INC 491, 691Optimization Techniques in Engineering **HW 1**
Assign: 12 Aug 2024 **Due : 20 Aug 2024**

Instruction:
Member:

1. Name:______________________________Code:_____________

There are two questions in the homework.

1. Determine whether the following quadratic form is positive definite (10 points):

$$f(x_1, x_2, x_3) = 2x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 4x_2x_3, \qquad x \in \mathbb{R}^3$$

   Solution: The matrix form of the function is:

$$f(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & -2 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

   Using submatrix test, we have

$$\begin{vmatrix} 2 \end{vmatrix} = 2 > 0, \quad \begin{vmatrix} 2 & -1 \\ -1 & 5 \end{vmatrix} = 10 - 1 = 9 > 0$$

$$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 5 & -2 \\ 0 & -2 & 3 \end{vmatrix} = 30 - 8 - 3 = 19 > 0$$

   Therefore the quadratic form is positive definite.

MATLAB code

```
syms x1 x2 x3

fx = 2*x1^2 + 5*x2^2 + 3*x3^2 - 2*x1*x2 - 4*x2*x3;

a11=(diff(diff(q,x1),x1))/2;
a22=(diff(diff(q,x2),x2))/2;
a33=(diff(diff(q,x3),x3))/2;
a12=(diff(diff(q,x1),x2))/2;
a13=(diff(diff(q,x1),x3))/2;
a23=(diff(diff(q,x2),x3))/2;
A=[a11,a12,a13;a12,a22,a23;a13,a23,a33];

% check the positive definite
d1 = [];
for i = 1:size(A,1)
    d1(i) = det(A(1:i,1:i));
end
if find(d1<0)
    disp("not positive definite")
else
    disp("positive definite")
end
```

□

2. Determine a cubic approximation based on the third-order Taylor series expansion, of $f(x,y) = e^{x^2+y}$ about $(x,y) = (0,0)$ (10 points).
Solution: From

$$\begin{aligned}\tilde{f}(x,y) = f(x_0,y_0) &+ f'_x(x,y)\big|_{x_0,y_0}(x-x_0) + f'_y(x,y)\big|_{x_0,y_0}(y-y_0)\\ &+ \frac{1}{2!}f''_{xx}(x,y)\Big|_{x_0,y_0}(x-x_0)^2 + \frac{1}{2!}f''_{yy}(x,y)\Big|_{x_0,y_0}(y-y_0)^2 + f''_{xy}\big|_{x_0,y_0}(x-x_0)(y-y_0)\\ &+ \frac{1}{3!}f'''_{xxx}(x,y)\Big|_{x_0,y_0}(x-x_0)^3 + \frac{1}{3!}f'''_{yyy}(x,y)\Big|_{x_0,y_0}(y-y_0)^3\\ &+ \frac{3}{3!}f'''_{xxy}(x,y)\Big|_{x_0,y_0}(x-x_0)^2(y-y_0) + \frac{3}{3!}f'''_{xyy}(x,y)\Big|_{x_0,y_0}(x-x_0)(y-y_0)^2\end{aligned}$$

We have

$$\begin{aligned}f(x_0,y_0) &= 1\\ f'_x(x,y)\big|_{x_0,y_0}(x-0) &= 2xe^{x^2+y}\Big|_{0,0}x = 0, \quad f'_y(x,y)\big|_{x_0,y_0}(y-0) = e^{x^2+y}\Big|_{0,0}y = y\\ \frac{1}{2!}f''_{xx}(x,y)\Big|_{x_0,y_0}(x-x_0)^2 &= \frac{1}{2}\,4x^2e^{x^2+y} + 2e^{x^2+y}\Big|_{0,0}(x-0)^2 = x^2\\ \frac{1}{2!}f''_{yy}(x,y)\Big|_{x_0,y_0}(y-y_0)^2 &= \frac{1}{2}\,e^{x^2+y}\Big|_{0,0}(y-0)^2 = \frac{1}{2}y^2\\ f''_{xy}(x,y)\big|_{x_0,y_0}(x-x_0)(y-y_0) &= 2xe^{x^2+y}\Big|_{0,0}(x-0)(y-0) = 0\\ \frac{1}{3!}f'''_{xxx}(x,y)\Big|_{x_0,y_0}(x-x_0)^3 &= \frac{1}{6}\left(8x^3e^{x^2+y} + 12xe^{x^2+y}\right)\Big|_{0,0}(x-0)^3 = 0\\ \frac{1}{3!}f'''_{yyy}(x,y)\Big|_{x_0,y_0}(y-y_0)^3 &= \frac{1}{6}\,e^{x^2+y}\Big|_{0,0}(y-0)^3 = \frac{1}{6}y^3\\ \frac{3}{3!}f'''_{xxy}(x,y)\Big|_{x_0,y_0}(x-x_0)^2(y-y_0) &= \frac{1}{2}\left(4x^2e^{x^2+y} + 2e^{x^2+y}\right)\Big|_{0,0}(x-0)^2(y-0) = x^2y\\ \frac{3}{3!}f'''_{xyy}(x,y)\Big|_{x_0,y_0}(x-x_0)(y-y_0)^2 &= \frac{1}{2}\left(2e^{x^2+y}\right)\Big|_{0,0}(x-0)(y-0)^2 = 0\end{aligned}$$

Hence,

$$\tilde{f}(x,y) = 1 + y + x^2 + \frac{1}{2}y^2 + \frac{1}{6}y^3 + x^2y$$

The approximate function is shown in Figure 1.

MATLAB code

```
syms x y

f = exp(x.^2 + y)
T = taylor(f,[x y], [0 0], 'Order',4)
% cubic approximate function
fn = matlabFunction(f)
Tn = matlabFunction(T)

% contour plot
x1 = -2:0.1:2; y1 = -2:0.1:2;
[X, Y] = meshgrid(x1,y1)
Z = fn(X,Y)
Z2 = Tn(X,Y)
contour(X,Y,Z,0:4:400)
xlabel("$x$", 'Interpreter','latex');
ylabel("$y$", 'Interpreter','latex');

hold on
contour(X,Y,Z,[0, 0.25, 0.5, 1, 2, 4])
contour(X,Y,Z2,[1 1], LineColor='r', linewidth=2)
hold off
```

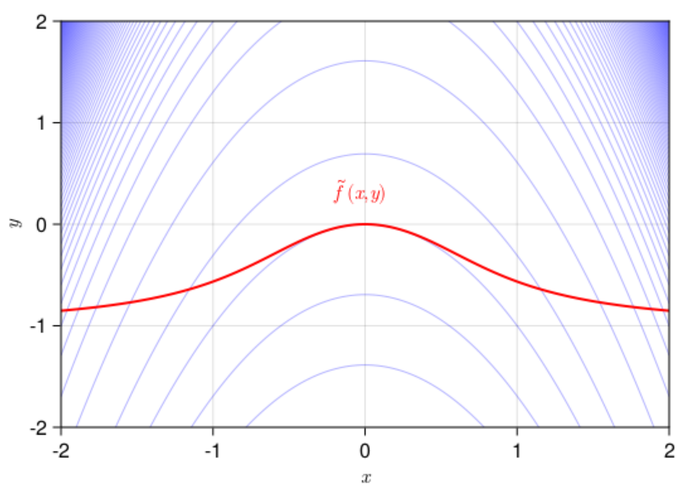

Figure 1: Comparison between the real and approximated funciton