Lecture 11: Linear Matrix Inequalities

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Linear Matrix Inequalities

A linear matrix inequality (LMI) has the form

$$M(p) = M_0 + p_1 M_1 + \cdots + p_N M_N < 0$$

where $M_0, M_1, \ldots, M_N$ are given symmetric matrices, $p = \begin{bmatrix} p_1 & p_2 & \ldots, p_N \end{bmatrix}^T$ is a column vector of real scalar variables.

- the matrix inequality $M(p) < 0$ means that the left hand side is negative definite.
- An important property of LMIs is that the set of all solutions p is convex.
- LMIs can be used as constraints for the minimization problem

$$\min_p c^T p \text{ subject to } M(p) < 0$$

  where the elements of the vector c in the linear cost function are weights on the individual decision variables.

Linear Matrix Inequalities

- the convex problem can be solved by efficient, polynomial-time interior-point methods.
- Several LMI constraints can be combined into a single constraint of type.
- for example the constraint

$$M_1(p) < 0 \quad \text{and} \quad M_2(p) < 0$$

is equivalent to the single LMI constraint

$$\begin{bmatrix} M_1(p) & 0 \\ 0 & M_2(p) \end{bmatrix} < 0$$

Pole Region Constraints

- the condition that the poles of a system are located within a given region in the complex plane can be formulated as an LMI constraint.
- the dynamic system $\dot{x}(t) = Ax(t)$. This system is stable if an only if the matrix A has all eigenvalues in the left half plane, which is true iff there exists a positive definite, symmetric matrix P that satisfies the Lyapunov inequality

$$PA^T + AP < 0$$

- This inequality is linear in the matrix variable P, and one can use LMI solvers to search for solutions.
- assume that A is a 2 by 2 matrix and write the symmetric matrix variable P as

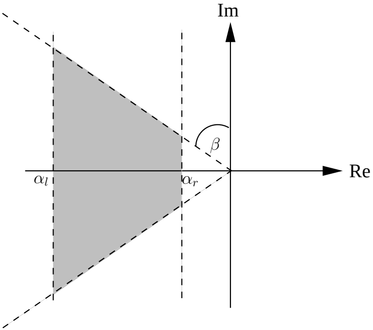
$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = p_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Pole Region Constraints

- the LMI represents a necessary and sufficient condition for the matrix A to have all eigenvalues in the left half plane.
- one can express an arbitrary region $\mathcal{D}$ in the complex plane in terms of two matrix $L = L^T$ and M as the set of all complex numbers that satisfy and LMI constraint.

$$\mathcal{D} = \{s \in \mathbb{C} : L + Ms + M^T \bar{s} < 0\}$$

  where $\bar{s}$ denotes the complex conjugate of s.
- Such a region is called an **LMI region**.



From $\mathrm{Re}\, s < \alpha_r$, we have

$$\frac{s + \bar{s}}{2} < \alpha_r$$

$$s + \bar{s} - 2\alpha_r < 0.$$

and $\operatorname{Im} s > \alpha_l$

$$\frac{s + \bar{s}}{2} > \alpha_l$$
$$-s - \bar{s} + 2\alpha_l < 0.$$

Thus

$$L_v = \begin{bmatrix} 2\alpha_l & 0 \\ 0 & -2\alpha_r \end{bmatrix}, \qquad M_v = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the conic sector, a complex number $s = x + jy$ lies in the conic sector if and only if

$$\left|\frac{x}{y}\right| < \tan\beta = \frac{\sin\beta}{\cos\beta} \quad \text{and} \quad x\cos\beta < 0.$$

Rewrite the above conditions in the form

$$\frac{x^2}{y^2} < \frac{\sin^2\beta}{\cos^2\beta}$$

Pole Region Constraints

Example: poles region constraint

we get

$$x^2 \cos^2 \beta - y^2 \sin^2 \beta < 0$$

$$x \cos \beta - \frac{y^2 \sin^2 \beta}{x \cos \beta} < 0$$

By Schur's complement we have

$$M_c s + M_c^* \bar{s} = \begin{bmatrix} 2x \cos \beta & 2jy \sin \beta \\ -2jy \sin \beta & 2x \cos \beta \end{bmatrix} < 0$$

Thus

$$L_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_c = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix}$$

The two constraints can be combined as

$$L = \begin{bmatrix} L_c & 0 \\ 0 & L_v \end{bmatrix}, \quad M = \begin{bmatrix} M_c & 0 \\ 0 & M_v \end{bmatrix}$$

Constraints on the $\mathcal{H}_\infty$ Norm

Consider the system with transfer function $T(s)$ as state space realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \qquad x(0) = 0 \\ z(t) &= Cx(t) + Dw(t) \end{aligned}$$

Assuming that $T(s)$ is stable, the $\mathcal{H}_\infty$ norm of the system is

$$\|T\|_\infty^2 = \max_{w \neq 0} \frac{\int_0^\infty z^T(t)z(t)dt}{\int_0^\infty w^T(t)w(t)dt}, \quad x(0) = 0.$$

It follows that $\|T\|_\infty < \gamma$ is equivalent to

$$\int_0^\infty (z^T(t)z(t) - \gamma^2 w^T(t)w(t))dt < 0$$

Holding true for all square integrable, non-zero $w(t)$.

Constraints on the $\mathcal{H}_\infty$ Norm

Introduce a Lyapunov function $V(x) = x^T P x$ with $P = P^T > 0$. Since $x(0) = x(\infty) = 0$, the constraint $\|T\|_\infty < \gamma$ is enforced by the existence of a matrix $P = P^T > 0$ such that

$$\frac{dV(x)}{dt} + \frac{1}{\gamma} z^T(t) z(t) - \gamma w^T(t) w(t) < 0$$

for all $x(t), w(t)$; to turn into a LMI, substitute

$$\frac{dV(x)}{dt} = x^T (A^T P + PA) x + x^T P B w + w^T B^T P x, \quad z = Cx + Dw$$

To obtain

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \frac{1}{\gamma} C^T C & PB + \frac{1}{\gamma} C^T D \\ B^T P + \frac{1}{\gamma} D^T C & -\gamma I + \frac{1}{\gamma} D^T D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

Constraints on the $\mathcal{H}_\infty$ Norm

For $\|T\|_\infty < \gamma$ the above must hold for all x and w, thus the block matrix must be negative definite. The condition can be rewritten as

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} < 0$$

By Schur complement, we have

Theorem (Bound real lemma)

$\|T\|_\infty < \gamma$ if and only if there exists a positive definite, symmetric matrix P that satisfies the linear matrix inequality

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

Constraints on the $\mathcal{H}_\infty$ Norm

Using the congruence transformation and $Q = P^{-1}$,

$$\begin{bmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \begin{bmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0.$$

An equivalent form

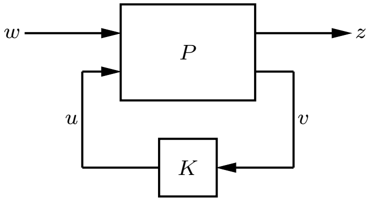
$$\begin{bmatrix} QA^T + AQ & B & QC^T \\ B^T & -\gamma I & D^T \\ CQ & D & -\gamma I \end{bmatrix} < 0$$

Constraints on the $\mathcal{H}_\infty$ Norm
Using cvx

```
sys = rss(3,3);
A = sys.a; B = sys.b; C = sys.c; D = sys.d;
n = size(A,1); nu = size(B,2); ny = size(D,1);

cvx_begin sdp
    variable P(n,n) symmetric
    variable gamma;

    minimize gamma;
    subject to
        P > 0;
        [A'*P + P*A, P*B, C';
           B'*P , -gamma*eye(nu), D';
           C, D, -gamma*eye(ny)] < 0;
cvx_end

display(P);
```

Controller Design Using LMIs

Generalized Plant



The generalized plant $P(s)$ has a state space realization

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_w w(t) + B_u u(t) \\
z(t) &= C_z x(t) + D_{zw} w(t) + D_{zu} u(t) \\
v(t) &= C_v x(t) + D_{vw} w(t)
\end{aligned}$$

Controller Design Using LMIs

$\mathcal{H}_\infty$ State Feedback

State feedback $u = Fx$ yields the closed-loop system

$$\dot{x}(t) = (A + B_u F)x(t) + B_w w(t)$$
$$z(t) = (C_z + D_{zu} F)x(t) + D_{zw} w(t)$$

Replacing the system matrices in Bounded real lemma by the closed-loop matrices and using the variable transformation $Y = FQ$ leads to the following result: a necessary and sufficient condition for a state feedback controller to achieve a $\mathcal{H}_\infty$-norm less than γ i s the existence of matrices $P = P^T > 0$ and Y that satisfy

$$\begin{bmatrix} QA^T + AQ + Y^T B_u^T + B_u Y & B_w & QC_z^T + Y^T D_{zu}^T \\ B_w^T & -\gamma I & D_{zw}^T \\ C_z Q + D_{zu} Y & D_{zw} & -\gamma I \end{bmatrix} < 0, \; F = YQ^{-1}$$

$$G(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ I & 0 & 0 \end{array}\right]$$

with (A, B_u) assumed to be stabilizable

minimize γ

subject to

$$Q = Q^T > 0$$

$$\begin{bmatrix} QA^T + AQ + Y^T B_u^T + B_u Y & B_w & QC_z^T + Y^T D_{zu}^T \\ B_w^T & -\gamma I & D_{zw}^T \\ C_z Q + D_{zu} Y & D_{zw} & -\gamma I \end{bmatrix} < 0$$

If this has a solution then

$$F = YQ^{-1}$$

```
G = ss(A, [Bw, Bu], [Cz; Cv], [Dzw, Dzu; Dvw, zeros(nv,nu)]);

cvx_begin sdp
    variable Q(n,n) symmetric;
    variable Y(nu,n);
    variable gamma;

    minimize gamma;
    subject to
        Q > 0;
        [Q*A' + A*Q + Bu*Y + Y'*Bu',    Bw,      Q*Cz' +Y'*Dzu';
                    Bw' ,          -gamma*eye(nu,nu),  Dzw';
           Cz*Q + Dzu*Y ,                Dzw,        -gamma*eye(nv,nv)] < 0;
cvx_end
    F = Y*inv(Q);

    % check closed-loop poles
    Aclp = A + Bu*F;
    disp(eig(Aclp));
```

The controller dynamics are represented by a state space model

$$\begin{aligned}\dot{\zeta}(t) &= A_K \zeta(t) + B_K v(t)\\ u(t) &= C_K \zeta(t) + D_K v(t)\end{aligned}$$

The state space realization of the closed loop-system is

$$\begin{aligned}\dot{x}_c(t) &= A_c x_c(t) + B_c w(t)\\ z(t) &= C_c x_c(t) + D_c w(t)\end{aligned}$$

where

$$A_c = \begin{bmatrix} A + B_u D_K C_v & B_u C_K \\ B_K C_v & A_K \end{bmatrix}, \quad B_c = \begin{bmatrix} B_w + B_u D_K D_{vw} \\ B_K D_{vw} \end{bmatrix},$$
$$C_c = \begin{bmatrix} C_z + D_{zu} D_K C_v & D_{zu} C_K \end{bmatrix}, \quad D_c = D_{zw} + D_{zu} D_K D_{vw}$$

$$P(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_v & D_{vw} & 0 \end{array}\right]$$

with (A, B_u) assumed to be stabilizable and (C_v, A) assumed to be detectable

for output feedback (assume $D_K = 0$):

$$u = K(s)y = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & 0 \end{array}\right] y$$

$$G(s) = \mathcal{F}_l(P(s), K(s)) = \left[\begin{array}{cc|c} A & B_u C_K & B_w \\ B_K C_v & A_K & B_K D_{vw} \\ \hline C_z & D_{zu} C_K & D_{zw} \end{array}\right]$$

$$G(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]$$

The LMI condition is :

$$\begin{bmatrix} A_c^T P + P A_c & P B_c & C_c^T \\ B^T c P & -\gamma I & D^T c \\ C_c & D_c & -\gamma I \end{bmatrix} < 0$$

Partition P as:

$$P = \begin{bmatrix} X & R \\ R^T & * \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} Y & S \\ S^T & * \end{bmatrix}$$

Define an inertia-preserving transform via:

$$P T_Y = T_X \text{ where } T_Y = \begin{bmatrix} Y & I \\ S^T & 0 \end{bmatrix}, \qquad T_X = \begin{bmatrix} I & X \\ 0 & R^T \end{bmatrix}$$

The matrices T_X and T_Y can be used to transform the nonlinear constraint into a linear one. This transformation is based on the fact that

$$T_Y^T P A_c T_Y = T_X^T A_c T_Y = \begin{bmatrix} AY + B_u\tilde{C}_K & A \\ \tilde{A}_K & XA + \tilde{B}_K C_v \end{bmatrix}$$

$$T_Y^T P B_c = \begin{bmatrix} B_w \\ XB_w + \tilde{B}_K D_{vw} \end{bmatrix}, \qquad C_c T_Y = \begin{bmatrix} C_z Y + D_{zu}\tilde{C}_K & C_z \end{bmatrix}$$

where

$$\begin{aligned} \tilde{A}_K &= RA_K S^T + RB_K C_v Y + XB_u C_K S^T + XAY \\ \tilde{B}_K &= RB_K \\ \tilde{C}_K &= C_K S^T \\ \tilde{D}_K &= D_K \\ T_Y^T P T_Y &= \begin{bmatrix} Y & I \\ I & X \end{bmatrix} \end{aligned}$$

The LMI condition is :

$$\begin{bmatrix} A_c^T P + PA_c & PB_c & C_c^T \\ B^T cP & -\gamma I & D^T c \\ C_c & D_c & -\gamma I \end{bmatrix} < 0, \text{ and } P > 0.$$

$$\begin{bmatrix} T_Y^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_c^T P + PA_c & PB_c & C_c^T \\ B_c^T P & -\gamma I & D_c^T \\ C_c & D_c & -\gamma I \end{bmatrix} \begin{bmatrix} T_Y & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} =$$

$$\begin{bmatrix} AY + YA^T + B_u\tilde{C}_K + \tilde{C}_K^T B_u^T & \tilde{A}_K^T + A & B_w & YC_z^T + \tilde{C}_K^T D_{zu}^T \\ * & A^T X + XA + \tilde{B}_K C_v + C_v^T \tilde{B}_K^T & XB_w + \tilde{B}_K D_{vw} & C_z^T \\ * & * & -\gamma I & D_{zw}^T \\ * & * & * & -\gamma I \end{bmatrix}$$

minimize γ

subject to : $\begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0$

$$\begin{bmatrix} AY + YA^T + B_u\tilde{C}_K + \tilde{C}_K^T B_u^T & \tilde{A}_K^T + A & B_w & YC_z^T + \tilde{C}_K^T D_{zu}^T \\ * & A^T X + XA + \tilde{B}_K C_v + C_v^T \tilde{B}_K^T & XB_w + \tilde{B}_K D_{vw} & C_z^T \\ * & * & -\gamma I & D_{zw}^T \\ * & * & * & -\gamma I \end{bmatrix} < 0$$

If this has a solution $\gamma, X, Y, \tilde{A}_K, \tilde{B}_K$ and $\tilde{C}_K$ then

$$PP^{-1} = I \quad \Longrightarrow \quad RS^T = I - YX \quad \text{(solve for } R \text{ and } S\text{)}$$

Solve for A_K, B_K and C_K from:

$$\begin{aligned} \tilde{A}_K &= RA_K S^T + RB_K C_v Y + XB_u C_K S^T + XAY \\ \tilde{B}_K &= RB_K \\ \tilde{C}_K &= C_K S^T \end{aligned}$$

```
G = ss(A, [Bw, Bu], [Cz; Cv], [Dzw, Dzu; Dvw, zeros(nv,nu)]);
cvx_begin sdp
    variable X(n,n) symmetric;
    variable Y(n,n) symmetric;
    variable Ah(n,n);                        % Ah is a tilde A
    variable Bh(n,n);                        % Bh is a tilde B
    variable Ch(nu,n);                       % Ch is a tilde C
    variable gamma;

    minimize gamma;
    subject to

        [Y, eye(n,n);
         eye(n,n), X] > 0;

        [A*Y + Bu*Ch + Y*A' + Ch'*Bu',  A+Ah',  Bw,  Y*Cz' + Ch'*Dzu';
         A'+Ah, X+A + A'*Y + Bh*Cv + Cv'*Bh', X*Bw + Bh*Dvw, Cz';
         Bw', Bw'*X + Dvw'*Bh', -gamma*eye(nw,nw), Dzw';
         Cz*Y + Dzu*Ch, Cz, Dzw,   -gamma*eye(nz,nz)] < 0;
cvx_end
```

$\mathcal{H}_\infty$ Output Feedback Design

Using cvx

```
% Reconstruct the controller by inverting the linearizing transform.
MNt = eye(n,n) - X*Y;            % not symmetric
[Umn, Smn, Vmn] = svd(MNt);

sSmn = sqrt(diag(Smn))           % take the square roots
isSmn = 1./sSmn;                   % their inverse

M = Umn*diag(sSmn);     N = Vmn*diag(sSmn);
% Calculate inverses
iM = diag(isSmn)*Umn';   iN = diag(isSmn)*Vmn';
% Real use you should check whether the inverse is succeeded or not


DK = zeros(nu,nv);
BK = iN*Bh;
CK = Ch*(iM)';
AK = iN*(Ah - Bh*Cv*X - Y*Bu*Ch - Y*A*X)*(iM)';

K = ss(AK,BK,CK,DK);
```

Reference

1. Herbert Werner "Lecture note on *Optimal and Robust Control*", 2012
2. Roy Smith "Lecture note on *Robust Control & Convex Optimization*", 2012