Linear Programming III : Simplex Method II

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Objective

- Linear programming (LP) problems occur in a diverse range of real-life applications in economic analysis and planning, operations research, computer science, medicine, and engineering.
- These prolems, it is known that nay minima occur at the vertices of the feasible region and can be determined through a “brute-force” or exhaustive approach by evaluating the objective function at all the vertices of the feasible region.
- The number of variables involved in practical LP problem is often vary large and an exhaustive approach would entail a considerable amount of computation.
- In 1947, Dantzig developed a method for solving LP problems known as the *simplex method*. He solved this problem because he came to the class late and thought an unsolved problem on a blackboard was homework.
- Named one of the “Top 10 algorithms of the 20th century” by *Computing in Science & Engineering magazine.* Full list at: `https://www.siam.org/pdf/news/637.pdf`
- The simplex method has been the primary method for solving LP problems since its introduction.

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Simplex Method

- At a degenerate vertex, $\mathbf{x}_k$, the number of active constraints is larger than n minus the dimension of the variable vector $\mathbf{x}$. Consequently, the number of rows in matrix $\mathbf{A}_{a_k}$ is larger than n.
- The Matrix $\mathbf{A}_{a_k}$ should be replaced in Step 2 and 3 of the Algorithm.
- The matrix $\mathbf{A}_{a_k}$ will replaced with $\hat{\mathbf{A}}_{ak}$ that is composed of n *linearly independent* rows of $\mathbf{A}_{a_k}$.
- The set of constraints corresponding to the rows in $\hat{\mathbf{A}}_{a_k}$ is called a *working set* of active constraints and in the literature $\hat{\mathbf{A}}_{a_k}$ is often referred to as a *working-set matrix*.
- Associated with $\hat{\mathbf{A}}_{a_k}$ is The *working index set* denoted as

$$\mathcal{W}_k = \{w_1, w_2, \ldots, w_n\}$$

- The index set $\mathcal{I}_k$ is redefined as

$$\mathcal{I}_k = \{i : i \notin \mathcal{W}_k \text{ and } \mathbf{a}_i^T \mathbf{d}_k > 0\}$$

Simplex Method

Simplex algorithm for the alternative-form LP problem, degenerate vertices

1. Input vertex $\mathbf{x}_0$, and form a working-set matrix $\hat{\mathbf{A}}_{a_0}$ and a working-index set $\mathcal{W}_0$. Set $k = 0$.
2. Solve $\hat{\mathbf{A}}_{a_k}^T \boldsymbol{\mu}_k = -\mathbf{c}$ for $\boldsymbol{\mu}_k$. If $\boldsymbol{\mu}_k \geq 0$, stop (vertex $\mathbf{x}_k$ is a minimizer); otherwise, select index l using $l = \min_{w_i \in \mathcal{W}_k, (\mu_k)_i < 0}(w_i)$
3. Solve $\hat{\mathbf{A}}_{a_k} \mathbf{d}_k = -\mathbf{e}_l$ for $\mathbf{d}_k$.
4. Form index set $\mathcal{I}_k$ using $\mathcal{I}_k = \{i : i \notin \mathcal{W}_k \text{ and } \mathbf{a}_i^T \mathbf{d}_k > 0\}$. If $\mathcal{I}_k$ is empty, stop (the objective function tends to $-\infty$ in the feasible region).
5. Compute the residual vector $\mathbf{r}_k = \mathbf{A}\mathbf{x}_k - \mathbf{b} = (r_i)_{i=1}^p$ parameter $\delta_i = \frac{-r_i}{\mathbf{a}_i^T \mathbf{d}_k}$ for $i \in \mathcal{I}_k$ and $\alpha_k = \min_{i \in \mathcal{I}_k}(\delta_i)$. Record index i^* as $i^* = \min_{\delta_i = \alpha_k}(i)$
6. Set $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$. Update $\hat{\mathbf{A}}_{a_{k+1}}$ by deleting row $\mathbf{a}_l^T$ and adding row $\mathbf{a}_{i^*}^T$ and update index set $\mathcal{W}_{k+1}$ accordingly. Set $k = k + 1$ and repeat for Step 2.

Simplex Method

Solve the LP problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = -2x_1 - 3x_2 + x_3 + 12x_4 \\ \text{subject to} \quad & -x_1 \leq 0, -x_2 \leq 0, -x_3 \leq 0, -x_4 \leq 0 \\ & -2x_1 - 9x_2 + x_3 + 9x_4 \leq 0 \\ & \frac{1}{3}x_1 + x_2 - \frac{1}{3}x_3 - 2x_4 \leq 0 \end{aligned}$$

- We start with $\mathbf{x}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ which is obviously a degenerate vertex. (6 active constraints) Appling the algorithm, the first iteration results in the following computations:

$$\hat{\mathbf{A}}_{a0} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathcal{W} = \{1, 2, 3, 4\}$$

Simplex Method

- $\hat{\mathbf{A}}_{a_0}^T \mu_0 = \begin{bmatrix} 2 & 3 & -1 & -12 \end{bmatrix}^T \Longrightarrow \boldsymbol{\mu}_0 = \begin{bmatrix} -2 & -3 & 1 & 12 \end{bmatrix}^T$. The lowest absolute value is at $l = 1$ (from 1 and 2).
- $\hat{\mathbf{A}}_{a_0} \mathbf{d}_0 = -\mathbf{e}_1 \Longrightarrow \mathbf{d}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$. We have $\mathbf{r}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. $\mathcal{I}_0 = \{i : i \notin \mathcal{W}_k$ and $\mathbf{a}_i^T \mathbf{d}_k > 0\}$ $i = 5$ or 6 that are not in $\mathcal{W}$, but only $\mathbf{a}_6^T \mathbf{d}_0 = \frac{1}{3}$ is positive. $\mathcal{I}_0 = \{6\}$. $\alpha_0 = 0$ and $i^* = 6$
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$$\mathbf{x}_1 = \mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{A}}_{a_1} = \begin{bmatrix} \frac{1}{3} & 1 & -\frac{1}{3} & -2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathcal{W}_1 = \{6, 2, 3, 4\}$$

- Note that although $\mathbf{x}_1 = \mathbf{x}_0$, $\hat{\mathbf{A}}_{a_1}$ differs from $\hat{\mathbf{A}}_{a_0}$. Repeating from Step 2, the second iteration ($k = 1$) gives $\boldsymbol{\mu}_1 = \begin{bmatrix} 6 & 3 & -1 & 0 \end{bmatrix}^T$, $l = 3$, $\mathbf{d}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T$, $\mathbf{r}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$

Simplex Method

- $i = 1, 5 \notin \mathcal{W}$ and

$$\mathbf{a}_1^T \mathbf{d}_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = -1$$

$$\mathbf{a}_5^T \mathbf{d}_1 = \begin{bmatrix} -2 & -9 & 1 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = -1$$

  we have $\mathcal{I}_1 = \{\varnothing\}$

- $\mathcal{I}_1$ is an empty set. Therefore, in the feasible region the objective function tends to $-\infty$.

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Simplex Method for Standard Form

Consider an example of the standard LP problem:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(\mathbf{x}) = x_1 - 2x_2 - x_4 \\ \text{subject to} \quad & 3x_1 + 4x_2 + x_3 \quad = 9 \\ & 2x_1 + x_2 + x_4 \quad = 6 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \end{aligned}$$

We have

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 9 \\ 6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x} \in \mathbb{R}^4, \quad p = 2$$

Simplex Method for Standard Form

The p equality constraints can be used to express p dependent variables in terms of $n - p$ independent variables. Assume $\mathbf{B}$ is a matrix that consists of p linearly independent column of $\mathbf{A}$. The we have

$$\mathbf{A}\mathbf{x} = \mathbf{b} \Longrightarrow \mathbf{A}\mathbf{x} \quad = \left[\mathbf{B} \;\middle|\; \mathbf{N}\right] \begin{bmatrix} \mathbf{x}_B \\ \hline \mathbf{x_N} \end{bmatrix} = \left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \hline x_3 \\ x_4 \end{bmatrix} = \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$$

- The variables contained in $\mathbf{x}_B$ and $\mathbf{x}_N$ are called basic and non basic variables, respectively.
- $\mathbf{B}$ is nonsingular, we can express the basic variables in terms of the nonbasic variables as

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$$

Simplex Method for Standard Form

- At vertex $\mathbf{x}_k$, there is at least n active constraints. In addition to the p equality constraints, there are at least $n-p$ inequality constraints that become active in $\mathbf{x}_k$.
- Therefore, for the standard-form LP problem a vertex contains at least $n-p$ zero components.

Theorem: Linear independence of columns in matrix $\mathbf{A}$

The columns of $\mathbf{A}$ corresponding to strictly positive of a vertex $\mathbf{x}_k$ are linearly independent.

Proof: Let $\hat{\mathbf{B}}$ be formed by the columns of $\mathbf{A}$ that correspond to strictly positive components of $\mathbf{x}_k$ $(\mathbf{x}_k \geq 0)$, and let $\hat{\mathbf{x}}_k$ be the collection of the positive components of $\mathbf{x}_k$. If $\hat{\mathbf{B}}\hat{\mathbf{w}} = 0$ for some nonzero $\hat{\mathbf{w}}$, then it follows that

$$\mathbf{A}\mathbf{x}_k = \hat{\mathbf{B}}\hat{\mathbf{x}}_k = \hat{\mathbf{B}}(\hat{\mathbf{x}} + \alpha\hat{\mathbf{w}}) = \mathbf{b} \text{ for any scalar } \alpha$$

Simplex Method for Standard Form

Since $\hat{\mathbf{x}}_k > 0$, there exists a sufficiently small $\alpha_+ > 0$ such that

$$\hat{\mathbf{y}}_k = \hat{\mathbf{x}}_k + \alpha\hat{\mathbf{w}} > 0 \text{ for } -\alpha_+ \leq \alpha \leq \alpha_+.$$

$\mathbf{y} \in \mathbb{R}^{n\times 1}$ be such that the components of $\mathbf{y}_k$ corresponding to $\hat{\mathbf{x}}_k$ are equal to the components of $\hat{\mathbf{y}}_k$ and the remaining correspondents of $\mathbf{y}_k$ are zero. Note that with $\alpha = 0$, $\mathbf{y}_k = \mathbf{x}_k$ is a vertex, and when α varies from $-\alpha_+$ to α_+ , vertex x_k would lie between two feasible points on a straight line, which is a contradiction. Hence $\hat{\mathbf{w}}$ must be zero and the columns of $\hat{\mathbf{B}}$ are linearly independent.

Simplex Method for Standard Form

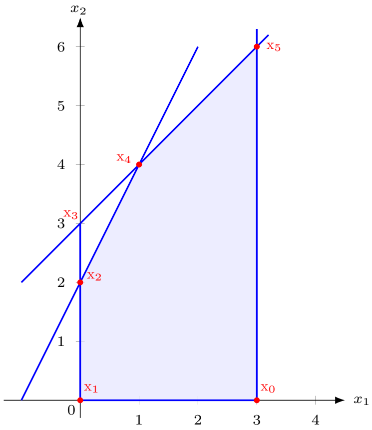
Consider

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = -x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 + x_3 = 2 \\ & -x_1 + x_2 + x_4 = 3 \\ & x_1 + x_5 = 3 \\ & \mathbf{x} \geq 0 \end{aligned}$$

There are five variables, however we need only three basic variables.

$$\mathbf{Ax} = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

Simplex Method for Standard Form



$$\mathbf{x}_3 = \begin{bmatrix} 0 & 3 & -1 & 0 & 3 \end{bmatrix}^T$$

$x_1, x_3 = 0$ for the basic infeasible solution

$$\mathbf{x}_1 = \begin{bmatrix} 0 & 0 & 2 & 3 & 3 \end{bmatrix}^T$$

$x_1, x_2 = 0$ for the basic feasible solution

$$\mathbf{x}_5 = \begin{bmatrix} 3 & 6 & 2 & 0 & 0 \end{bmatrix}^T$$

$x_4, x_5 = 0$ for the basic feasible solution

Simplex Method for Standard Form

- Using above theorem, we can use the columns of $\hat{\mathbf{B}}$ as a set of core basis vectors to construct a nonsingular square matrix $\mathbf{B}$. If $\hat{\mathbf{B}}$ already contains p columns, we assume that $\mathbf{B} = \hat{\mathbf{B}}$, otherwise, we augment $\hat{\mathbf{B}}$ with additional columns of $\mathbf{A}$ to obtain a square nonsingular $\mathbf{B}$.
- Let the index set associated with $\mathbf{B}$ at $\mathbf{x}_k$ be denoted as $\mathcal{I}_\beta = \{\beta_1, \beta_2, \ldots, \beta_p\}$. With matrix $\mathbf{B}$ so formed, matrix $\mathbf{N}$ can be constructed with those $n - p$ columns of $\mathbf{A}$ that are not in $\mathbf{B}$. Let $\mathcal{I}_N = \{v_1, v_2, \ldots, v_{n-p}\}$ be the index set for the columns of $\mathbf{N}$ and let $\mathbf{I}_N$ be the $(n - p) \times n$ matrix composed of rows $v_1, v_2, \ldots, v_{n-p}$ of the $n \times n$ identity matrix.
- It is clear that at vertex $\mathbf{x}_k$ the active constrain matrix $\mathbf{A}_{a_k}$ contains the working-set matrix

$$\hat{\mathbf{A}}_{a_k} = \begin{bmatrix} \mathbf{A} \\ \mathbf{I}_N \end{bmatrix}$$

as an $n \times n$ submatrix.

Simplex Method for Standard Form

- It can be shown that matrix $\hat{\mathbf{A}}_{a_k}$ is nonsingular. If $\hat{\mathbf{A}}_{a_k}\mathbf{x} = 0$ for some $\mathbf{x}$, then we have

$$\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = 0 \text{ and } \mathbf{x}_N = 0 \quad \Longrightarrow \quad \mathbf{x}_B = -\mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = 0$$
$$\mathbf{x} = \begin{bmatrix}\mathbf{x}_B & \mathbf{x}_N\end{bmatrix}^T = 0.$$

Therefore, $\hat{\mathbf{A}}_{a_k}$ is nonsingular. In summary, at a vertex $\mathbf{x}_k$ a working set of active constraints for the application of the simplex method can be obtained with three simple steps as follows:

1. Select the columns in matrix $\mathbf{A}$ that correspond to the strictly positive components of $\mathbf{x}_k$ to form matrix $\hat{\mathbf{B}}$.
2. If the number of columns in $\hat{\mathbf{B}}$ is equal to p, take $\mathbf{B} = \hat{\mathbf{B}}$; otherwise, $\hat{\mathbf{B}}$ is augmented with additional columns of $\mathbf{A}$ to form a square nonsingular matrix $\mathbf{B}$.
3. Determine the index set $\mathcal{I}_n$ and form matrix $\mathbf{I}_N$.

Simplex Method for Standard Form

Identify working sets of active constraints at vertex $\mathbf{x} = [3\ 0\ 0\ 0]^T$ for the LP problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(\mathbf{x}) = x_1 - 2x_2 - x_4 \\ \text{subject to} \quad & 3x_1 + 4x_2 + x_3 \quad = 9 \\ & 2x_1 + x_2 + x_4 \quad = 6 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \end{aligned}$$

Solution Using $\mathbf{r} = \mathbf{A}\mathbf{x} - \mathbf{b}$, we can verify that the point $\mathbf{x} = [3\ 0\ 0\ 0]^T$ is a degenerate vertex at which there are five active constraints. (count the zero element in $\mathbf{r}$). Since x_1 is the only strictly positive component, $\hat{\mathbf{B}}$ contains only the first column of $\mathbf{A}$, i.e., $\mathbf{B} = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$. Matrix $\hat{\mathbf{B}}$ can be augmented, by using the second column of $\mathbf{A}$ to generate a nonsingular $\hat{\mathbf{B}} = \mathbf{B}$ as

$$\mathbf{B} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

Simplex Method for Standard Form

This leads to

$$\mathcal{I}_N = \{3, 4\} \quad \text{and} \quad \hat{\mathbf{A}}_a = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The vertex $\mathbf{x}$ is degenerate, matrix $\hat{\mathbf{A}}_a$ is not unique. There are two possibilities for augmenting $\hat{\mathbf{B}}$. Using the third column of $\mathbf{A}$ for the augmentation, we have

$$\mathbf{B} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}, \ \mathcal{I}_N = \{2,\ 4\},\ \hat{\mathbf{A}}_a = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Alternatively, augmenting $\hat{\mathbf{B}}$ with the fourth column of $\mathbf{A}$ yields

$$\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \ \mathcal{I}_N = \{2, 3\}, \text{ and } \hat{\mathbf{A}}_a = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

It can be easily verified that all three $\hat{\mathbf{A}}_a$'s are nonsingular.

Simplex Method for Standard Form

We could change steps 2 and 3 of the previous simplex algorithm to reduce the computational complexity.

- At a vertex $\mathbf{x}_k$, the nonsingularity of the working-set matrix $\hat{\mathbf{A}}_{a_k}$ given by $\hat{\mathbf{A}}_{a_k} = \begin{bmatrix} \mathbf{A} \\ \mathbf{I}_N \end{bmatrix}$ implies that there exist $\boldsymbol{\lambda}_k \in \mathbb{R}^{p \times 1}$ and $\hat{\boldsymbol{\mu}}_k \in \mathbb{R}^{(n-p) \times 1}$ such that

$$\mathbf{c} = \hat{\mathbf{A}}_{a_k}^T \begin{bmatrix} -\boldsymbol{\lambda}_k \\ \hat{\mu}_k \end{bmatrix} = -\mathbf{A}^T \boldsymbol{\lambda}_k + \mathbf{I}_N^T \hat{\boldsymbol{\mu}}_k$$

  If $\boldsymbol{\mu}_k \in \mathbb{R}^{n \times 1}$ is the vector with zero basic variables and the components of $\hat{\boldsymbol{\mu}}_k$ as its nonbasic variables, then the above equation can be expressed as

$$\mathbf{c} = -\mathbf{A}^T \boldsymbol{\lambda}_k + \boldsymbol{\mu}_k$$

  The vertex $\mathbf{x}_k$ is a minimizer if and only if $\hat{\boldsymbol{\mu}}_k \geq 0$.

Simplex Method for Standard Form

- If we use a permutation matrix $\mathbf{P}$ to rearrange the components of $\mathbf{c}$ in accordance with the partition of $\mathbf{x}_k$ into basic and nonbasic variables then

$$\mathbf{Pc} = \begin{bmatrix} \mathbf{c}_B \\ \mathbf{c}_N \end{bmatrix} = -\mathbf{PA}^T \boldsymbol{\lambda}_k + \mathbf{PI}_N^T \hat{\boldsymbol{\mu}}_k = - \begin{bmatrix} \mathbf{B}^T \\ \mathbf{N}^T \end{bmatrix} \boldsymbol{\lambda}_k + \begin{bmatrix} 0 \\ \hat{\boldsymbol{\mu}}_k \end{bmatrix}$$

  It follows that

$$\mathbf{B}^T \boldsymbol{\lambda}_k = -\mathbf{c}_B \text{ and } \hat{\boldsymbol{\mu}}_k = \mathbf{c}_N + \mathbf{N}^T \boldsymbol{\lambda}_k$$

  Since $\mathbf{B}$ is nonsingular, $\boldsymbol{\lambda}_k$ and $\hat{\boldsymbol{\mu}}_k$ can be computed. The size of the matrix is $p \times p$, which is much smaller than $n \times n$ of the simplex method for the non-standard form.

Simplex Method for Standard Form

- If some entry in $\hat{\boldsymbol{\mu}}_k$ is negative, then $\mathbf{x}_k$ is not a minimizer and a search direction $\mathbf{d}_k$ needs to be determined. Note the Lagrange multipliers $\hat{\boldsymbol{\mu}}_k$ are not related to the equality constraints in $\mathbf{A}\mathbf{x} = \mathbf{b}$ but are related to those bound constraints $\mathbf{x} \geq 0$ that are active and are associated with the nonbasic variables.
- If the search direction $\mathbf{d}_k$ is partitioned according to the basic and nonbasic variables, $\mathbf{x}_B$ and $\mathbf{x}_N$, into $\mathbf{d}_k^{(\mathbf{B})}$ and $\mathbf{d}_k^{(\mathbf{N})}$, respectively, and if $(\hat{\boldsymbol{\mu}}_k)_l < 0$, then assigning

  $\mathbf{d}_k^{(\mathbf{N})} = \mathbf{e}_l$ where $\mathbf{e}_l$ is the lth column of the $(n-p) \times (n-p)$ identity matrix.

  $\mathbf{d}_k$ makes the v_lth constraint inactive without affecting other bound constraints that are associated with the nonbasic variables.
- In order to assure the feasibility of $\mathbf{d}_k$, it is also required that $\mathbf{A}\mathbf{d}_k = 0$. This requirement can be described as

$$\mathbf{A}\mathbf{d}_k = \mathbf{B}\mathbf{d}_k^{(\mathbf{B})} + \mathbf{N}\mathbf{d}_k^{(\mathbf{N})} = \mathbf{B}\mathbf{d}_k^{(\mathbf{B})} + \mathbf{N}\mathbf{e}_l = 0$$

Simplex Method for Standard Form

- $\mathbf{d}_k^{(\mathbf{B})}$ can determined by solving the system of equations

$$\mathbf{B}\mathbf{d}_k^{(\mathbf{B})} = -\mathbf{a}_{v_l} \text{ where } \mathbf{a}_{v_l} = \mathbf{N}\mathbf{e}_l$$

  Altogether we can determine the search direction d_k. It follows that

$$\mathbf{c}^T\mathbf{d}_k = -\boldsymbol{\lambda}_k^T\mathbf{A}\mathbf{d}_k + \hat{\boldsymbol{\mu}}_k^T\mathbf{I}_N d_k = \hat{\boldsymbol{\mu}}_k^T\mathbf{d}_k^{(\mathbf{N})} = \hat{\boldsymbol{\mu}}_k^T\mathbf{e}_l = (\hat{\boldsymbol{\mu}}_k)_l < 0$$

  Therefore, $\mathbf{d}_k$ is a feasible descent direction.

- To determine the step size α_k, we note that a point $\mathbf{x}_k + \alpha\mathbf{d}_k$ with any α satisfies the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$, i.e.

$$\mathbf{A}(\mathbf{x}_k + \alpha\mathbf{d}_k) = \mathbf{A}\mathbf{x}_k + \alpha\mathbf{A}\mathbf{d}_k = \mathbf{b}$$

  The only constraints that are sensitive to step size α_k are those that are associated with the basic variables and are decreasing along direction $\mathbf{d}_k$.

Simplex Method for Standard Form

- When limited to the basic variables, $\mathbf{d}_k$ becomes $\mathbf{d}_k^{(\mathbf{B})}$. Since the normals of the constraints in $\mathbf{x} \geq 0$ are simply coordinate vectors, a bound constraint associated with a basic variable is decreasing along $\mathbf{d}_k$ if the associated component in $\mathbf{d}_k^{(\mathbf{B})}$ is negative.
- The special structure of the inequality constraints in $\mathbf{x} \geq 0$ implies that the residual vector, when limited to basic variables in $\mathbf{x}_B$, is $\mathbf{x}_B$ itself.
- The above analysis lead to a simple step that can be used to determine the index set

$$\mathcal{I}_k = \{i : (\mathbf{d}_k^{(\mathbf{B})})_i < 0\} \text{ and, if } \mathcal{I} \text{ is not empty}$$

$$\alpha_k = \min_{i \in \mathcal{I}_k} \left[\frac{(\mathbf{x}_k^{(\mathbf{B})})_i}{(-\mathbf{d}_k^{(\mathbf{B})})_i} \right]$$

where $\mathbf{x}_k^{(\mathbf{B})}$ denotes the vector for the basic variables of $\mathbf{x}_k$.

Simplex Method for Standard Form

- If i^* is the index in $\mathcal{I}_k$ that achieves α_k, then the i^*th component of $\mathbf{x}_k^{(\mathbf{B})} + \alpha_k \mathbf{d}_k^{(\mathbf{B})}$ is zero. This zero component is then interchanged with the lth component of $\mathbf{x}_k^{(\mathbf{N})}$, which is now not zero but α_k.
- The vector $\mathbf{x}_k^{(\mathbf{B})} + \alpha \mathbf{d}_k^{(\mathbf{B})}$ after this updating becomes $\mathbf{x}_{k+1}^{(\mathbf{B})}$ and $\mathbf{x}_{k+1}^{(\mathbf{N})}$ remains a zero vector. Matrices $\mathbf{B}$ and $\mathbf{N}$ as well as the associated index sets $\mathcal{I}_B$ and $\mathcal{I}_N$ also need to be updated accordingly.

Simplex Method for Standard Form

Simplex algorithm for the standard-form LP problem

1. Input vertex $\mathbf{x}_0$ set $k = 0$, and form $\mathbf{B}, \mathbf{N}, \mathbf{x}_0^{(\mathbf{B})}$, $\mathcal{I}_B = \{\beta_1^{(0)}, \beta_2^{(0)}, \ldots, \beta_p^{(0)}$, and $\mathcal{I}_N = \{v_1^{(0)}, v_2^{(0)}, \ldots, v_{n-p}^{(0)}\}$.
2. Partition vector $\mathbf{c}$ into $\mathbf{c}_B$ and $\mathbf{c}_N$. Solve $\mathbf{B}^T \lambda_k = -\mathbf{c}_B$ for $\boldsymbol{\lambda}_k$ and compute $\hat{\boldsymbol{\mu}}_k$ using

$$\hat{\boldsymbol{\mu}}_k = \mathbf{c}_N + \mathbf{N}^T \boldsymbol{\lambda}_k$$

   If $\hat{\boldsymbol{\mu}}_k \geq 0$, stop ($\mathbf{x}_k$ is a vertex minimizer); otherwise, select the index l that corresponds to the most negative component in $\hat{\boldsymbol{\mu}}_k$.
3. Solve $\mathbf{B}\mathbf{d}_k^{(\mathbf{B})} = -\mathbf{a}_{v_l}$ for $\mathbf{d}_k^{(\mathbf{B})}$ where $\mathbf{a}_{v_l}$ is the $v_l^{(k)}$th column of $\mathbf{A}$.
4. Form index set $\mathcal{I}_k$ in $\mathcal{I}_k = \{i : (\mathbf{d}_k^{(\mathbf{B})})_i < 0\}$. If $\mathcal{I}_k$ is empty then stop (the objective function tends to $-\infty$ in the feasible region); otherwise, compute α_k using $\alpha_k = \min_{i \in \mathcal{I}_k} \left[\frac{(\mathbf{x}_k^{(\mathbf{B})})_i}{(-\mathbf{d}_k^{(\mathbf{B})})_i} \right]$

Simplex Method for Standard Form

4. (cont.) and record the index i^* with $\alpha_k = \frac{(\mathbf{x}_k^{(B)})_i^*}{(-\mathbf{d}_k^{(B)})_i^*}$
5. Compute $\mathbf{x}_{k+1}^{(\mathbf{B})} = \mathbf{x}_k^{(\mathbf{B})} + \alpha_k \mathbf{d}_k^{(\mathbf{B})}$ and replace its i^*th zero component by α_k. Set $\mathbf{x}_{k+1}^{(\mathbf{N})} = 0$. Update $\mathbf{B}$ and $\mathbf{N}$ by interchanging the lth column of $\mathbf{N}$ with the i^*th column of $\mathbf{B}$.
6. Update $\mathcal{I}_B$ and $\mathcal{I}_N$ by interchanging index $v_l^{(k)}$ of $\mathcal{I}_N$ with index $\beta_{i^*}^{(\mathbf{B})}$ of $\mathcal{I}_B$. Use the $\mathbf{x}_{k+1}^{(\mathbf{B})}$ and $\mathbf{x}_{k+1}^{(\mathbf{N})}$ obtained in Step 5 in conjunction with $\mathcal{I}_B$ and $\mathcal{I}_N$ to form $\mathbf{x}_{k+1}$. Set $k = k + 1$ and repeat form Step 2.

Simplex Method for Standard Form

Solve the standard-form LP problem

$$\begin{aligned}\underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = 2x_1 + 9x_2 + 3x_3 \\ \text{subject to} \quad & -2x_1 + 2x_2 + x_3 - x_4 = 1 \\ & x_1 + 4x_2 - x_3 - x_5 = 1 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0\end{aligned}$$

Solution: We have

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & 1 & -1 & 0 \\ 1 & 4 & -1 & 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} 2 & 9 & 3 & 0 & 0 \end{bmatrix}^T$$

To identify a vertex, we set $x_1 = x_3 = x_4 = 0$ and solve the system

$$\begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ for } x_2 \text{ and } x_5.$$

Simplex Method for Standard Form

We have $x_2 = 1/2$ and $x_5 = 1$; hence $\mathbf{x}_0 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}^T$ is a vertex.
Associated with $\mathbf{x}_0$ are $\mathcal{I}_B = \{2, 5\}$, $\mathcal{I}_N = \{1, 3, 4\}$

$$\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \text{ and } x_0^{(\mathbf{B})} = \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix}^T$$

Partitioning c into

$$\mathbf{c}_B = \begin{bmatrix} 9 & 0 \end{bmatrix}^T \text{ and } \mathbf{c}_N = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}^T$$

and solving $\mathbf{B}^T \boldsymbol{\lambda}_0 = -\mathbf{c}_B$ for $\boldsymbol{\lambda}_0$, we obtain $\boldsymbol{\lambda}_0 = \begin{bmatrix} -\frac{9}{2} & 0 \end{bmatrix}^T$. Hence

$$\hat{\boldsymbol{\mu}}_0 = \mathbf{c}_N + \mathbf{N}^T \boldsymbol{\lambda}_0 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{9}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ -\frac{2}{3} \\ \frac{9}{2} \end{bmatrix}$$

Simplex Method for Standard Form

Since $(\hat{\mu}_0)_2 < 0$, x_0 is not a minimizer, and $l = 2$. Next, we solve $\mathbf{B}\mathbf{d}_0^{(B)} = -\mathbf{a}_{v_2}$ for $\mathbf{d}_0^{(\mathbf{B})}$ with $v_2^{(0)} = 3$ and $\mathbf{a}_3 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, which yields

$$\mathbf{d}_0^{(\mathbf{B})} = \begin{bmatrix} -\frac{1}{2} \\ -3 \end{bmatrix} \text{ and } \mathcal{I}_0 = \{1, 2\}$$

Hence

$$\alpha_0 = \min\left(1, \frac{1}{3}\right) = \frac{1}{3} \text{ and } i^* = 2$$

To find $\mathbf{x}_1^{(\mathbf{B})}$, we compute

$$\mathbf{x}_0^{(\mathbf{B})} + \alpha_0 \mathbf{d}_0^{(\mathbf{B})} = \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$$

Replace i^*th component by α_0, i.e.,

$$\mathbf{x}_1^{(\mathbf{B})} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \text{ with } \mathbf{x}_1^{(\mathbf{N})} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Update $\mathbf{B}$ and $\mathbf{N}$ as

$$\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \text{ and } \mathbf{N} = \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

and update $\mathcal{I}_B$ and $\mathcal{I}_N$ as $\mathcal{I}_B = \{2, 3\}$ and $\mathcal{I}_N = \{1, 5, 4\}$. The vertex obtained is $\mathbf{x}_1 = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix}^T$ to compute the first iteration.
The second iteration starts with the partitioning of $\mathbf{c}$ into

$$\mathbf{c}_B = \begin{bmatrix} 9 \\ 3 \end{bmatrix} \text{ and } \mathbf{c}_N = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Solving $\mathbf{B}^T \boldsymbol{\lambda}_1 = -\mathbf{c}_B$ for $\boldsymbol{\lambda}_1$, we obtain $\boldsymbol{\lambda}_1 = \begin{bmatrix} -\frac{7}{2} & -\frac{1}{2} \end{bmatrix}^T$ which leads to

$$\hat{\boldsymbol{\mu}}_1 = \mathbf{c}_N + \mathbf{N}^T \boldsymbol{\lambda}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}^T \begin{bmatrix} -\frac{7}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{17}{2} \\ \frac{1}{2} \\ \frac{7}{2} \end{bmatrix}$$

Since $\hat{\boldsymbol{\mu}}_1 > 0$, $\mathbf{x}_1$ is the unique vertex minimizer.

Tabular Form of the Simplex Method

For LP problems of very small size, the simple method can be applied in terms of a tabular form in which the input data such as $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$ are used to form a table. Consider the standard form LP problem:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

- Assume that at vertex $\mathbf{x}_k$ the equality constraints are expressed as

$$\mathbf{x}_k^{(\mathbf{B})} + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_k^{(\mathbf{N})} = \mathbf{B}^{-1}\mathbf{b}$$

From $\mathbf{c} = -\mathbf{A}^T\boldsymbol{\lambda}_k + \boldsymbol{\mu}_k$, the objective function is given by

$$\mathbf{c}^T\mathbf{x}_k = \boldsymbol{\mu}_k^T\mathbf{x}_k - \boldsymbol{\lambda}_k^T\mathbf{A}\mathbf{x}_k = 0^T\mathbf{x}_k^{(\mathbf{B})} + \hat{\boldsymbol{\mu}}_k^T\mathbf{x}_k^{(\mathbf{N})} - \boldsymbol{\lambda}_k^T\mathbf{b}$$

Tabular Form of the Simplex Method

The important data at the kth iteration can be put together in a tabular form as a table. ($\mathbf{B}^T\boldsymbol{\lambda}_k = -\mathbf{c}_B, \hat{\boldsymbol{\mu}}_k = \mathbf{c}_N + \mathbf{N}^T\boldsymbol{\lambda}_k$)

$\mathbf{x}_B^T$	$\mathbf{x}_N^T$	
$\mathbf{I}$	$\mathbf{B}^{-1}N$	$\mathbf{B}^{-1}\mathbf{b}$
0^T	$\hat{\boldsymbol{\mu}}_k^T$	$\boldsymbol{\lambda}_k^T\mathbf{b}$

- If $\hat{\boldsymbol{\mu}} \geq 0$, $\mathbf{x}_k$ is a minimizer.
- Otherwise, and appropriate rule can be used to choose a negative component in $\hat{\boldsymbol{\mu}}_k$, say $(\hat{\boldsymbol{\mu}})_l < 0$. The column in $\mathbf{B}^{-1}\mathbf{N}$ gives $-\mathbf{d}_k^{(\mathbf{B})}$. This column will be referred to as the pivot column. The variable in $\mathbf{x}_N^T$ that corresponds to $(\hat{\boldsymbol{\mu}})_l$ is the variable chosen as a basic variable.
- Since $\mathbf{x}_k^{(\mathbf{N})} = 0$, $\mathbf{x}_k^{(\mathbf{B})} + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_k^{(\mathbf{N})} = \mathbf{B}^{-1}\mathbf{b}$ implies that $\mathbf{x}_k^{(\mathbf{B})} = \mathbf{B}^{-1}\mathbf{b}$. Therefore, the far-right p-dimensional vector gives $\mathbf{x}_k^{(\mathbf{B})}$.
- Since $\mathbf{x}_k^{(\mathbf{N})} = 0$, $\mathbf{c}^T x_k = 0^T\mathbf{x}_k^{(\mathbf{B})} + \hat{\boldsymbol{\mu}}_k^T\mathbf{x}_k^{(\mathbf{N})} - \boldsymbol{\lambda}_k^T\mathbf{b}$ implies that the number in the lower-right corner of the table is equal to $-f(\mathbf{x}_k)$.

Tabular Form of the Simplex Method

The important data at the kth iteration can be put together in a tabular form as a table.

Basic variables		Nonbasic variables				
x_2	x_5	x_1	x_3	x_4	$\mathbf{B}^{-1}\mathbf{b}$	
1	0	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	
0	1	-5	3	-2	1	
0	0	11	$-\frac{3}{2}$	$\frac{9}{2}$	$-\frac{9}{2}$	$\leftarrow \boldsymbol{\lambda}_k^T\mathbf{b}$

- From the previous example with x_0, since $(\hat{\mu})_2 < 0$, x_0 is not a minimizer. x_3 is the variable in $\mathbf{x}_0^{(\mathbf{N})}$ that will become a basic variable, and the vector above $(\hat{\mu})_2$, $\begin{bmatrix} \frac{1}{2} & 3 \end{bmatrix}^T$ is the pivot column $-\mathbf{d}_0^{(\mathbf{B})}$.
- From $\mathcal{I}_k = \{i : (\mathbf{d}_k^{(\mathbf{B})})_i < 0\}$, , the positive components of the pivot column should be used to compute the ratio $(\mathbf{x}_0^{(\mathbf{B})})_i/(-\mathbf{d}_0^{(\mathbf{B})})_i$ where $\mathbf{x}_0^{(\mathbf{B})}$ is the far-right column ($\mathbf{B}^{-1}\mathbf{b}$) in the table. The minimum ratio is $i^* = 2$. The second basic variable, x_5, should be exchanged with x_3 to become a nonbasic variable.

Tabular Form of the Simplex Method

Basic variables		Nonbasic variables				
x_2	x_5	x_1	x_3	x_4	$\mathbf{B}^{-1}\mathbf{b}$	
1	$-\frac{1}{6}$	$-\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{1}{3}$	
0	$\frac{1}{3}$	$-\frac{5}{3}$	1	$-\frac{2}{3}$	$\frac{1}{3}$	
0	0	11	$-\frac{3}{2}$	$\frac{9}{2}$	$-\frac{9}{2}$	$\leftarrow \boldsymbol{\lambda}_k^T\mathbf{b}$

- To transform x_3 into the second basic variable, we use row operations to transform the pivot column into the i^*th coordinate vector. Here we can add $-1/6$ times the second row to the first row, and then multiply the second row by $1/3$

Tabular Form of the Simplex Method

Basic variables		Nonbasic variables				
x_2	x_3	x_1	x_5	x_4	$\mathbf{B}^{-1}\mathbf{b}$	
1	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	
0	1	$-\frac{5}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	
0	0	$\frac{17}{2}$	$\frac{1}{2}$	$\frac{7}{2}$	-4	$\leftarrow \boldsymbol{\lambda}_k^T\mathbf{b}$

- We interchange the columns associated with variable x_3 and x_5 to form the updated basic and nonbasic variables, and then add $3/2$ times the second row to the last row to eliminate the nonzero Lagrange multiplier associated with variable x_3. Then swap x_3 and x_5.
- The Lagrange multipliers $\hat{\boldsymbol{\mu}}_1$ in the last row of the tale are all positive and hence $\mathbf{x}_1$ is the unique minimizer. Vector $\mathbf{x}_1$ is specified by $\mathbf{x}_1^{(\mathbf{B})} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3}\end{bmatrix}^T$ in the far-right column and $\mathbf{x}_1^{(\mathbf{N})} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T$.

Tabular Form of the Simplex Method

- In the conjunction with the composition of the basic and nonbasic variables, $\mathbf{x}_1^{(\mathbf{B})}$ and $\mathbf{x}_1^{(\mathbf{N})}$ yield

$$\mathbf{x}_1 = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix}^T$$

  At $\mathbf{x}_1$, the lower-right corner of the table gives the minimum of the objective function as $f(\mathbf{x}_1) = -\boldsymbol{\lambda}_k^T b = 4$.

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