Linear Programming II: Simplex Method I

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Objective

- Linear programming (LP) problems occur in a diverse range of real-life applications in economic analysis and planning, operations research, computer science, medicine, and engineering.
- These prolems, it is known that nay minima occur at the vertices of the feasible region and can be determined through a "brute-force" or exhaustive approach by evaluating the objective function at all the vertices of the feasible region.
- The number of variables involved in practical LP problem is often vary large and an exhaustive approach would entail a considerable amount of computation.
- In 1947, Dantzig developed a method for solving LP problems known as the *simplex method*. He solved this problem because he came to the class late and thought an unsolved problem on a blackboard was homework.
- Named one of the "Top 10 algorithms of the 20th century" by *Computing in Science & Engineering magazine.* Full list at: `https://www.siam.org/pdf/news/637.pdf`
- The simplex method has been the primary method for solving LP problems since its introduction.

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A general constrained optimization problems:

$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) \\
\text{subject to} \quad & g_i(\mathbf{x}) \leq 0, && \text{for } i = 1, 2, \ldots, q \\
& h_j(\mathbf{x}) = 0, && \text{for } j = 1, 2, \ldots, p \\
& x_{L_k} \leq x_k \leq x_{U_k}, && \text{for } k = 1, 2, \ldots, n,
\end{aligned}
$$

where x_L and x_U are lower bound and upper bound, respectively.

Regular point

A point $\mathbf{x}$ is called a *regular point* of the equality constraints if $\mathbf{x}$ satisfies $h_j(\mathbf{x}) = 0$ and column vector $\nabla h(\mathbf{x})$ are linearly independent.

- $\mathbf{x}$ is a regular point of the equality constraints if it is a solution of $h_j(\mathbf{x}) = 0$ and the Jacobian $J = \left[\nabla_{h_1}(\mathbf{x}) \quad \nabla_{h_2}(\mathbf{x}), \ldots, \nabla_{h_p}(\mathbf{x})\right]^T$ has full row rank.

General Constrained Optimization Problem

Consider the equality constraints

$$-x_1 + x_3 - 1 = 0$$
$$x_1^2 + x_2^2 - 2x_1 = 0$$

The Jacobian of the constraints is given by

$$\mathbf{J} = \begin{bmatrix} -1 & 0 & 1 \\ 2x_1 - 2 & 2x_2 & 0 \end{bmatrix}$$

- The Jacobian has rank 2 except $\mathbf{x} = \begin{bmatrix} 1 & 0 & x_3 \end{bmatrix}^T$
- $\mathbf{x} = \begin{bmatrix} 1 & 0 & x_3 \end{bmatrix}^T$ does not satisfy the second constraint.
- Any points satisfying both constraints is regular.

General Constrained Optimization Problem: Inequality constraints

Consider the constraints

$$g_1(\mathbf{x}) \leq 0, \quad g_2(\mathbf{x}) \leq 0, \quad \cdots \quad g_q(\mathbf{x}) \leq 0$$

- For the feasible point $\mathbf{x}$, these inequalities can be divided into two classes.
- The set of constraints with $g_i(\mathbf{x}) = 0$ are called ***active constraints***.
- The set of constraints with $g_i(\mathbf{x}) < 0$ is called ***inactive constraints***.
- We can convert inequality constraints into equality constraints by adding *slack variable* $s \geq 0$ as

$$\hat{g}_i(\mathbf{x}) = g_i(\mathbf{x}) + s_i = 0$$

The standard-form LP problem:

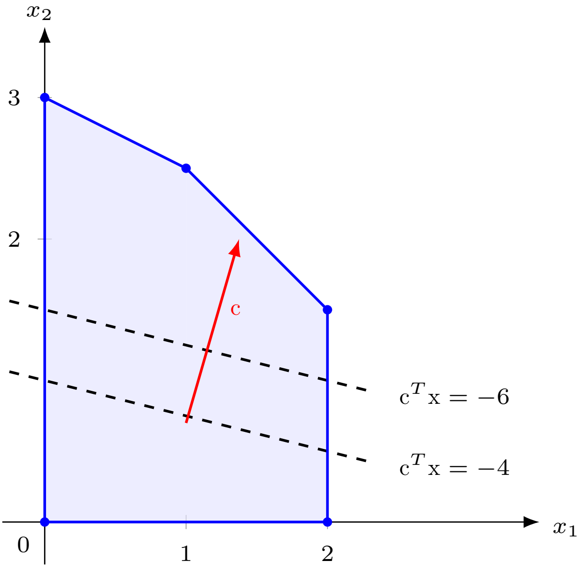
$$\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{c}^T\mathbf{x} \tag{4.1a}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \tag{4.1b}$$

$$\mathbf{x} \geq 0 \tag{4.1c}$$

where $\mathbf{c} \in \mathbb{R}^{n\times 1}$ with $\mathbf{c} \neq 0$, $\mathbf{A} \in \mathbb{R}^{p\times n}$, and $\mathbf{b} \in \mathbb{R}^{p\times 1}$ are given. We assume that $\mathbf{A}$ is of full row rank, i.e., $\text{rank}(\mathbf{A}) = p$. To be meaningful LP problem, full row rank in $\mathbf{A}$ implies that $p < n$.

- For $n = 2, \mathbf{c}^T\mathbf{x} = \beta$ represents a linea and $\mathbf{c}^T\mathbf{x} = \beta$ for $\beta = \beta_1, \beta_2, \ldots,$ represents a family of parallel lines.
- The normal of these lines is $\mathbf{c}$, and the vector $\mathbf{c}$ is often referred to as the *normal vector* of the objective function.


x_2
3
2
c
$c^T x = -6$
$c^T x = -4$
0
1
2
x_1

Another LP problem, which is often encountered in practice, involves minimizing a linear function subject to inequality constraints, i.e.,

$$\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{c}^T\mathbf{x} \tag{4.2a}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{4.2b}$$

where $\mathbf{c} \in \mathbb{R}^{n\times 1}$ with $\mathbf{c} \neq 0$, $\mathbf{A} \in \mathbb{R}^{p\times n}$, and $\mathbf{b} \in \mathbb{R}^{p\times 1}$ are given. This will be referred to as the *alternative-form* LP problem hereafter. If we let

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_p^T \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix}, \text{ then } a_i^T\mathbf{x} \leq b_i, \quad \text{for } i = 1, 2, \ldots, p$$

where vector $\mathbf{a}_i$ is the normal of the ith inequality constraint, and $\mathbf{A}$ is usually referred to as the *constraint matrix.*

General Properties

- by introducing a p-dimensional slack vector variable $\mathbf{s}$, the LP problem can be reformaulated as

$$\mathbf{A}\mathbf{x} + \mathbf{s} = \mathbf{b} \qquad \text{for } \mathbf{s} \geq 0$$

The vector variable $\mathbf{x}$ can be decomposed as

$$\mathbf{x} = \mathbf{x}' - \mathbf{x}'' \quad \text{with} \quad \mathbf{x}' \geq 0 \quad \text{with} \quad \mathbf{x}'' \geq 0$$

Hence if we let

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x}' & \mathbf{x}'' & \mathbf{s} \end{bmatrix}^T, \quad \hat{\mathbf{c}} = \begin{bmatrix} \mathbf{c} & -\mathbf{c} & 0 \end{bmatrix}^T, \quad \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & -\mathbf{A} & \mathbf{I}_p \end{bmatrix}$$

General Properties

- The non-standard LP can be reformulated as a standard-form LP problem , the increase in problem size leasd to reduced computational efficiency which can sometimes be a serious problem particularly when the number of inequality constraints is large.
- To solve each form LP will be described separately to enable us to solve each of these problems directly without the need of converting the one form into the other.

General Properties: KKT Conditions

- **Lagrange Multipliers** use to convert a constrained problem into a form such that the derivative test of an unconstrained problem can be applied.
- To find the maximum or minimum of a function $f(\mathbf{x})$ subjected to the equality constraint $g(\mathbf{x}) = 0$, we can introduce the Lagrangian function

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T g(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{p} \lambda_i a_i(\mathbf{x})$$

- At the extremal point we need

$$\nabla_x \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0$$
$$\nabla_\lambda \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0$$

- The Lagrangian incorporates the constraints into a modified objective function in such a way that a constrained minimizer $\mathbf{x}^*$ is connected to an unconstrainted minimizer $\{\mathbf{x}^*, \boldsymbol{\lambda}^*\}$ for the augmented objective function $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$

General Properties: KKT Conditions

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T\mathbf{p} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

$\mathbf{H} > 0$ and $\mathbf{A} \in \mathbb{R}^{p\times n}$ has full row rank.

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{x}^T p + \boldsymbol{\lambda}^T(\mathbf{A}\mathbf{x} - \mathbf{b})$$

$$\nabla\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \begin{bmatrix} \mathbf{H}\mathbf{x} + \mathbf{p} + \mathbf{A}^T\boldsymbol{\lambda} \\ \mathbf{A}\mathbf{x} - \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{H} & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{p} \\ -\mathbf{b} \end{bmatrix} = 0$$

$$\begin{bmatrix} \mathbf{x}^* \\ \boldsymbol{\lambda}^* \end{bmatrix} = \begin{bmatrix} \mathbf{H} & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{p} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} -\mathbf{H}^{-1}(\mathbf{A}^T\boldsymbol{\lambda}^* - \mathbf{p}) \\ -(\mathbf{A}\mathbf{H}^{-1}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{H}^{-1}\mathbf{p} + \mathbf{b}) \end{bmatrix}$$

General Properties: KKT Conditions

Karush-Kuhn-Tucker (KKT) conditions for standard LP

If $\mathbf{x}^*$ is regular for the constraints that are active at $\mathbf{x}^*$, then it is a global solution of the LP problem in the standard LP if an only if

- $\mathbf{A}\mathbf{x}^* = \mathbf{b}$, (4.3a)
- $\mathbf{x}^* \geq 0$, (4.3b)
- there exist Lagrange multipliers $\boldsymbol{\lambda}^* \in \mathbb{R}^{p\times 1}$ and $\boldsymbol{\mu}^* \in \mathbb{R}^{n\times 1}$ such that $\boldsymbol{\mu}^* \geq 0$ and

$$\mathbf{c} + \mathbf{A}^T\boldsymbol{\lambda}^* - \boldsymbol{\mu}^* = 0 \tag{4.3c}$$

- $\mu_i^* x_i^* = 0$ for $1 \leq i \leq n$ (4.3d)

- The first two condition simply say that solution $\mathbf{x}^*$ must be a feasible point. The constraint matrix $\mathbf{A}$ and vector $\mathbf{c}$ are related through the Lagrange multipliers $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$.

General Properties

- From (4.3a)-(4.3d), in most cases solution $\mathbf{x}^*$ cannot be strictly feasible.
- The term *strictly feasible points* is the points that satisfy the equality constraints with $x_i^* > 0$ for $1 \leq i \leq n$
- From (4.3d), $\boldsymbol{\mu}^*$ must be a zero vector for a strictly feasible point $\mathbf{x}^*$ to be a solution ($x_i^* > 0$). Hence (4.1c) becomes

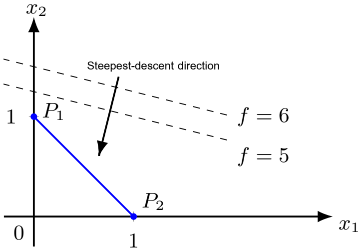
$$\mathbf{c} + \mathbf{A}^T \boldsymbol{\lambda}^* = 0$$

- For strictly feasible point to be a minimizer of the standard-form LP problem, the n-dimensional vector $\mathbf{c}$ must lie in the p-dimensional subspace spanned by the p columns of $\mathbf{A}^T$. Since $p < n$, the probability that $\mathbf{c} + \mathbf{A}^T \boldsymbol{\lambda}^* = 0$ is very small.
- Any solutions of the problem are very likely to be located on the *boundary* of the feasible region.

Solve the LP problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + x_2 = 1 \\ & x_i \geq 0, \quad i = 1, 2 \end{aligned}$$

- The feasible region of the problem is the segment of the line $x_1 + x_2 = 1$ in the first quadrant.
- The dashed lines are contours of the form $f(\mathbf{x}) =$ constant, and the arrow points to the steepest-descent direction of $f(\mathbf{x})$



We have

$$\mathbf{c} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \mathbf{A}^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since $\mathbf{c}$ and $\mathbf{A}^T$ are linearly independent, $\mathbf{c} = \mathbf{A}^T \boldsymbol{\lambda}^*$ cannot be satisfied and no interior feasible point can be a solution.

From the figure, the unique minimizer is $\mathbf{x}^* = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. At $\mathbf{x}^*$ the constraint $x_1 + x_2 = 1$ and $x_2 = 0$ are active. The Jacobian of these constraints,

$$\mathbf{J} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is nonsingular, $\mathbf{x}^*$ is a regular point. From $\mu_i^* x_i^* = 0$ and $x_1^* = 1$, then $\mu_1^* = 0$

$$\mathbf{c} + \mathbf{A}^T \boldsymbol{\lambda}^* - \boldsymbol{\mu}^* = 0$$
$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \boldsymbol{\lambda}^* - \begin{bmatrix} 0 \\ \mu_2^* \end{bmatrix} = 0$$
$$\boldsymbol{\lambda}^* = -1 \text{ and } \mu_2^* = 3$$

This is confirm that $\mathbf{x}^* = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is indeed a global solution (KKT condition).

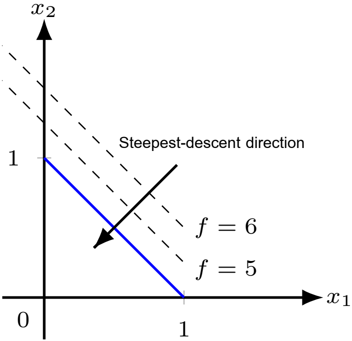
General Properties

Note: if the objective function is changed to

$$f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} = 4x_1 + 4x_2$$

We can have

$$\mathbf{c} + \mathbf{A}^T\boldsymbol{\lambda}^* - \boldsymbol{\mu}^* = 0 \quad \Longrightarrow \quad \boldsymbol{\lambda}^* = -4, \boldsymbol{\mu}^* = 0$$



Any feasible point becomes a global solution. The objective function remains constant ($x_1 + x_2 = 1$)in the feasible region, i.e.,

$$f(\mathbf{x}) = 4(x_1 + x_2) = 4, \text{ for } \mathbf{x} \in \mathbb{R}^2$$

General Properties

Consider an alternative LP

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

Necessary and sufficient conditions for a minimum in alternative form LP problem

If $\mathbf{x}^*$ is regular for the constraints in (4.2b) that are active at $\mathbf{x}^*$, then it is a global solution of the problem in (4.2a) if and only if

1. $\mathbf{A}\mathbf{x}^* \leq \mathbf{b}$ (4.4a)
2. there exists a $\boldsymbol{\mu}^* \in \mathbb{R}^{p\times 1}$ such that $\boldsymbol{\mu}^* \geq 0$ and

$$\mathbf{c} + \mathbf{A}^T\boldsymbol{\mu}^* = 0 \tag{4.4b}$$

3. $\boldsymbol{\mu}_i^*(\mathbf{a}_i^T\mathbf{x}^* - b_i) = 0$ for $1 \leq i \leq p$ (4.4c)
   where $\mathbf{a}_i^T$ is the ith row of $\mathbf{A}$

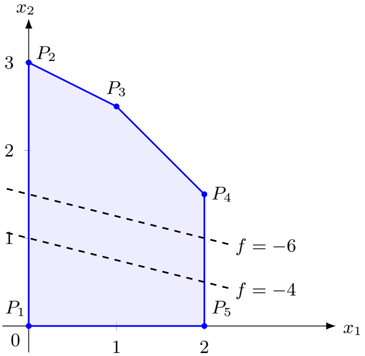
General Properties

- The theorem show that the solutions of the problem must be located on the boundary of the feasible region.
- If $\mathbf{x}^*$ is a strictly feasible point satisfying $\mu_i^*(\mathbf{a}_i^T\mathbf{x} - b_i) = 0$, then $\mathbf{A}\mathbf{x}^* < \mathbf{b}$ and the complementarity condition in (4.4c) implies that $\boldsymbol{\mu}^* = 0$. Hence (4.4b) cannot be satisfied unless $\mathbf{c} = 0$
- If $\mathbf{c} = 0$, it would lead to a meaningless LP problem.
- In other word, any solutions of (4.4a)-(4.4c) can only occur on the boundary of the feasible region.

General Properties

Solve the LP problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = -x_1 - 4x_2 \\ \text{subject to} \quad & -x_1 \le 0 \\ & x_1 \le 2 \\ & -x_2 \le 0 \\ & x_1 + x_2 - 3.5 \le 0 \\ & x_1 + 2x_2 - 6 \le 0 \end{aligned}$$



The five constraints can be expressed as $\mathbf{Ax} \le \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3.5 \\ 6 \end{bmatrix}$$

the feasible region is the polygon shown above.

General Properties

- The solution cannot be inside the polygon, we consider the five edges of the polygon. At any point of $\mathbf{x}$ on an edge other than the five vertices P_i only one constraints is active. This mean that only one of the five μ_i's is nonzero.
- At such an x_i, (which is on the edge.), (4.4b) becomes

$$\mathbf{c} = \begin{bmatrix} -1 & -4 \end{bmatrix}^T = -\mu_i \mathbf{a}_i$$

where $\mathbf{a}_i$ is the transpose of the ith row in $\mathbf{A}$.

- Since each $\mathbf{a}_i$ is linearly independent of $\mathbf{c}$, no μ_i exists that satisfies $\mathbf{c} = -\mu_i \mathbf{a}_i$
- We have five vertices for verification. At $P_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, the constraints $-x_1 = 0$, and $-x_2 = 0$ are active. Then $\mathbf{c} = -\mathbf{A}^T \boldsymbol{\mu}$ is

$$\begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_3 \end{bmatrix}, \Longrightarrow \quad \mu_1 = -1, \mu_3 = -4$$

- Since $\mu_i \leq 0$, then P_1 is not a solution.

General Properties

- At the point $P_2 = \begin{bmatrix} 0 & 3 \end{bmatrix}^T$, the constraints $-x_1 = 0$ and $x_1 + 2x_2 - 6 = 0$ are active. Then $\mathbf{c} = -\mathbf{A}^T \boldsymbol{\mu}$ is

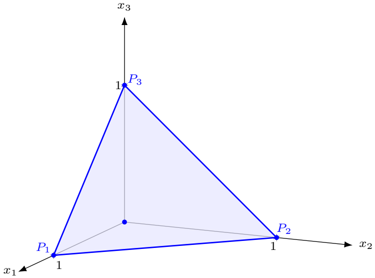
$$\begin{bmatrix} -1 \\ -4 \end{bmatrix} = -\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_5 \end{bmatrix}$$

$$\mu_1 = 1, \quad \mu_5 = 2$$

$$\boldsymbol{\mu} = \boldsymbol{\mu}^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \end{bmatrix}^T \geq 0$$

- $P_2 = \begin{bmatrix} 0 & 3 \end{bmatrix}^T$ is a minimizer, i.e., $\mathbf{x} = \mathbf{x}^* = P_2$.
- By checking the other vertex point, the point P_2 is the unique solution to the problem.

Facets, Edges, and Vertices



$$x_1 + x_2 + x_3 \le 1$$

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

$$Ax \le b$$

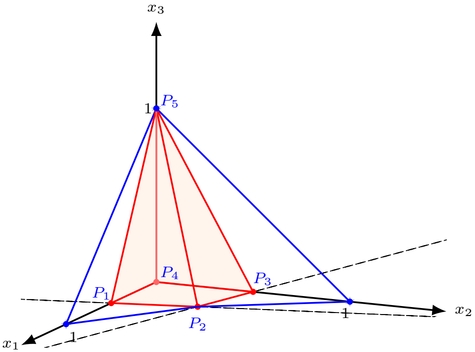
$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The polyhedron is three-dimension face, which has four facets, six edges, and four vertices.

A vertex is a feasible point P at which there exist at least n active constraints which contain n linearly independent constraints where n is the dimension of $\mathbf{x}$. Vertex P is said to be ***nondegenerate*** if exactly n constraints are active at P or ***degenerative*** if more than n constraints are active at P.

P_1, P_2, P_3, P_4 are nondegenerate vertices.

Facets, Edges, and Vertices



$$x_1 + x_2 + x_3 \leq 1$$

$$0.5x_1 + 2x_2 + x_3 \leq 1$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0.5 & 2 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- The convex polyhedron has five facets, eight edges, and five vertices.
- At vertex P_5 four constraints are active but since $n = 3$, P_5 is degenerate.
- The other four vertices, namely, P_1, P_2, P_3, and P_4, are nondegenerate.

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Feasible Descent Directions

Start from an initial point, we need to find a better new point:

Feasible direction

Let $\boldsymbol{\delta} = \alpha\mathbf{d}$ be a change in $\mathbf{x}$ where α is a positive constant and $\mathbf{d}$ is a direction vector. If $\mathcal{R}$ is the feasible region and a constant $\hat{\alpha} > 0$ exists such that

$$\mathbf{x} + \alpha\mathbf{d} \in \mathcal{R}$$

for all α in the range $0 \leq \alpha \leq \hat{\alpha}$, then $\mathbf{d}$ is said to be a *feasible direction* at point $\mathbf{x}$.

- A vector $\mathbf{d} \in \mathbb{R}^{n\times 1}$ is said to be a ***feasible descent direction*** at a feasible point $\mathbf{x} \in \mathbb{R}^{n\times 1}$ if $\mathbf{d}$ is a feasible direction and the linear objective function strictly decreases along $\mathbf{d}$, i.e., $f(\mathbf{x} + \alpha\mathbf{d}) < f(\mathbf{x})$ for $\alpha > 0$, where $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$.
- This implies that

$$f(\mathbf{x} + \alpha\mathbf{d}) = \mathbf{c}^T(\mathbf{x} + \alpha\mathbf{d}) = \mathbf{c}^T\mathbf{x} + \alpha\mathbf{c}^T\mathbf{d}$$
$$\frac{1}{\alpha}\left[f(\mathbf{x} + \alpha\mathbf{d}) - f(\mathbf{x})\right] = \frac{1}{\alpha}\left[\mathbf{c}^T\mathbf{x} + \alpha\mathbf{c}^T\mathbf{d} - \mathbf{c}^T\mathbf{x}\right] = \mathbf{c}^T\mathbf{d} < 0$$

Feasible Descent Directions

- For LP, we denote $\mathbf{A}_a$ as the matrix whose rows are the rows of $\mathbf{A}$ associated with the constraints active at $\mathbf{x}$.
- We can construct a matrix $\mathbf{A}_a$ the ***active constraint matrix*** at $\mathbf{x}$. If $\mathcal{J} = [j_1, j_2, \ldots, j_k]$ is the set of indices that identify active constraints at $\mathbf{x}$ then

$$\mathbf{A}_a = \begin{bmatrix} \mathbf{a}_{j_1}^T \\ \mathbf{a}_{j_2}^T \\ \vdots \\ \mathbf{a}_{j_k}^T \end{bmatrix}, \quad \mathbf{a}_j^T \mathbf{x} = b_j \qquad \text{for } j \in \mathcal{J}$$

- If $\mathbf{d}$ is a feasible direction, we must have

$$\mathbf{A}_a(\mathbf{x} + \alpha \mathbf{d}) \leq \mathbf{b}_a$$

where $\mathbf{b}_a = \begin{bmatrix} b_{j_1} & b_{j_2} & \cdots & b_{j_k} \end{bmatrix}^T$

Feasible Descent Directions

- $\mathbf{A}_a(\mathbf{x} + \alpha\mathbf{d}) \leq \mathbf{b}_a \Rightarrow \mathbf{A}_a\mathbf{x} + \alpha\mathbf{A}_a\mathbf{d} \leq \mathbf{b}_a$ since $\mathbf{A}_a\mathbf{x} = \mathbf{b}_a$, we must have $\mathbf{A}_a\mathbf{d} \leq 0$
- So the characterizes of a feasible descent direction $\mathbf{d}$ is

$$\mathbf{A}_a\mathbf{d} \leq 0 \quad \text{and} \quad \mathbf{c}^T\mathbf{d} < 0$$

The point $\mathbf{x}^*$ is a solution of the problem in (4.2a) and (4.2b) if and only if there is **no feasible descent directions exist at $\mathbf{x}^*$**.

Necessary and sufficient conditions for a minimum in alternative form LP problem

Point $\mathbf{x}^*$ is a solution of the problem in (4.2a) and (4.2b) if and only if it is feasible and

$$\mathbf{c}^T\mathbf{d} \geq 0 \quad \text{for all } \mathbf{d} \text{ with } \mathbf{A}_{a^*}d \leq 0$$

where $\mathbf{A}_{a^*}$ is the active constraint matrix at $\mathbf{x}^*$.

The theorem shows that we could not find the feasible descent directions.

Feasible Descent Directions

- For the standard form LP problem in (4.1a)-(4.1c), a feasible descent direction $\mathbf{d}$ at a feasible point $\mathbf{x}^*$ satisfies the constraints $\mathbf{A}\mathbf{d} = 0$ and $d_j \geq 0$ for $j \in \mathcal{J}_*$ and $\mathbf{c}^T\mathbf{d} \leq 0$, where $\mathcal{J}_* = [j_1, j_2, \ldots, j_k]$ is the set of indices for the constraints in (4.1c) that are active at $\mathbf{x}^*$.

Necessary and sufficient conditions for a minimum in standard form LP problem

Point $\mathbf{x}^*$ is a solution of the problem in (4.1a)-(4.1c) if and only if it is feasible and

$$\mathbf{c}^T\mathbf{d} \geq 0 \quad \text{for all } \mathbf{d} \text{ with } \mathbf{d} \in \mathcal{N}(\mathbf{A}) \text{ and } d_j \geq 0 \text{ for } j \in \mathcal{J}_*$$

where $\mathcal{N}(\mathbf{A})$ denotes the null space of $\mathbf{A}$.

$\mathbf{d} \in \mathcal{N}(\mathbf{A})$ means the set of $\mathbf{d}$ such that $\mathbf{A}\mathbf{d} = 0$.

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Feasible Descent Directions

Finding a Vertex

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Finding a Vertex (not the LP solution)

- We know that the solution of the LP problems can occur at vertex points. Under some conditions a vertex minimizer always exists.
- We need to have a strategy that can be used to find a minimizer vertex for the LP problem starting with a feasible point $\mathbf{x}_0$.
- In the kth iteration, if the active constraint matrix at $\mathbf{x}_k$, $\mathbf{A}_{a_k}$, has rank n, then $\mathbf{x}_k$ itself is already a vertex.
- Assume that $\text{rank}(\mathbf{A}_{a_k}) < n$. We will generate a feasible point $\mathbf{x}_{k+1}$ such that the active constraint matrix at $\mathbf{x}_{k+1}$, $\mathbf{A}_{a_{k+1}}$, is an *augmented* version of $\mathbf{A}_{a_k}$ with $\text{rank}(\mathbf{A}_{a_{k+1}})$ increased by one.
- $\mathbf{x}_{k+1}$ is a point such that (a) it is feasible, (b) all the constraints that are active at $\mathbf{x}_k$ remain active at $\mathbf{x}_{k+1}$, and (c) there is a new active constraint at $\mathbf{x}_{k+1}$, which was inactive at $\mathbf{x}_k$. A vertex can be identified in a finite number of steps.
- Let $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$. To make sure that all active constraints at $\mathbf{x}_k$ remain active at $\mathbf{x}_{k+1}$, we must have

$$\mathbf{A}_{a_k} \mathbf{x}_{k+1} = \mathbf{b}_{a_k}$$

Finding a Vertex

- Since $\mathbf{A}_{a_k}\mathbf{x}_k = \mathbf{b}_{a_k}$,

$$\mathbf{A}_{a_k}\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \alpha_k \mathbf{A}_{a_k}\mathbf{d}_k = \mathbf{b}_{a_k}$$

  it follows that $\mathbf{A}_{a_k}\mathbf{d}_k = 0$ (no more feasible direction).

- Since $\mathrm{rank}(\mathbf{A}_{a_k}) < n$, the solutions of $\mathbf{A}_{a_k}\mathbf{d}_k = 0$ form the null space of $\mathbf{A}_{a_k}$ of dimension $n - \mathrm{rank}(\mathbf{A}_{a_k})$. For a fixed $\mathbf{x}_k$ and $\mathbf{d}_k \in \mathcal{N}(\mathbf{A}_{a_k})$. We call an inactive constraint $\mathbf{a}_i^T\mathbf{x}_k - b_i < 0$ *increasing* with respect to $\mathbf{d}_k$ if $\mathbf{a}_i^T\mathbf{d}_k > 0$. (not in the null space.)

- If the ith constraint is an increasing constraint with respect to $\mathbf{d}_k$, then moving from $\mathbf{x}_k$ to $\mathbf{x}_{k+1}$ along $\mathbf{d}_k$, the constraint becomes

$$\begin{aligned}\mathbf{a}_i^T\mathbf{x}_{k+1} - b_i &= \mathbf{a}_i^T(\mathbf{x}_k + \alpha_k\mathbf{d}_k) - b_i \\ &= (\mathbf{a}_i^T\mathbf{x}_k - b_i) + \alpha_k\mathbf{a}_i^T\mathbf{d}_k = 0\end{aligned}$$

  with $\mathbf{a}_i^T\mathbf{x}_k - b_i < 0$ and $\mathbf{a}_i^T\mathbf{d}_k > 0$.

Finding a Vertex

- A positive α_k that makes the ith constraint active at point $\mathbf{x}_{k+1}$ can be identified as (= 0)

$$\alpha_k = -\frac{\mathbf{a}_i^T \mathbf{x}_k - b_i}{\mathbf{a}_i^T \mathbf{d}_k}$$

- The moving point along $\mathbf{d}_k$ also affects other inactive constraints and care must be taken to ensure that the value of α_k used does not lead to an infeasible $\mathbf{x}_{k+1}$.
- Two problems need to be addressed. (1) how to find a direction $\mathbf{d}_k$ in the null space $\mathcal{N}(\mathbf{A}_{a_k})$ such that there is at least one decreasing inactive constraint with respect to $\mathbf{d}_k$. (2) if $\mathbf{d}_k$ is found, how to determine the step size α_k.
- Given $\mathbf{x}_k$ and $\mathbf{A}_{a_k}$, we can find an inactive constraint whose normal $\mathbf{a}_i^T$ is linearly independent of the rows of $\mathbf{A}_{a_k}$. It follows that the system of equations

$$\begin{bmatrix} \mathbf{A}_{a_k} \\ \mathbf{a}_i^T \end{bmatrix} \mathbf{d}_k = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

has a solution $\mathbf{d}_k$ with $\mathbf{d}_k \in \mathcal{N}(\mathbf{A}_{a_k})$ and $\mathbf{a}_i^T \mathbf{d}_k > 0$.

Finding a Vertex

- The set of indices corresponding to increasing active constraints with respect to $\mathbf{d}_k$ can be defined as

$$\mathcal{I}_k = \left\{ i : \mathbf{a}_i^T \mathbf{x}_k - b_i < 0, \mathbf{a}_i^T \mathbf{d}_k > 0 \right\}$$

- The value of α_k can be determined as the value for which $\mathbf{x}_k + \alpha_k \mathbf{d}_k$ intersects the nearest new constraint. Hence

$$\alpha_k = \min_{i \in \mathcal{I}_k} \left(-\frac{(\mathbf{a}_i^T \mathbf{x}_k - b_i)}{\mathbf{a}_i^T \mathbf{d}_k} \right)$$

- If $i = i^*$ is an index in $\mathcal{I}_k$ that yields the α_k , then it is quite clear that at point $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$ the active constraint matrix becomes

$$\mathbf{A}_{a_{k+1}} = \begin{bmatrix} \mathbf{A}_{a_k} \\ \mathbf{a}_{i^*}^T \end{bmatrix}$$

where $\text{rank}(\mathbf{A}_{a_{k+1}}) = \text{rank}(\mathbf{A}_{a_k}) + 1$.

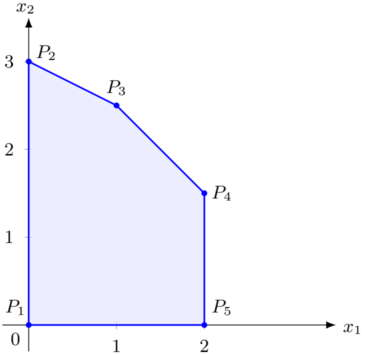
Finding a Vertex

- By repeating the above steps, a feasible point $\mathbf{x}_k$ with $\mathrm{rank}(\mathbf{A}_{a_k}) = n$ will eventually be reached, and point $\mathbf{x}_k$ is then deemed to be a vertex.

Finding a Vertex

Solve the LP problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = -x_1 - 4x_2 \\ \text{subject to} \quad & -x_1 \le 0 \\ & x_1 \le 2 \\ & -x_2 \le 0 \\ & x_1 + x_2 - 3.5 \le 0 \\ & x_1 + 2x_2 - 6 \le 0 \end{aligned}$$



The five constraints can be expressed as $\mathbf{A}\mathbf{x} \le \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3.5 \\ 6 \end{bmatrix}$$

the feasible region is the polygon shown above.

Finding a Vertex

- Starting from point $\mathbf{x}_0 = [1\ 1]^T$, apply the iterative procedure to find a vertex for the LP problem. Since the components of the residual vector at $\mathbf{x}_0$ is

$$\mathbf{r}_0 = \mathbf{A}\mathbf{x}_0 - \mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3.5 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1.5 \\ -3 \end{bmatrix} \text{ are all negative .}$$

- There are no active constraints at $\mathbf{x}_0$. If the first constraint (whose residual is the smallest) is chosen to form

$$\begin{bmatrix} \mathbf{A}_{a_k} \\ \mathbf{a}_i^T \end{bmatrix} \mathbf{d}_k = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies \mathbf{a}_1^T \mathbf{d}_0 = \begin{bmatrix} -1 & 0 \end{bmatrix} \mathbf{d}_0 = -d_{01} + (0)d_{02} = 1$$

$$\mathbf{d}_0 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T.$$

Finding a Vertex

- The set $\mathcal{I}_0$ in this case contains only one index, i.e., $\mathcal{I}_0 = \{1\}$.

$$\alpha_0 = -\frac{(\mathbf{a}_1^T \mathbf{x}_0 - b_1)}{\mathbf{a}_1^T \mathbf{d}_0} = -\left(\frac{-1-0}{1}\right) = 1$$

- Hence

$$\mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 \mathbf{d}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ with } \mathbf{A}_{a_1} = \begin{bmatrix} -1 & 0 \end{bmatrix}.$$

- At point $\mathbf{x}_1$

$$\mathbf{r}_1 = \mathbf{A}\mathbf{x}_1 - \mathbf{b} = \begin{bmatrix} 0 \\ -2 \\ -1 \\ -2.5 \\ -4 \end{bmatrix} \text{ Only } -x_1 \leq 1 \text{ is active.}$$

Finding a Vertex

- The third constraint (whose residual is the smallest and inactive) is chosen to be active:

$$\begin{bmatrix} \mathbf{A}_{a_1} \\ \mathbf{a}_3^T \end{bmatrix} \mathbf{d}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} d_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  we obtain $\mathbf{d}_1 = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$. It follows that $\mathcal{I}_1 = \{3\}$.

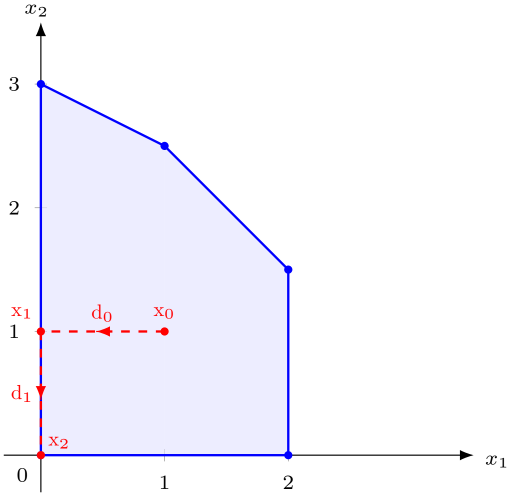
-

$$\alpha_1 = -\frac{\mathbf{a}_3^T \mathbf{x}_1 - b_3}{\mathbf{a}_3^T \mathbf{d}_1} = -\left(\frac{-1-0}{1}\right) = 1 \text{ with } i^* = 3$$

$$\mathbf{x}_2 = \mathbf{x}_1 + \alpha_1 \mathbf{d}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \mathbf{A}_{a_2} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Since $\operatorname{rank} \mathbf{A}_{a_2} = 2 = n$, $\mathbf{x}_2$ is a vertex.

Finding a Vertex



Finding a Vertex

Find the vertex for the convex polygon $x_1 + x_2 + x_3 = 1$ such that $x \geq 0$ starting with $\mathbf{x}_0 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}^T$. (For LP, we need $\mathbf{x} \geq 0$ to be $-\mathbf{x} \leq 0$.)

- We have

$$\mathbf{r}_0 = \mathbf{A}\mathbf{x}_0 - \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

  The problem is standard form.

- We select the first inequality constraint (they are equal) so

$$\begin{bmatrix} \mathbf{A}_{a_0} \\ \mathbf{a}_1^T \end{bmatrix} \mathbf{d}_0 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \mathbf{d}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  Since $d_{01} + d_{02} + d_{03} = 0$ and $-d_{01} = 1$, we have $d_{02} + d_{03} = 1$. Here we select $d_{02} = 1$ and $d_{03} = 0$. Then $\mathbf{d}_0 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T, \quad \mathcal{I}_0 = \{2\}$.

Finding a Vertex

- We have

$$\alpha_0 = -\frac{\mathbf{a}_1^T \mathbf{x}_0 - b_1}{\mathbf{a}_1^T \mathbf{d}_0} = -\frac{-\frac{1}{3} - 0}{1} = \frac{1}{3}, \text{ with } i^* = 2$$

$$\mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 \mathbf{d}_0 = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}^T$$

- $\mathbf{r}_1 = \begin{bmatrix} 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}^T$. Choosing the fourth inequality constraint , we have

$$\begin{bmatrix} \mathbf{A}_a \\ \mathbf{a}_4^T \end{bmatrix} \mathbf{d}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \implies \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{d}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{d}_1 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T, \text{ with } \mathcal{I}_1 = \{4\}$$

$$\alpha_1 = -\frac{\mathbf{a}_4^T \mathbf{x}_1 - b_4}{\mathbf{a}_4^T \mathbf{d}_1} = -\frac{-\frac{1}{3} - 0}{1} = \frac{1}{3}, \text{ with } i^* = 4$$

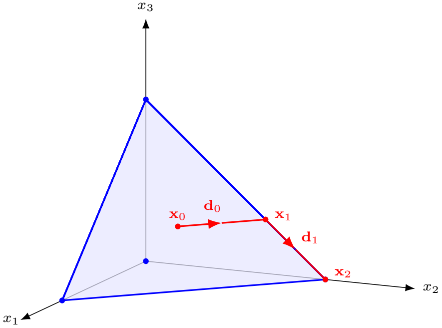
$$\mathbf{x}_2 = \mathbf{x}_1 + \alpha_1 \mathbf{d}_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

Finding a Vertex

We have

$$\mathbf{A}\mathbf{x}_2 - \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \Longrightarrow \mathbf{A}_{a_2} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ with } \operatorname{rank}(\mathbf{A}_{a_2}) = 3.$$

The point $\mathbf{x}_2$ is a vertex.



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Vertex Minimizers

The iterative method for finding a vertex described in the previous section does not involve the objective function $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$. The vertex obtained may not be a minimizer. If we start the iterative step at a minimizer, a vertex would eventually be reached without increasing the objective function, which is a vertex minimizer.

Existence of a vertex minimizer in alternative-form LP problem

If the minimum of $f(\mathbf{x})$ in the alternative-form LP problem is finite, then there is a vertex minimizer.

Proof: If $\mathbf{x}_0$ is a minimizer, then $\mathbf{x}_0$ is finite and satisfies the condition $\mathbf{A}\mathbf{x}_0 \leq \mathbf{b}$ and there exists a $\boldsymbol{\mu}^* \geq 0$ such that

$$\mathbf{c} + \mathbf{A}^T\boldsymbol{\mu}^* = 0 \quad \Rightarrow \quad \mathbf{c} + \mathbf{A}_{a_0}^T\boldsymbol{\mu}_a^* = 0,$$

where $\mathbf{A}_{a_0}$ is the active constraint matrix at $\mathbf{x}_0$ and $\boldsymbol{\mu}_a^*$ is composed of the entries of $\boldsymbol{\mu}^*$ that correspond to the active constraints.

Vertex Minimizers

- If $\mathbf{x}_0$ is not a vertex, the method described in the previous section can be applied to yield a point

$$\mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 \mathbf{d}_0$$

  which is closer to a vertex, where $\mathbf{d}_0$ is a feasible direction that satisfies the condition $\mathbf{A}_{a_0}\mathbf{d}_0 = 0$.
- It follows that at $\mathbf{x}_1$ the objective function remains the same as at $\mathbf{x}_0$, i.e.,

$$\begin{aligned} f(\mathbf{x}_1) = \mathbf{c}^T\mathbf{x}_1 = \mathbf{c}^T(x_0 + \alpha_0\mathbf{d}_0) = \mathbf{c}^T\mathbf{x}_0 - \alpha_0\mathbf{c}^T\mathbf{d}_0 \\ = \mathbf{c}^T\mathbf{x}_0 - \alpha_0(\boldsymbol{\mu}_a^*)^T\mathbf{A}_{a_0}\mathbf{d}_0 = \mathbf{c}^T\mathbf{x}_0 = f(\mathbf{x}_0) \end{aligned}$$

- If $\mathbf{x}_1$ is not yet a vertex, then the process is continued to generate minimizers $\mathbf{x}_2, \mathbf{x}_3, \ldots$ until a vertex minimizer is reached.

Vertex Minimizers

- To apply the theorem to the standard form, it follow that

$$\mathbf{c} = -\mathbf{A}^T\boldsymbol{\lambda}^* + \boldsymbol{\mu}^* = -\mathbf{A}^T\boldsymbol{\lambda}^* + \mathbf{I}_0^T\boldsymbol{\mu}_a^*$$

  where $\mathbf{I}_0$ consists of the rows of the $n \times n$ identity matrix that are associated with the inequality constraints $\mathbf{x} \geq 0$ that are active at $\mathbf{x}_0$, and $\boldsymbol{\mu}_a^*$ is composed of the entries of $\boldsymbol{\mu}^*$ that correspond to the active (inequality) constraints.

- At $\mathbf{x}_0$, the active constraint matrix $\mathbf{A}_{a0}$ is given by

$$\mathbf{A}_{a0} = \begin{bmatrix} -\mathbf{A} \\ \mathbf{I}_0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{c} = \mathbf{A}_{a0}^T \begin{bmatrix} \boldsymbol{\lambda}^* \\ \boldsymbol{\mu}_a^* \end{bmatrix}$$

  It can show that the objective function is not change.

Existence of a vertex minimizer in standard-form LP problem

If the minimum of $f(\mathbf{x})$ in the standard LP problem is finite, then a vertex minimizer exists.

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Simplex Method: Nondegenerate

- If the minimum value of the objective function in the feasible region is finite, then a vertex minimizer exists.
- Let $\mathbf{x}_0$ be a vertex and assume that it is not a minimizer. The simplex method generates an adjacent vertex $\mathbf{x}_1$ with $f(\mathbf{x}_1) < f(\mathbf{x}_0)$ and continues doing so until a vertex minimizer is reached.
- Given a vertex $\mathbf{x}_k$, a vertex $\mathbf{x}_{k+1}$ is *adjacent* to $\mathbf{x}_k$ if $\mathbf{A}_{a_{k+1}}$ is different from $\mathbf{A}_{a_k}$ by only one row.

$$\mathbf{A}_{a_k} = \begin{bmatrix} \mathbf{a}_{j_1}^T \\ \mathbf{a}_{j_2}^T \\ \vdots \\ \mathbf{a}_{j_n}^T \end{bmatrix}, \quad \mathcal{J}_k = \{j_1, j_2, \dots, j_n\}$$

- If $\mathcal{J}_k$ and $\mathcal{J}_{k+1}$ have exactly $(n-1)$ members, vertices $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ are adjacent.

Simplex Method: Nondegenerate

- At vertex $\mathbf{x}_k$, the simplex method verifies whether $\mathbf{x}_k$ is a vertex minimizer, and if it is not, it finds an adjacent vertex $\mathbf{x}_{k+1}$ that yields a *reduced* value of the objective function.
- Since a vertex minimizer exists and there is only a finite number of vertices, the method will find the solution using a finite number of iterations.
- Under the nondegeneracy assumption, $\mathbf{A}_{ak}$ is square and nonsingular. There exists a $\boldsymbol{\mu}_k \in \mathbb{R}^{n\times 1}$ such that

$$\mathbf{c} + \mathbf{A}_{a_k}^T \boldsymbol{\mu}_k = 0$$

Since $\mathbf{x}_k$ is a feasible point, we conclude that $\mathbf{x}_k$ is a vertex minimizer if and only if

$$\boldsymbol{\mu}_k \geq 0$$

$\mathbf{x}_k$ is not a vertex minimizer if and only if at least one component of $\boldsymbol{\mu}_k$ or $(\boldsymbol{\mu}_k)_l$ is negative.

Simplex Method: Nondegenerate

- Assume that $\mathbf{x}_k$ is not a vertex minimizer and let $(\boldsymbol{\mu}_k)_l < 0$.
- The simplex method finds an edge as a feasible descent direction $\mathbf{d}_k$ that points from $\mathbf{x}_k$ to an adjacent vertex $\mathbf{x}_{k+1}$ given by $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$.
- A feasible descent direction $\mathbf{d}_k$ is characterized by

$$\mathbf{A}_{a_k} \mathbf{d}_k \leq 0 \quad \text{and} \quad \mathbf{c}^T \mathbf{d}_k < 0 \qquad (*)$$

- To find the edge that satisfies $(*)$, we denote the lth coordinate vector (the lth column of the $n \times n$ identify matrix as $\mathbf{e}_l$ and examine vector $\mathbf{d}_k$ that solves the equation $\mathbf{A}_{a_k} \mathbf{d}_k = -\mathbf{e}_l$
- We note that $\mathbf{A}_{a_k} \mathbf{d}_k \leq 0$. We have

$$\mathbf{c}^T + \boldsymbol{\mu}_k^T \mathbf{A}_{a_k} = 0 \quad \Rightarrow \quad \mathbf{c}^T \mathbf{d}_k + \boldsymbol{\mu}_k^T \mathbf{A}_{ak} \mathbf{d}_k = 0$$
$$\mathbf{c}^T \mathbf{d}_k = -\boldsymbol{\mu}_k^T \mathbf{A}_{ak} \mathbf{d}_k = \boldsymbol{\mu}_k^T \mathbf{e}_l = (\boldsymbol{\mu}_k)_l < 0$$

Hence $\mathbf{d}_k$ satisfies $(*)$ and it is a feasible descent direction.

Simplex Method: Nondegenerate

- For $i \neq l$, $\mathbf{A}_{a_k}\mathbf{d}_k = -\mathbf{e}_l$ implies that

$$\mathbf{a}_{j_i}^T(\mathbf{x}_k + \alpha\mathbf{d}_k) = \mathbf{a}_{j_i}^T\mathbf{x}_k + \alpha\mathbf{a}_{j_i}^T\mathbf{d}_k = b_{j_i}$$

- There are exactly $n-1$ constraints that are active at $\mathbf{x}_k$ and remain active at $\mathbf{x}_k + \alpha\mathbf{d}_k$. This means that $\mathbf{x}_k + \alpha\mathbf{d}_k$ with $\alpha > 0$ is an edge that connects $\mathbf{x}_k$ to an adjacent vertex $\mathbf{x}_{k+1}$ with $f(\mathbf{x}_k + 1) < f(\mathbf{x}_k)$. The right step size α_k can be identified as

$$\alpha_k = \min_{i \in \mathcal{I}_k}\left(\frac{-(\mathbf{a}_i^T\mathbf{x}_k - b_i)}{\mathbf{a}_i^T\mathbf{d}_k}\right) = \frac{-(\mathbf{a}_{i^*}^T\mathbf{x}_k - b_{i^*})}{\mathbf{a}_{i^*}^T\mathbf{d}_k}$$

  where $\mathcal{I}_k$ contains the indices of the constraints that are inactive at $\mathbf{x}_k$ with $\mathbf{a}_i^T\mathbf{d}_k > 0$.

- The vertex $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$. Then at $\mathbf{x}_{k+1}$ the i^*th constraint becomes active.

Simplex Method: Nondegenerate

- Substituting j_lth constraint with the active i^*th constraint in $\mathbf{A}_{a_{k+1}}$, there are exactly n active constraints at $\mathbf{x}_{k+1}$ and $\mathbf{A}_{a_{k+1}}$ is given by

$$\mathbf{A}_{a_{k+1}} = \begin{bmatrix} \mathbf{a}_{j_1}^T \\ \vdots \\ \mathbf{a}_{j_{l-1}}^T \\ \mathbf{a}_{i^*}^T \\ \mathbf{a}_{j_{l+1}}^T \\ \vdots \\ \mathbf{a}_{j_n}^T \end{bmatrix}$$

- The index set is given by

$$\mathcal{J}_{k+1} = \{j_1, \ldots, j_{l-1}, i^*, j_{l+1}, \ldots, j_n\}$$

Simplex Method: Nondegenerate

Simplex algorithm for the alternative-form LP problem, nondegenerate vertices

1. Input vertex $\mathbf{x}_0$, and form $\mathbf{A}_{a_0}$ and $\mathcal{J}_0$. Set $k = 0$
2. Solve $\mathbf{A}_{a_k}^T \boldsymbol{\mu}_k = -\mathbf{c}$ for $\boldsymbol{\mu}_k$. If $\boldsymbol{\mu}_k \geq 0$, stop ($\mathbf{x}_k$ is a vertex minimizer): otherwise, select the index l that corresponds to the most negative component in $\boldsymbol{\mu}_k$.
3. Solve $\mathbf{A}_{a_k} \mathbf{d}_k = -\mathbf{e}_l$, where $\mathbf{e}_l$ is a unit vector at l index for $\mathbf{d}_k$.
4. Compute the residual vector $\mathbf{r}_k = \mathbf{A}\mathbf{x}_k - \mathbf{b} = (r_i)_{i=1}^p$ If the index set

$$\mathcal{I}_k = \{i : r_i < 0 \text{ and } \mathbf{a}_i^T \mathbf{d}_k > 0\} \text{ is empty, stop}$$

   (The objective function tends to $-\infty$ in the feasible region);

Simplex Method

Simplex algorithm for the alternative-form LP problem, nondegenerate vertices cont.

4. (cont.) otherwise, compute

$$\alpha_k = \min_{i \in \mathcal{I}_k} \left(\frac{-r_i}{\mathbf{a}_i^T \mathbf{d}_k} \right)$$

   and record the index i^* with $\alpha_k = -r_{i^*}/(\mathbf{a}_{i^*}^T \mathbf{d}_k)$. **Note:** $\mathbf{a}_i^T$ is the row of $\mathbf{A}_{a_k}$ such that $r_i < 0$.

5. Set $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$. Update $\mathbf{A}_{a_{k+1}}$ and $\mathcal{J}_{k+1}$ using

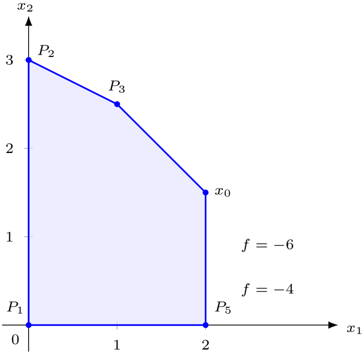
$$\mathbf{A}_{a_{k+1}} = \begin{bmatrix} \mathbf{a}_{j1} & \cdots & \mathbf{a}_{j_{l-1}} & \mathbf{a}_{i^*} & \mathbf{a}_{j_{l+1}} & \cdots & \mathbf{a}_{j_n} \end{bmatrix}^T$$
$$\mathcal{J}_{k+1} = \{j_1, \ldots, j_{l-1}, i^*, j_{l+1}, \ldots, j_n\}$$

   Set $k = k + 1$ and repeat from Step 2.

Simplex Method

Solve the LP problem with initial vertex $x_0 = \begin{bmatrix} 2 & 1.5 \end{bmatrix}^T$

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f(\mathbf{x}) = -x_1 - 4x_2 \\
\text{subject to} \quad & -x_1 \leq 0 \\
& x_1 \leq 2 \\
& -x_2 \leq 0 \\
& x_1 + x_2 - 3.5 \leq 0 \\
& x_1 + 2x_2 - 6 \leq 0
\end{aligned}
$$



The five constraints can be expressed as $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ with

$$
\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3.5 \\ 6 \end{bmatrix}
$$

the feasible region is the polygon shown above.

Simplex Method

- With $\mathbf{x}_0$ the second and fourth constraints are active and hence

$$\mathbf{A}_{a_0} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathcal{J} = \{2, 4\}$$

- Solving $\mathbf{A}_{a_0}^T \boldsymbol{\mu}_0 = -\mathbf{c}$ for $\boldsymbol{\mu}_0$ where $\mathbf{c} = \begin{bmatrix} -1 & -4 \end{bmatrix}^T$, we obtain $\boldsymbol{\mu}_0 = \begin{bmatrix} -3 & 4 \end{bmatrix}^T$. Since $\boldsymbol{\mu}_{0_1}$ is negative, $\mathbf{x}_0$ is not a minimizer and $l = 1$. Next we solve

$$\mathbf{A}_{a_0} \mathbf{d}_0 = -\mathbf{e}_1 \quad \Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{d}_0 = -\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\mathbf{d}_0 = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$$

- The residual vector at $\mathbf{x}_0$ is given by

$$\mathbf{r}_0 = \mathbf{A}\mathbf{x}_0 - \mathbf{b} = \begin{bmatrix} -2 & 0 & -1.5 & 0 & -1 \end{bmatrix}^T$$

- $\mathbf{r}_0$ shows that the first, third, and fifth constrains are inactive at $\mathbf{x}_0$.

$$\begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_3^T \\ \mathbf{a}_5^T \end{bmatrix} \mathbf{d}_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathcal{I}_0 = \{1, 5\}$$

$$\alpha_0 = \min \left(\frac{-r_{01}}{\mathbf{a}_1^T \mathbf{d}_0}, \frac{-r_{05}}{\mathbf{a}_5^T \mathbf{d}_0} \right) = 1$$

$$\mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 \mathbf{d}_0 = \begin{bmatrix} 2 \\ 1.5 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$$

- Since $l = 1$, we have (by swapping $\mathbf{a}_1^T$ and $\mathbf{a}_5^T$),

$$\mathbf{A}_{a1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } \mathcal{J}_1 = \{5, 4\}$$

- End of the first iteration.

- The second iteration starts by solving $\mathbf{A}_{a_1}^T \boldsymbol{\mu}_1 = -\mathbf{c}$ for $\boldsymbol{\mu}_1$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{\mu}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \Longrightarrow \boldsymbol{\mu}_1 = (-1) \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

  Then the point $\mathbf{x}_1$ is not a minimizer and $l = 2$.

- By solving $\mathbf{A}_{a_1} \mathbf{d}_1 = -\mathbf{e}_2$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{d}_1 = - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{d}_1 = (-1) \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- We compute the residual vector at $\mathbf{x}_1$ as

$$\mathbf{r}_1 = \mathbf{A}\mathbf{x}_1 - \mathbf{b} = \begin{bmatrix} -1 & -1 & -2.5 & 0 & 0 \end{bmatrix}^T$$

  It indicates that the first three constraints are inactive at $\mathbf{x}_1$.

Simplex Method

- By evaluating

$$\begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \mathbf{a}_3^T \end{bmatrix} \mathbf{d}_1 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, \quad \mathcal{I}_1 = \{1\}, \quad \alpha_1 = \frac{-r_{11}}{\mathbf{a}_1^T \mathbf{d}_1} = \frac{1}{2}$$

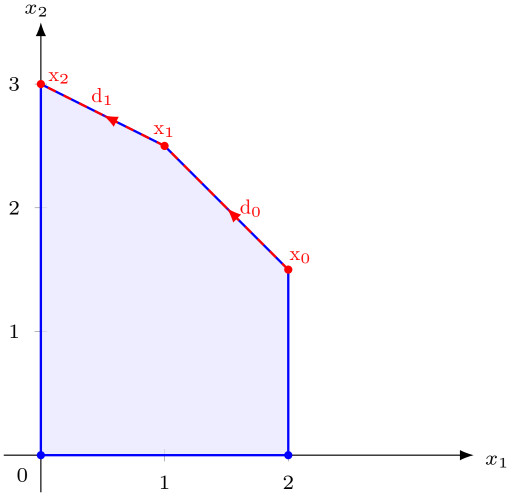
- This leads to $\mathbf{x}_2 = \mathbf{x}_1 + \alpha_1 \mathbf{d}_1 = \begin{bmatrix} 0 & 3 \end{bmatrix}^T$ with $l = 2$ (by swapping $\mathbf{a}_2^T$ and $\mathbf{a}_1^T$)

$$\mathbf{A}_{a_2} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \text{ and } \mathcal{J}_2 = \{5, 1\}$$

  Which is complete the second iteration.

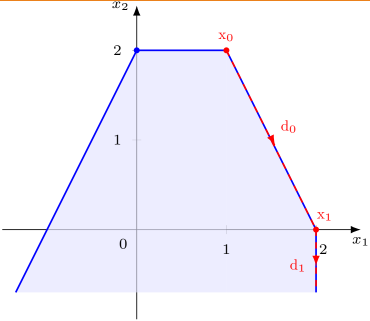
- Vertex x_2 is confirmed to be a minimizer at the beginning of the third iteration since the equation $\mathbf{A}_{a_2}^T \boldsymbol{\mu}_2 = -\mathbf{c}$ and yields nonnegative Lagrange multipliers $\boldsymbol{\mu}_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$.

Simplex Method



Simplex Method

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = x_1 + x_2 \\ \text{subject to} \quad & x_1 \leq 2 \\ & x_2 \leq 2 \\ & -2x_1 + x_2 \leq 2 \\ & 2x_1 + x_2 \leq 4 \end{aligned}$$



the constraints can be written as $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 1 \\ 2 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

Note: The feasible region is unbounded.

Simplex Method

- We start with the vertex $\mathbf{x}_0 = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$. At $\mathbf{x}_0$

$$\mathbf{r}_0 = \mathbf{A}\mathbf{x}_0 - \mathbf{b} = \begin{bmatrix} -1 & 0 & -2 & 0 \end{bmatrix}^T, \text{ second and fourth constraints are active.}$$

$$\mathbf{A}_{a_0} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \text{ and } \mathcal{J}_0 = \{2, 4\}$$

- From $\mathbf{A}_{a_0}^T \boldsymbol{\mu}_0 = -\mathbf{c}$, we have

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \mu_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \implies \boldsymbol{\mu}_0 = -\frac{1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$\mathbf{x}_0$ is not a minimizer.

- Since both components of $\boldsymbol{\mu}_0$ are negative, we can choose index l to be either 1 or 2.

Simplex Method

- Choosing $l = 1$,

$$\mathbf{A}_{a_0}\mathbf{d}_0 = -\mathbf{e}_1 \implies \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{d}_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \implies \mathbf{d}_0 = -\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$$

- The residual vector at $\mathbf{x}_0$ is given by

$$\mathbf{r}_0 = \mathbf{A}\mathbf{x}_0 - \mathbf{b} = \begin{bmatrix} -1 & 0 & -2 & 0 \end{bmatrix}^T$$ the first and third constraints are inactive at $\mathbf{x}_0$.

- We compute

$$\begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_3^T \end{bmatrix} \mathbf{d}_0 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -2 \end{bmatrix}, \quad \mathcal{I}_0 = \{1\}$$

$$\alpha_0 = \frac{-r_{01}}{\mathbf{a}_1^T \mathbf{d}_0} = \frac{1}{\frac{1}{2}} = 2$$

Simplex Method

- the next vertex is $\mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 \mathbf{d}_0 = \begin{bmatrix} 2 & 0 \end{bmatrix}^T$, with

$$\mathbf{A}_{a_1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \text{ and } \mathcal{J}_1 = \{1, 4\}$$

- Check whether x_1 is a minimizer by solving

$$\mathbf{A}_{a_1}^T \boldsymbol{\mu}_1 = -\mathbf{c} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \boldsymbol{\mu}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{\mu}_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

  indicating that $\mathbf{x}_1$ is not a minimizer and $l = 2$

- Solving

$$\mathbf{A}_{a_1} \mathbf{d}_1 = -\mathbf{e}_2 \quad \Longrightarrow \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \mathbf{d}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \Longrightarrow \mathbf{d}_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Simplex Method

- The residual vector at x_1 is

$$\mathbf{r}_1 = \mathbf{A}\mathbf{x}_1 - \mathbf{b} = \begin{bmatrix} 0 & -2 & -6 & 0 \end{bmatrix}^T$$

- The second and third constraints are inactive. We evaluate

$$\begin{bmatrix} \mathbf{a}_2^T \\ \mathbf{a}_3^T \end{bmatrix} \mathbf{d}_1 = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

- Since $\mathcal{I}_1$ is empty, we conclude that the solution of this LP problem is un-bounded.

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