INC 691 Linear and Nonlinear System Identification **Homework 4**
Assign: Fri 27 Jan 2012 **Wed 1 Feb 2012**

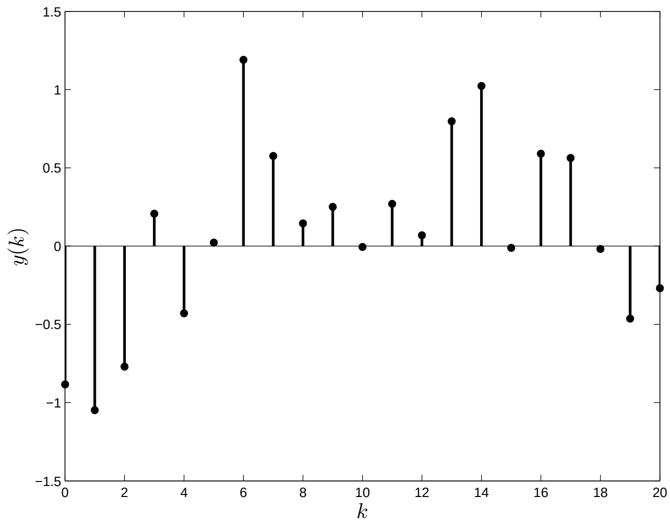
Instruction: Hand in your work in the mail box labeled INC691 by 4 pm. or submit it via email. DO NOT copy homework from your classmates or lend it to others. Anyone who violates this regulation will be given zero for the homework.

1. A moving average (MA) random process defined as

$$y(k) = \frac{1}{2}(x(k) + x(k-1)), \quad -\infty < k < \infty,$$

where $x(k)$ is a IID white gaussian noise with variance σ^2.

(a) plot $y(k)$ using Matlab,



MATLAB code

```
randn('state',0);
u =randn(21,1);

for i=1:21,
    if i==1
        x(i,1) = 0.5*(u(1)+randn(1,1)); % needed to initialize sequence
    else
        x(i,1) = 0.5*(u(i)+u(i-1));
    end
end
kk = 0:20;
stem(kk,x,'linewidth',2,'markersize',4,'MarkerFaceColor','k');
ylabel('$y(k)$','interpreter','latex','fontsize',16);
xlabel('$k$','interpreter','latex','fontsize',16);
```

(b) determine the covariance function $C_y(k, l)$.

The mean of $y(k)$ is zero since $\mathrm{E}[x(k)] = 0$ for all k. The covariance is

$$
\begin{aligned}
C_y(k,l) = \mathrm{E}[y(k), y(l)] &= \frac{1}{4}\mathrm{E}[(x(k) + x(k-1))(x(l) + x(l-1))] \\
&= \frac{1}{4}\{\mathrm{E}[x(k)x(l)] + \mathrm{E}[x(k)x(l-1)] + \mathrm{E}[x(k-1)x(l) + \mathrm{E}[x(k-1)x(l-1)]]\} \\
&= \begin{cases} \frac{1}{2}\sigma^2 & \text{if } k = l \\ \frac{1}{4}\sigma^2 & \text{if } |k-l| = 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Note $\mathrm{E}[x(k)x(k-1)]$ for example is zero. $|k-l| = 1$ means two cases: $k = j+1$ and $k = j-1$.

2. A random phased sinusoid signal defined by

$$x(k) = \cos(2\pi(0.1)k + \Phi), \ -\infty < k < \infty,$$

where $\Phi \in [0, 2\pi]$.

(a) Find the mean of $x(k)$.

$$\mu_x(k) = \mathrm{E}[\cos(2\pi(0.1)k + \Phi)] = \frac{1}{2\pi}\int_0^{2\pi} \cos(2\pi(0.1)k + \phi)d\phi = 0.$$

(b) Find the auto-correlation and auto-covariance of $x(k)$.

$$
\begin{aligned}
C_x(k,l) = R_x(k,l) &= \frac{1}{2\pi}\int_0^{2\pi} \cos(2\pi(0.1)k + \phi)\cos(2\pi(0.1)l + \phi)d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\{\cos(2\pi(0.1)(k-l)) + \cos(2\pi(0.1)(k+l) + 2\phi)\}\, d\phi \\
&= \frac{1}{2\pi}\left[\frac{1}{2}\cos(2\pi(0.1)(k-l)\phi\Big|_0^{2\pi} + \frac{1}{4}\sin(2\pi(0.1)(k+l) + 2\phi)\Big|_0^{2\pi}\right] \\
&= \frac{1}{2}\cos(2\pi(0.1)(k-l))
\end{aligned}
$$

(c) Find power spectral density of $x(k)$.

$$
\begin{aligned}
P_x(\omega) &= \frac{1}{2}\mathcal{F}(\cos(2\pi(0.1)(k-l))) \\
&= \frac{1}{4}\delta(\omega - \omega_0) + \frac{1}{4}\delta(\omega + \omega_0),
\end{aligned}
$$

where $\omega_0 = 2\pi(0.1)$.

3. The mean sequence is defined as

$$\mu_x(k) = \mathrm{E}[y(k)] = \mathrm{E}\left[\frac{1}{2}(x(k) + x(k-1)\right] = 0, \ -\infty < k < \infty,$$

where $x(k)$ is a white Gaussian noise, which has a zero mean for all k.

(a) Find the covariance sequence $C_x(k,l)$ of $y(k)$.

$$
\begin{aligned}
C_x(k,l) &= \mathrm{E}[(y(k) - \mu_y(k))(y(l) - \mu_y(l)] \\
&= \mathrm{E}[y(k)y(l)] \quad \text{since } \mu_y(k) = 0 \\
&= \frac{1}{4}\mathrm{E}[(y(k) + y(k-1)(y(l) + y(l-1))] \\
&= \frac{1}{4}(\mathrm{E}[y(k)y(l)] + \mathrm{E}[y(k)y(l-1)] + \mathrm{E}[y(k-1)y(l)] + \mathrm{E}[y(k-1)y(l-1)])
\end{aligned}
$$

But $\mathrm{E}\left[y(k)y(l)\right] = \sigma_y^2\sigma(l-k)$ since $y(k)$ is WGN, and as a result

$$\begin{aligned} C_x(k,l) &= \frac{1}{4}\left(\sigma_y^2\delta(l-k) + \sigma_y^2(l-1-k) + \sigma_y^2\delta(l-k+1) + \sigma_y^2\delta(l-k)\right) \\ &= \frac{\sigma_y^2}{2}\delta(l-k) + \frac{\sigma_y^2}{4}\delta(l-k-1) + \frac{\sigma_y^2}{4}(l-k+1). \end{aligned}$$

(b) Plot $C_x(k,l)$. In summary, we have that

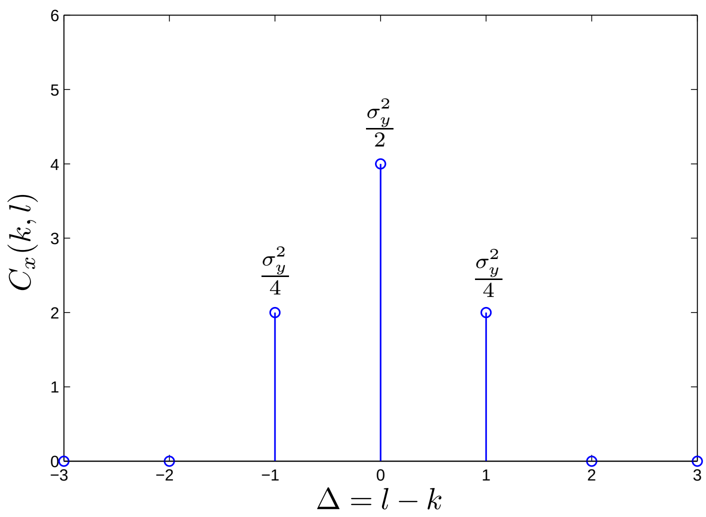


Figure 1: Covariance sequence for moving average random process

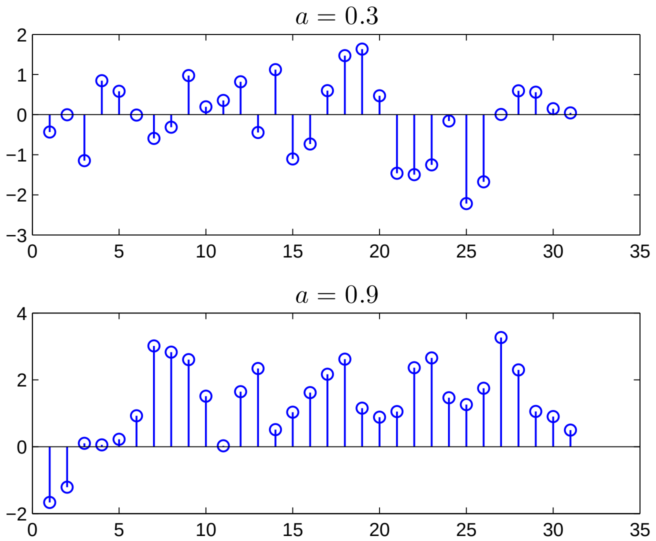
$$\begin{aligned} \mu_x(k) &= 0 \\ C_x(k,l) &= \begin{cases} \frac{\sigma_y^2}{2} & k = l \\ \frac{\sigma_y^2}{4} & |l-k| = 1 \\ 0 & |l-k| > 1. \end{cases} \end{aligned}$$

4. An auto-regressive (AR) random process $x(k)$ defined to be a WSS random process with zero mean as follow:

$$x(k) = ax(k-1) + e(k),$$

where $|a| < 1$ and $e(k)$ is white Gaussian noise, with zero mean and variance σ_e^2 for all k.

(a) Plot $x(k)$ for $a = 0.3$ and $a = 0.9$ with Matlab.



MATLAB code

```
clear all;
randn('state',0);
a1 = 0.3; a2 = 0.9;
x1(1) = randn(1,1);
x2(1) = randn(1,1);
for n=2:31
    x1(n) = a1*x1(n-1) + randn(1,1);
    x2(n) = a2*x2(n-1) + randn(1,1);
end
nn = 1:31;
subplot(211); stem(nn,x1,'linewidth',1);
title('$a=0.3$','interpreter','latex','fontsize',14);
subplot(212); stem(nn,x2,'linewidth',1);
title('$a=0.9$','interpreter','latex','fontsize',14);
```

(b) Determine $R_x(\tau)$

$$\begin{aligned} R_x(\tau) &= \mathrm{E}\left[x(k)x(k+\tau)\right] \\ &= \mathrm{E}\left[x(k)(ax(k+\tau-1)+e(k+\tau))\right] \\ &= a\mathrm{E}\left[x(\tau)x(k+\tau-1)\right] \\ &= aR_x(\tau-1) \end{aligned}$$

$$\begin{aligned} R_x(1) &= aR_x(0) \\ R_x(2) &= aR_x(1) = a^2R_x(0) \\ &\vdots \\ R_x(\tau) &= \sigma_x^2 a^\tau, \end{aligned}$$

where $R_x(0) = \sigma_x^2$.

(c) Determine the power spectrum of the system.

$$
\begin{aligned}
R_x(\tau) &= \sigma_x^2 a^{|\tau|} \quad -\infty < \tau < \infty \\
P_x(\omega) &= \sum_{\tau=-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau T} \\
&= \sigma_x^2 \sum_{k=-\infty}^{\infty} a^{|\tau|} e^{-j\omega\tau T} \\
&= \sigma_x^2 \left[\sum_{\tau=-\infty}^{-1} a^{-\tau} e^{-j\omega\tau T} + \sum_{k=0}^{\infty} a^{\tau} e^{-j\omega\tau T} \right] \\
&= \sigma_x^2 \left[\sum_{\tau=1}^{\infty} [a e^{j\omega T}]^{\tau} + \sum_{\tau=0}^{\infty} [a e^{-j\omega T}]^{\tau} \right]
\end{aligned}
$$

Since $\sum_{k=k_0}^{\infty} z^k = z^{k_0}/(1-z)$, then

$$
\begin{aligned}
P_x(\omega) &= \sigma_x^2 \left(\frac{a e^{j\omega T}}{1 - a e^{j\omega T}} + \frac{1}{1 - a e^{-j\omega T}} \right) \\
&= \sigma_x^2 \frac{a e^{j\omega T}(1 - a e^{-j\omega T}) + (1 - a e^{j\omega T})}{(1 - a e^{j\omega T})(1 - a e^{-j\omega T})} \\
&= \frac{\sigma_x^2 (1 - a^2)}{1 - 2a\cos(\omega T) + a^2}, \quad -\pi/T \le \omega \le \pi/T
\end{aligned}
$$

(d) Plot the power spectrum of the system for $a = 0.3$ and $a = 0.9$.