1. Given the scalar function
$$f(x) = \|Ax - b\|_2^2,$$
with $A \in \mathbb{R}^{N\times n}(N \ll n), b \in \mathbb{R}^N$, and $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$,

(a) prove that
$$\frac{d}{dx} f(x) = 2A^T Ax - 2A^T b$$

(b) Prove that a solution to the minimization problem
$$\min_x f(x) \tag{1}$$
satisfies the linear set of equations (which are called the normal equations)
$$A^T Ax = A^T b. \tag{2}$$

(c) When is the solution to (1) unique?

(d) Let $Q \in \mathbb{R}^{N\times N}$ be an orthogonal matrix, that is , $Q^T Q = QQ^T = I_N$. Show that
$$\|Ax - b\|_2^2 = \|Q^T (Ax - b)\|_2^2.$$

(e) Assume that the matrix A has full column rank, and that its QR factorization is given by
$$A = Q \begin{bmatrix} R \\ 0 \end{bmatrix}.$$
Show, without using (2), that the solution to (1) is
$$x = R^{-1} b_1, \tag{3}$$
where $b_1 \in \mathbb{R}^n$ come from the partitioning
$$Q^T b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$
with $b_2 \in \mathbb{R}^{N-n}$. Show also that
$$\min \|Ax - b\|_2^2 = \|b_2\|_2^2.$$

(f) Assume that the matrix A has full column rank and that its SVD is given by $A = U\Sigma V^T$ with
$$\Sigma = \begin{bmatrix} \Sigma_n \\ 0 \end{bmatrix}, \quad \Sigma_n = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \sigma_n \end{bmatrix}.$$

Show, without using (2), that the solution to (1) is

$$x = \sum_{i=1}^{n} \frac{u_i^T b}{\sigma_i} v_i, \tag{4}$$

where u_i and v_i represent the ith column vector of the matrix U and the matrix V, respectively. Show also that

$$\min \|Ax - b\|_2^2 = \sum_{i=n+1}^{N} (u_i^T b)^2.$$

(g) Show that, if A has full column rank, the solutions (3) and (4) are equivalent to the solution obtained by solving (2).

2. Compute the inverse DTFT of

   (a) $X(e^{j\omega T}) = j\frac{\pi}{T}\delta\left(\omega + \frac{a}{T}\right) - j\frac{\pi}{T}\delta\left(\omega - \frac{a}{T}\right),\ a \in \mathbb{R}.$

   (b) $X(e^{j\omega T}) = \frac{\pi}{T}\delta\left(\omega + \frac{a}{T}\right) + \frac{\pi}{T}\delta\left(\omega - \frac{a}{T}\right),\ a \in \mathbb{R}.$

3. Prove the Parseval's theorem

$$\|x\|_2^2 = \sum_{k=-\infty}^{\infty} |x(k)|^2 = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} |X(e^{j\omega T})|^2 d\omega.$$