Linear Programming III

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Objective

- Understand the interior-Point Methods

The Primal-Dual Advantage

- The Simplex method is a cornerstone of optimization theory. It operates by navigating the boundary of the feasible region (a geometry shape called a polytope). It begins at one corner (vertex) and systematically moves along the edges to adjacent corners that improve the objective function, continuing until an optimal vertex is found.
- The number of vertices in a plytope can be astronomically large. In the worst-case scenario, the Simplex method might visit an exponential number of these vertices, making it prohibitively slow for certain large-scale problems
- The primal-dual interior-point method takes a fundamentally different approach. Instead of crawling along the outside edges, it cuts directly through the middle (the interior) of the feasible region.
- It follows a smooth, curved route called the central path that leads from a starting point inside the region directly toward the optimal solution.

The Primal-Dual Advantage

- **Superior Scalability:** The number of iterations required by a primal-dual method is not as sensitive to the number of vertices. Its performance scales much better for problems with hundreds of thousands of variables and constraints, often making it dramatically faster than Simplex.
- **Polynomial-Time Complexity:** From a theoretical standpoint, interior-point methods are polynomial-time algorithms. This provides a mathematical guarantee that their runtime does not explode exponentially as the problem size increases, unlike the worst-case behavior of the Simplex method.
- **Foundation of Modern Optimization:** The core idea of using Newton's method to solve relaxed optimality conditions extends beautifully to more complex problems, such as Second-Order Cone Programming (SOCP) and Semidefinite Programming (SDP). This makes the primal-dual framework the foundation for a wide range of state-of-the-art optimization solvers.

Duality : The Lagrangian

Consider an optimization problem in the standard form:

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Where $\mathbf{A} \in \mathbb{R}^{p \times n}$, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^p$.

To derive the dual, we form the *Lagrangian*, which incorporates the constraints into the objective function using Lagrange multipliers $\boldsymbol{\lambda} \in \mathbb{R}^p$ (for $\mathbf{A}\mathbf{x} = \mathbf{b}$) and $\boldsymbol{\mu} \in \mathbb{R}^n$ (for $\mathbf{x} \geq 0$, also known as dual slack variables):

Definition: Lagrangian of primal function

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \mathbf{c}^T\mathbf{x} + \boldsymbol{\lambda}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) - \boldsymbol{\mu}^T\mathbf{x}, \text{ with } \boldsymbol{\mu} \geq 0$$

Note: $(\mathbf{x} \geq 0 \quad \Rightarrow -\mathbf{x} \leq 0)$

Duality : The Lagrangian

The Lagrange dual function is defined as

$$q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$$

To find this infimum, we take the gradient with respect to $\mathbf{x}$ and set it to zero:

$$\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \mathbf{c} + \mathbf{A}^T \boldsymbol{\lambda} - \boldsymbol{\mu} = 0$$

- Lagrangian is

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \mathbf{c}^T \mathbf{x} + \boldsymbol{\lambda}^T (\mathbf{A}\mathbf{x} - \mathbf{b}) - \boldsymbol{\mu}^T \mathbf{x} = -\mathbf{b}^T \boldsymbol{\lambda} + (\mathbf{c} + \mathbf{A}^T \boldsymbol{\lambda} - \boldsymbol{\mu})^T \mathbf{x}$$

- $\mathcal{L}$ is affine in $\mathbf{x}$, hence (The linear function is bounded from below only when it is identically zero. Then $q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = -\infty$ except when $\mathbf{c} + \mathbf{A}^T \boldsymbol{\lambda} - \boldsymbol{\mu} = 0$)

$$q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \begin{cases} -\mathbf{b}^T \boldsymbol{\lambda}, & \mathbf{c} + \mathbf{A}^T \boldsymbol{\lambda} - \boldsymbol{\mu} = 0 \\ -\infty, & \text{otherwise} \end{cases}$$

Duality : Standard form LP

The condition $\mathbf{c} + \mathbf{A}^T\boldsymbol{\lambda} - \boldsymbol{\mu} = 0$, must hold for the infimum to be bounded. Rearranging, we get $-\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} = \mathbf{c}$. When this hold, the Lagrangian becomes $\mathcal{L}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = -\mathbf{b}^T\lambda$. The dual problem is to maximize this function subject to the derived constraints.

Dual Problem

$$\begin{aligned} \underset{\boldsymbol{\lambda},\boldsymbol{\mu}}{\text{maximize}} \quad & -\mathbf{b}^T\boldsymbol{\lambda} \\ \text{subject to} \quad & -\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} = \mathbf{c} \\ & \boldsymbol{\mu} \geq 0 \end{aligned}$$

The standard-form LP problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0 \end{aligned} \tag{1}$$

The dual problem is

$$\begin{aligned} \underset{\boldsymbol{\lambda}}{\text{maximize}} \quad & \mathbf{h}(\boldsymbol{\lambda}) = -\mathbf{b}^T\boldsymbol{\lambda} \\ \text{subject to} \quad & -\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} = \mathbf{c}, \quad \boldsymbol{\lambda} \geq 0 \text{ (or) } \mathbf{A}^T\boldsymbol{\lambda} + \mathbf{c} \geq 0 \end{aligned} \tag{2}$$

- Under what conditions will the solutions of these problems exist?
- How are the feasible points and solutions of the primal and dual related?
- $\boldsymbol{\mu} \geq 0$

- An LP problem is said to be **feasible** if its feasible region is not empty. The problem in (1) is said to be **strictly feasible** if there exists and $\mathbf{x}$ that satisfies $-\boldsymbol{\lambda}^T\mathbf{A} + \boldsymbol{\mu} = \mathbf{c}$ with $\mathbf{x} \geq 0$
- The LP problem in (2) is said to be strictly feasible if there exist $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ that satisfy $-\boldsymbol{\lambda}^T\mathbf{A} + \boldsymbol{\mu} = \mathbf{c}$ with $\boldsymbol{\mu} \geq 0$.
- It is known that $\mathbf{x}^*$ is a minimizer of the problem in (1) if and only if there exist $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^* \geq 0$ such that

$$
\begin{aligned}
-\mathbf{A}^T\boldsymbol{\lambda}^* + \boldsymbol{\mu}^* &= \mathbf{c} \\
\mathbf{A}\mathbf{x}^* &= \mathbf{b} \\
x_i^*\mu_i^* &= 0 \text{ for } 1 \leq i \leq n \\
\mathbf{x}^* \geq 0, \quad \boldsymbol{\mu}^* &\geq 0
\end{aligned}
\tag{3}
$$

- A set $\{\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*\}$ satisfying (3) is called a **primal-dual solution**. The set $\{\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*\}$ is a primal-dual solution if and only if $\mathbf{x}^*$ solves the primal and $\{\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*\}$ solves the dual.

Theorem: Existence of a primal-dual solution

A primal-dual solution exists if the primal and dual problems are both feasible.

Proof: If point $\mathbf{x}$ is feasible for the LP problem and $\{\boldsymbol{\lambda}, \boldsymbol{\mu}\}$ is feasible for the LP problem, then set

$$
\begin{aligned}
-\boldsymbol{\lambda}^T\mathbf{b} \leq -\boldsymbol{\lambda}^T\mathbf{b} + \boldsymbol{\mu}^T\mathbf{x} = -\boldsymbol{\lambda}^T\mathbf{A}\mathbf{x} + \boldsymbol{\mu}^T\mathbf{x} \\
= (-\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu})^T\mathbf{x} = \mathbf{c}^T\mathbf{x}
\end{aligned}
$$

Since $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ has a finite lower bound in the feasible region, there exists a set $\{\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*\}$ that satisfies (3). This $\mathbf{x}^*$ solves the problem in (1). From above condition $\mathbf{h}(\boldsymbol{\lambda})$ has a finite upper bound and $\{\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*\}$ solves the problem in (2). Consequently, the set $\{\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*\}$ is a primal-dual solution.

Primal-Dual Solutions and Central Path: Primal-Dual Solutions

Theorem: Strict feasibility of primal-dual solutions

If the primal and dual problems are both feasible, then

1. solutions of the primal problem are bounded if the dual is strictly feasible:
2. solutions of the dual problem are bounded if the primal is strictly feasible:
3. primal-dual solutions are bounded if the primal and dual are both strictly feasible.

Proof: see reference 5.

Primal-Dual Solutions and Central Path: Primal-Dual Solutions

Duality gap From (3), we observe that

$$\mathbf{c}^T\mathbf{x}^* = [(\boldsymbol{\mu}^*)^T - (\boldsymbol{\lambda}^*)^T\mathbf{A}]\mathbf{x}^* = -(\boldsymbol{\lambda}^*)^T\mathbf{A}\mathbf{x}^* = -(\boldsymbol{\lambda}^*)^T\mathbf{b} \quad \Rightarrow \quad f(\mathbf{x}^*) = \mathbf{h}(\boldsymbol{\lambda}^*)$$

If we define the **duality gap** as

$$\delta(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{c}^T\mathbf{x} + \mathbf{b}^T\boldsymbol{\lambda}$$

Then the above equations imply that $\delta(\mathbf{x}, \boldsymbol{\lambda})$ is always nonnegative with $\delta(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0$. For any feasible x and $\boldsymbol{\lambda}$, we have

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{c}^T\mathbf{x}^* \geq -\mathbf{b}^T\boldsymbol{\lambda}^* \geq -\mathbf{b}^T\boldsymbol{\lambda}$$
$$\mathbf{c}^T\mathbf{x} - \mathbf{c}^T\mathbf{x}^* \geq 0 \geq -\mathbf{b}^T\boldsymbol{\lambda}^* - \mathbf{c}^T\mathbf{x}^* \geq -\mathbf{b}^T\boldsymbol{\lambda} - \mathbf{c}^T\mathbf{x}^* \Longrightarrow \quad 0 \leq \mathbf{c}^T\mathbf{x} - \mathbf{c}^T\mathbf{x}^* \leq \delta(\mathbf{x}, \boldsymbol{\lambda})$$

It indicates that the duality gap can serve as a bound on the closeness of $f(\mathbf{x})$ to $f(\mathbf{x}^*)$.

Primal-Dual Solutions and Central Path: Central Path

One of the important concept related to the primal-dual solutions is central path. By using (3), set $\{\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}\}$ with $\mathbf{x} \in \mathbb{R}^n$, $\boldsymbol{\lambda} \in \mathbb{R}^p$, and $\boldsymbol{\mu} \in \mathbb{R}^n$ is a primal-dual solution if it satisfies the KKT conditions

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \quad \text{with } \mathbf{x} \geq 0 \\ -\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} &= \mathbf{c} \quad \text{with } \boldsymbol{\mu} \geq 0 \\ \mathbf{X}\boldsymbol{\mu} &= 0 \end{aligned} \tag{4}$$

where $\mathbf{X} = \mathrm{diag}\{x_1, x_2, \dots, x_n\}$ The centeral path for a standard form LP problem is defined as a set of vectors $\{\mathbf{x}(\tau), \boldsymbol{\lambda}(\tau), \boldsymbol{\mu}(\tau)\}$ that satisfy the conditions

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \quad \text{with } \mathbf{x} > 0 \\ -\mathbf{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} &= \mathbf{c} \quad \text{with } \boldsymbol{\mu} > 0 \\ \mathbf{X}\boldsymbol{\mu} &= \tau\mathbf{e} \qquad (x_i\mu_i = \tau) \end{aligned} \tag{5}$$

where τ is a strictly positive scalar parameter, and $\mathbf{e} = [1\ 1\ \cdots\ 1]^T$

Primal-Dual Solutions and Central Path: Central Path

- For each fixed $\tau > 0$, the vectors in the set $\{\mathbf{x}(\tau), \boldsymbol{\lambda}(\tau), \boldsymbol{\mu}(\tau)\}$ satisfying (5) can be viewed as sets of points in $\mathbb{R}^n$, $\mathbb{R}^p$, and $\mathbb{R}^n$, respectively.
- When τ varies, the corresponding points form a set of trajectories called the **central path**.
- By comparing (5) with (3), it is obvious that the centeral path is closely related to the primal-dual solutions. Every point on the central path is strictly feasible.
- The central path lies in the interior of the feasible regions of the problems in (1) and (2) and it approaches a primal-dual solution as $\tau \to 0$.
- Given $\tau > 0$, let $\{\mathbf{x}(\tau), \boldsymbol{\lambda}(\tau), \boldsymbol{\mu}(\tau)\}$ be on the central path. From (5), the duality gap $\delta[\mathbf{x}(\tau, \boldsymbol{\lambda}(\tau)]$ is given by

$$\begin{aligned}\delta[\mathbf{x}(\tau), \boldsymbol{\lambda}(\tau)] &= \mathbf{c}^T\mathbf{x}(\tau) + \mathbf{b}^T\boldsymbol{\lambda}(\tau) = [-\boldsymbol{\lambda}^T(\tau)\mathbf{A} + \boldsymbol{\mu}^T(\tau)]\mathbf{x}(\tau) + \mathbf{b}^T\boldsymbol{\lambda}(\tau)\\ &= \boldsymbol{\mu}^T(\tau)\mathbf{x}(\tau) = n\tau\end{aligned}$$

The central path converges linearly to zero a $\tau \to 0$. The objective function $\mathbf{c}^T\mathbf{x}(\tau)$, and $\mathbf{b}^T\boldsymbol{\lambda}(\tau)$ approach the same optimal value.

Primal-Dual Solutions and Central Path: Central Path

Example:

Sketch the central path of the LP problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = -2x_1 + x_2 - 3x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

Solution: With $\mathbf{c} = [-2\ 1\ \ -3]^T$, $\mathbf{A} = [1\ 1\ 1]$, and $\mathbf{b} = 1$, (5) become

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -\lambda + \mu_1 &= -2 \\ -\lambda + \mu_2 &= 1 \\ -\lambda + \mu_3 &= -3 \\ x_1\mu_1 = \tau,\ x_2\mu_2 &= \tau,\ x_3\mu_3 = \tau \end{aligned}$$

where $x_i > 0$ and $\mu_i > 0$ for $i = 1, 2, 3$.

From above equations, we have

$$\mu_1 = -2 + \lambda \quad \mu_2 = 1 + \lambda \quad \mu_3 = -3 + \lambda$$

Hence $\mu_i > 0$ for $1 \le i \le 3$ if $\lambda > 3$. If we assume that $\lambda > 3$, then

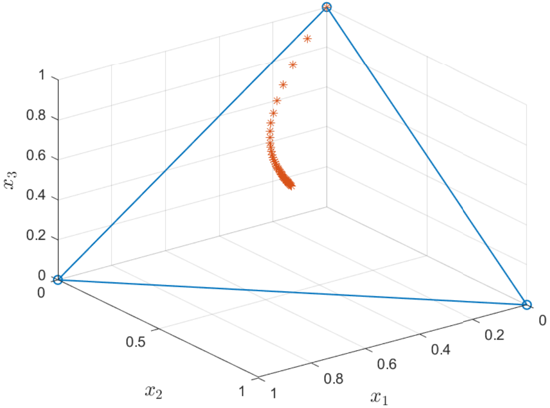
$$\frac{1}{\lambda - 2} + \frac{1}{\lambda + 1} + \frac{1}{\lambda - 3} = \frac{1}{\tau}$$

i.e.,

$$\frac{1}{\tau}\lambda^3 - \left(\frac{4}{\tau} + 3\right)\lambda^2 + \left(\frac{1}{\tau} + 8\right)\lambda + \left(\frac{6}{\tau} - 1\right) = 0$$

The central path can be constructed by finding a root of above equation that satisfies $\lambda > 3$.

Primal-Dual Solutions and Central Path : Central Path



We cannot set $\tau = 0$ directly. When we set $\tau = 0$, the key KKT condition becomes:

$$\mathbf{X}\boldsymbol{\mu} = 0 \quad \Longrightarrow \quad x_i \mu_i = 0 \quad \forall i$$

- **It Creates a Discontinuous Jump:** A step tries to jump from the current interior point directly to a solution on the boundary where many x_i and μ_i are zero. This is a huge, highly non-linear jump. The step will almost certainly overshoot, resulting in negative values for some x_i or μ_i, whichis infeasible and breaks the algorithm.
- **The Problem Becomes Non-Smooth:** The conditions $x_i \mu_i = 0$ represents a set of axes–a shap with a *sharp corner* at the origin. The method relies on using derivatives to find the next step, and this doesn't work well at shap corners.

Primal-Dual Interior Methods: Path-Following Method

We need to find τ_k to make $\{\mathbf{x}_k, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_k\}$ approach the minimizer vertex. In this lecture, we introduce the method that simultaneously solves the primal and dual LP problems and has emerged as *the modest efficient interior-point method for the LP problems.*

- Consider the standard form LP problem in (1) and its dual (2). Let $\mathbf{w}_k = \{\mathbf{x}_k, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_k\}$ where $\mathbf{x}_k$ is strictly feasible for the primal and $\{\boldsymbol{\lambda}_k, \boldsymbol{\mu}_k\}$ is strictly feasible for the dual.
- We need to find the increment vector $\boldsymbol{\delta}_w = \{\boldsymbol{\delta}_x, \boldsymbol{\delta}_\lambda, \boldsymbol{\delta}_\mu\}$ such that the next iterate $\mathbf{w}_{k+1} = \{\mathbf{x}_{k+1}, \boldsymbol{\lambda}_{k+1}, \boldsymbol{\mu}_{k+1}\} = \{\mathbf{x}_k + \boldsymbol{\delta}_x, \boldsymbol{\lambda}_k + \boldsymbol{\delta}_\lambda, \boldsymbol{\mu}_k + \boldsymbol{\delta}_\mu\}$ remains strictly feasible and approaches the central path defined by (2) with $\tau = \tau_{k+1} > 0$.
- The path-following method, a suitable $\boldsymbol{\delta}_w$ is obtained as a first-order approximate solution of (5).

Primal-Dual Interior Methods: Path-Following Method

If $\mathbf{w}_{k+1}$ satisfies (5) with $\tau = \tau_{k+1}$, then

Perturbation:

$$\begin{aligned} \mathbf{A}(\mathbf{x}_k + \boldsymbol{\delta}_x) &= \mathbf{b} \\ -\mathbf{A}^T(\boldsymbol{\lambda}_k + \boldsymbol{\delta}_\lambda) + (\boldsymbol{\mu}_k + \boldsymbol{\delta}_\mu) &= \mathbf{c} \\ \tilde{\mathbf{X}}(\boldsymbol{\mu}_k + \boldsymbol{\delta}_\mu) &= \tau_{k+1}\mathbf{e} \end{aligned}$$

Condition for $\mathbf{w}_{k+1}$

$$\begin{aligned} \mathbf{A}\boldsymbol{\delta}_x &= 0 \\ -\mathbf{A}^T\boldsymbol{\delta}_\lambda + \boldsymbol{\delta}_\mu &= 0 \\ \Delta\mathbf{X}\boldsymbol{\mu}_k + X\boldsymbol{\delta}_\mu + \Delta\mathbf{X}\boldsymbol{\delta}_\mu &= \tau_{k+1}\mathbf{e} - \mathbf{X}\boldsymbol{\mu}_k \end{aligned}$$

where

$$\mathbf{X} = \mathrm{diag}\{x_1, x_2, \ldots, x_n\}, \tilde{\mathbf{X}} = \mathrm{diag}\{x_1 + (\boldsymbol{\delta}_x)_1, x_2 + (\boldsymbol{\delta}_x)_2, \ldots, x_n + (\boldsymbol{\delta}_x)_n\},$$
$$\Delta\mathbf{X} = \mathrm{diag}\{(\boldsymbol{\delta}_x)_1, (\boldsymbol{\delta}_x)_2, \ldots, (\boldsymbol{\delta}_x)_n\}, \quad \tilde{\mathbf{X}} = \mathbf{X} + \Delta\mathbf{X}$$

By neglecting the $\Delta\mathbf{X}\boldsymbol{\delta}_\mu$ and let $\Delta\mathbf{X}\boldsymbol{\mu}_k = \mathbf{M}\boldsymbol{\delta}_x$ (we need to find $\boldsymbol{\delta}_x$) , where $\mathbf{M} = \mathrm{diag}\{(\boldsymbol{\mu}_k)_1, (\boldsymbol{\mu}_k)_2, \ldots, (\boldsymbol{\mu}_k)_n\}$, we have

$$\mathbf{A}\boldsymbol{\delta}_x = 0, \quad -\mathbf{A}^T\boldsymbol{\delta}_\lambda + \boldsymbol{\delta}_\mu = 0, \quad \mathbf{M}\boldsymbol{\delta}_x + \mathbf{X}\boldsymbol{\delta}_\mu = \tau_{k+1}\mathbf{e} - \mathbf{X}\boldsymbol{\mu}_k \tag{6}$$

Solving (6) for $\boldsymbol{\delta}_w$, we obtain

$$\begin{aligned}
\boldsymbol{\delta}_\lambda &= -\mathbf{YAy} \\
\boldsymbol{\delta}_\mu &= \mathbf{A}^T\boldsymbol{\delta}_\lambda \\
\boldsymbol{\delta}_x &= -\mathbf{y} - \mathbf{D}\boldsymbol{\delta}_\mu \\
\mathbf{D} = \mathbf{M}^{-1}\mathbf{X}, &\quad \mathbf{Y} = (\mathbf{ADA}^T)^{-1}, \quad \mathbf{y} = \mathbf{x}_k - \tau_{k+1}\mathbf{M}^{-1}\mathbf{e}
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\mathbf{M}^{-1}\mathbf{X}\boldsymbol{\mu}_k &= \begin{bmatrix} 1/(\boldsymbol{\mu}_k)_1 & & \\ & \ddots & \\ & & 1/(\boldsymbol{\mu}_k)_n \end{bmatrix} \begin{bmatrix} (\mathbf{x}_k)_1 & & \\ & \ddots & \\ & & (\mathbf{x}_k)_n \end{bmatrix} \begin{bmatrix} (\boldsymbol{\mu}_k)_1 \\ \vdots \\ (\boldsymbol{\mu}_k)_n \end{bmatrix} \\
&= \mathbf{x}_k
\end{aligned}$$

Primal-Dual Interior Methods : Path-Following Method

Primal-dual path-following algorithm for the LP problem

1. Input $\mathbf{A}$ and a strictly feasible $\mathbf{w}_0 = \{\mathbf{x}_0, \boldsymbol{\lambda}_0, \boldsymbol{\mu}_0\}$. Set $k = 0$ and $\rho > \sqrt{n}$ (n is the dimension of $\mathbf{x}$), and initialize the tolerance ε for the duality gap.
2. If $\boldsymbol{\mu}_k^T \mathbf{x}_k \leq \varepsilon$, output solution $\mathbf{w}^* = \mathbf{w}_k$ and stop; otherwise, continue with Step 3
3. Set $\tau_{k+1} = \frac{\boldsymbol{\mu}_k^T \mathbf{x}_k}{n+\rho}$ and compute $\boldsymbol{\delta}_w = \{\boldsymbol{\delta}_x, \boldsymbol{\delta}_\lambda, \boldsymbol{\delta}_\mu\}$ using (7).
4. compute step size α_k as follow:

$$\alpha_k = (1 - 10^{-6})\alpha_{\max}, \quad \alpha_{\max} = \min(\alpha_p, \alpha_d)$$

where

$$\alpha_p = \min_{i \text{ with } (\boldsymbol{\delta}_x)_i < 0} \left[-\frac{(\mathbf{x}_k)_i}{(\boldsymbol{\delta}_x)_i} \right], \quad \alpha_d = \min_{i \text{ with } (\boldsymbol{\delta}_\mu)_i < 0} \left[-\frac{(\boldsymbol{\mu}_k)_i}{(\boldsymbol{\delta}_\mu)_i} \right]$$

Primal-Dual Interior Methods: Path-Following Method Example

Example:

Sketch the central path of the LP problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = -2x_1 + x_2 - 3x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

Solution: Find an initial $\mathbf{w}_0$ on the central path by using the method described in the previous example with $\tau_0 = 5$. The vector $\mathbf{w}_0$ obtained is $\{\mathbf{x}_0, \boldsymbol{\lambda}_0, \boldsymbol{\mu}_0\}$ with

$$\mathbf{x}_0 = \begin{bmatrix} 0.344506 \\ 0.285494 \\ 0.370000 \end{bmatrix}, \quad \boldsymbol{\lambda}_0 = 16.513519, \quad \boldsymbol{\mu}_0 = \begin{bmatrix} 14.513519 \\ 17.513519 \\ 13.513519 \end{bmatrix}$$

With $\rho = 7\sqrt{n}$ and $\varepsilon = 10^{-6}$, the algorithm will converges after eight iterations to the solution $\mathbf{x}^* = \begin{bmatrix} 0.000000 & 0.000000 & 1.000000 \end{bmatrix}$

A nonfeasible-Initialization: Path-Following Method

If $\mathbf{w}_{k+1}$ satisfies (5) with $\tau = \tau_{k+1}$, then

Perturbation:

$$\begin{aligned} \mathbf{A}(\mathbf{x}_k + \boldsymbol{\delta}_x) &= \mathbf{b} \\ -\mathbf{A}^T(\boldsymbol{\lambda}_k + \boldsymbol{\delta}_\lambda) + (\boldsymbol{\mu}_k + \boldsymbol{\delta}_\mu) &= \mathbf{c} \\ \tilde{\mathbf{X}}(\boldsymbol{\mu}_k + \boldsymbol{\delta}_\mu) &= \tau_{k+1} e \end{aligned}$$

Condition for $\mathbf{w}_{k+1}$

$$\begin{aligned} \mathbf{A}\boldsymbol{\delta}_x &= \mathbf{r}_p \\ -\mathbf{A}^T\boldsymbol{\delta}_\lambda + \boldsymbol{\delta}_\mu &= \mathbf{r}_d \\ \mathbf{M}\boldsymbol{\delta}_x + \mathbf{X}\boldsymbol{\delta}_\mu &= \tau_{k+1}\mathbf{e} - \mathbf{X}\boldsymbol{\mu}_k \end{aligned}$$

where $\mathbf{r}_p = \mathbf{b} - \mathbf{A}\mathbf{x}_k$ and $\mathbf{r}_d = \mathbf{c} + \mathbf{A}^T\boldsymbol{\lambda}_k - \boldsymbol{\mu}_k$ are the residuals for the primal and dual constraints, respectively.

$$\begin{aligned} \boldsymbol{\delta}_\lambda &= -\mathbf{Y}(\mathbf{A}\mathbf{y} + \mathbf{A}\mathbf{D}\mathbf{r}_d + \mathbf{r}_p) \\ \boldsymbol{\delta}_\mu &= \mathbf{A}^T\boldsymbol{\delta}_\lambda + r_d \\ \boldsymbol{\delta}_x &= -\mathbf{y} - \mathbf{D}\boldsymbol{\delta}_\mu \end{aligned} \tag{8}$$

Nonfeasible-Initialization Primal-Dual Path-Following Method

Nonfeasible-initialization Primal-dual path-following algorithm for the LP problem

1. Input $\mathbf{A}, \mathbf{b}, \mathbf{c}$, and $w_0 = \{x_0, \lambda_0, \mu_0\}$. Set $k = 0$ and $\rho > \sqrt{n}$ (n is a dimension of x), and initialize the tolerance ε for the duality gap.
2. If $\boldsymbol{\mu}_k^T \mathbf{x}_k \leq \varepsilon$, output solution $\mathbf{w}^* = \mathbf{w}_k$ and stop; otherwise, continue with Step 3
3. Set $\tau_{k+1} = \frac{\boldsymbol{\mu}_k^T \mathbf{x}_k}{n+\rho}$ and compute $\boldsymbol{\delta}_w = \{\boldsymbol{\delta}_x, \boldsymbol{\delta}_\lambda, \boldsymbol{\delta}_\mu\}$ using (8).
4. compute step size α_k as follow:

$$\alpha_k = (1 - 10^{-6})\alpha_{\max} \quad \alpha_{\max} = \min(\alpha_p, \alpha_d)$$

where

$$\alpha_p = \min_{i \text{ with } (\boldsymbol{\delta}_x)_i < 0} \left[-\frac{(\mathbf{x}_k)_i}{(\boldsymbol{\delta}_x)_i} \right], \quad \alpha_d = \min_{i \text{ with } (\boldsymbol{\delta}_\mu)_i < 0} \left[-\frac{(\boldsymbol{\mu}_k)_i}{(\boldsymbol{\delta}_\mu)_i} \right]$$

Nonfeasible-Initialization Primal-Dual Path-Following Method Example

Example:

Sketch the central path of the LP problem

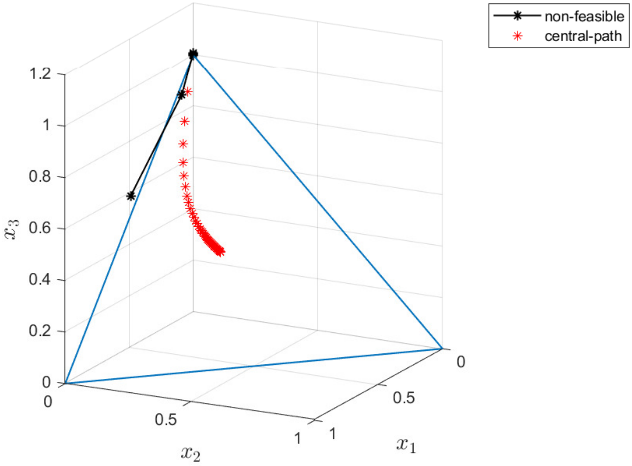
$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = -2x_1 + x_2 - 3x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

Solution: The vector $\mathbf{w}_0$, which is not feasible, is $\{\mathbf{x}_0, \boldsymbol{\lambda}_0, \boldsymbol{\mu}_0\}$ with

$$\mathbf{x}_0 = \begin{bmatrix} 0.4 \\ 0.3 \\ 0.4 \end{bmatrix}, \quad \boldsymbol{\lambda}_0 = -0.5, \quad \boldsymbol{\mu}_0 = \begin{bmatrix} 1.0 \\ 0.5 \\ 1.0 \end{bmatrix}$$

With $\rho = 7\sqrt{n}$ and $\varepsilon = 10^{-6}$, the algorithm will converges after eight iterations to the solution $\mathbf{x}^* = \begin{bmatrix} 0.000000 & 0.000000 & 1.000000 \end{bmatrix}$

Nonfeasible-Initialization Primal-Dual Path-Following Method Example



Nonfeasible-Initialization Primal-Dual Path-Following Method Example

```
% min -2x1 + x2 - 3x3
% s.t. x1 + x2 + x3 = 1
%      x1, x2, x3 >= 0

c = [-2; 1; -3];      % Objective function vector (c'x)
Aeq = [1 1 1]         % Equality constraint matrix (Aeq*x = beq)
beq = 1;              % Equality constraint vector
lb = [0; 0; 0];       % Lower bounds (x >= 0)
ub = [];              % No upper bounds

options = optimoptions('linprog', 'Algorithm',...
     'interior-point', 'Display', 'iter');

[x_opt, fval, exitflag, output] = linprog(c, [], [],...
     Aeq, beq, lb, ub, options);

disp(x_opt');
```

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