2 have nominal values 12 and 5, respectively, and relative uncertainty 15%, and the time constants T_1 and T_2 have nominal values 0.2 and 0.7, respectively, and relative uncertainty 20%



DK-iteration
Example

The closed-loop system is described by

$$z = T_{zw} w, \quad z = \begin{bmatrix} z_S \\ z_K \end{bmatrix}, \quad w = \begin{bmatrix} r \\ d \end{bmatrix}$$

The performance weighting and control weighting functions are

$$W_S(s) = \begin{bmatrix} w_S(s) & 0 \\ 0 & w_S(s) \end{bmatrix}, \quad W_K(s) = \begin{bmatrix} w_K(s) & 0 \\ 0 & w_K(s) \end{bmatrix},$$

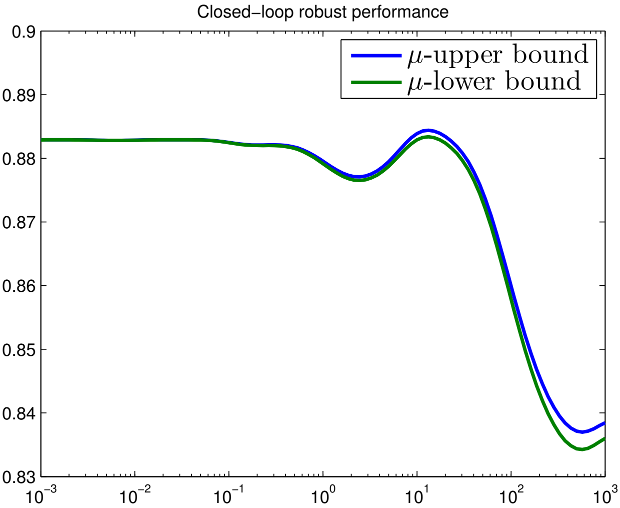
where

$$w_S(s) = 0.5 \frac{s + 10}{s + 0.3}, \quad w_K(s) = 0.1 \frac{0.001s + 1}{0.0001s + 1}.$$

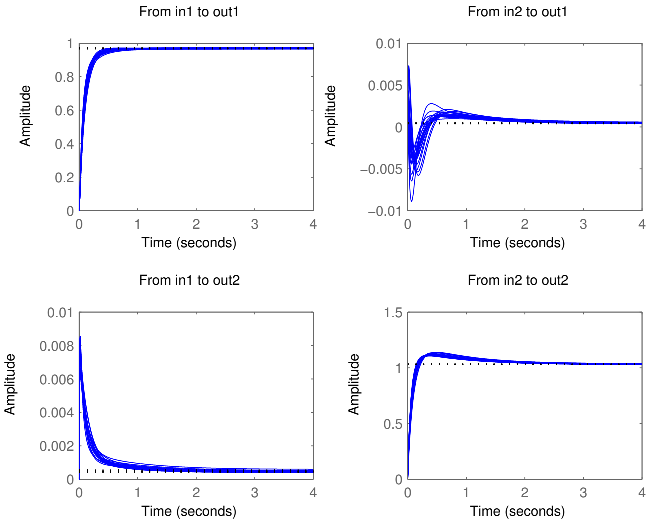
```
clc; clf;
s = tf('s');
k1 = ureal('k1',12,'Percentage',15);
k2 = ureal('k2',5,'Percentage',15);
T1 = ureal('T1',0.2,'Percentage',20);
T2 = ureal('T2',0.7,'Percentage',20);

G = [ k1/(T1*s+1), -0.05/(0.1*s+1);
      0.1/(0.3*s+1), k2/(T2*s-1)];
ws = 0.5*(s+10)/(s+0.3);
wk = 0.1*(0.001*s+1)/(0.0001*s+1);
WS = [ws 0 ; 0 ws];
WK = [wk 0 ; 0 wk];
```

```
systemnames = ' G WS WK';
inputvar = '[r{2}; d{2}; u{2}]';
outputvar = '[WS; WK; r-G-d]';  % e = r-G-d
input_to_G = '[ u ]';
input_to_WS = '[ r-G-d ]';
input_to_WK = '[ u ]';
sysIC = sysic;

nmeas = 2;
ncont = 2;
fv = logspace(-3,3,100);
opt = dkitopt('FrequencyVector', fv, ...
       'DisplayWhileAutoIter','on', ...
       'NumberOfAutoIterations',3)
[K,CL,BND,INFO] = dksyn(sysIC,nmeas,...
                         ncont,opt);
```

```
Iteration Summary
-------------------------------------------------
Iteration #                  1         2         3
Controller Order             8        20        22
Total D-Scale Order          0        12        14
Gamma Acheived           1.682     0.988     0.884
Peak mu-Value            1.567     0.987     0.884
```



Closed−loop robust performance
0.9
0.89
0.88
0.87
0.86
0.85
0.84
0.83
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
μ-upper bound
μ-lower bound



From in1 to out1
From in2 to out1
From in1 to out2
From in2 to out2
Amplitude
Time (seconds)
1
0.8
0.6
0.4
0.2
0
0.01
0.005
−0.005
−0.01
0.008
0.006
0.004
0.002
1.5
0.5
2
3
4

Reference

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2. Ryozo Nagamune "Lecture not on *Multivariable Feedback Control*", 2009
3. Sigurd Skogestad and Ian Postlethwaite, "Multivariable Feedback Control", 2008