Due date 21 Dec 2011 16:00 pm.

1. Given a square, symmetric, and invertible matrix $B \in \mathbb{R}^{(n+1)\times(n+1)}$ that can be partitioned as

$$B = \begin{bmatrix} A & v \\ v^T & \sigma \end{bmatrix}$$

express the inverse of the matrix B in terms of the inverse of the matrix $A \in \mathbb{R}^{n\times n}$.
Solution It is clear that A and σ are invertible. For the right inverse, we have

$$\begin{bmatrix} A & v \\ v^T & \sigma \end{bmatrix} \begin{bmatrix} X & y \\ y^T & z \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$$

$$AX + vy^T = I, \qquad Ay + vz = 0$$

$$v^T X + \sigma y^T = 0, \qquad v^T y + \sigma z = 1.$$

Then,

$$y = -A^{-1}vz$$

$$(\sigma - v^T A^{-1} v)z = 1$$

$$z = (\sigma - v^T A^{-1} v)^{-1}$$

$$y = -A^{-1}v(\sigma - v^T A^{-1} v)^{-1} = -\frac{1}{k}A^{-1}v, \text{ where } k = 1/(\sigma - v^T A^{-1} v),$$

since A is symmetric $A^T = A$

$$y^T = -\frac{1}{k}v^T A^{-1}$$

$$X = A^{-1} - A^{-1}vy^T = A^{-1} + \frac{1}{k}A^{-1}vv^T A^{-1}$$

Finally, we obtain

$$B^{-1} = \begin{bmatrix} A & v \\ v^T & \sigma \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + \frac{1}{k}A^{-1}vv^T A^{-1} & -\frac{1}{k}A^{-1}v \\ -\frac{1}{k}v^T A^{-1} & \frac{1}{k} \end{bmatrix}$$

A more elegant way can be done as follow (A does not need to be symmetric.):

$$\begin{bmatrix} A & v \\ v^T & \sigma \end{bmatrix} = \begin{bmatrix} I & 0 \\ v^T A^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \sigma - v^T A^{-1} v \end{bmatrix} \begin{bmatrix} I & A^{-1}v \\ 0 & 1 \end{bmatrix}$$

This decomposition is easy to invert,

$$\begin{bmatrix} A & v \\ v^T & \sigma \end{bmatrix}^{-1} = \begin{bmatrix} I & A^{-1}v \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A^{-1} & 0 \\ 0 & (\sigma - v^T A^{-1} v)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T A^{-1} & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} I & -A^{-1}v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & (\sigma - v^T A^{-1} v)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -v^T A^{-1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} + \frac{1}{k}A^{-1}vv^T A^{-1} & -\frac{1}{k}A^{-1}v \\ -\frac{1}{k}v^T A^{-1} & \frac{1}{k} \end{bmatrix}$$