Introduction to Optimization I

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Definition: Numerical (Mathematical) Optimization

- **Optimization (in everyday language):** Improvement of a good solution by intuitive, brute-force or heuristics-based decision-making.
- **Numerical (Mathematical) Optimization:** Finding the best possible solution using a mathematical problem formulation and a rigorous/ heuristic numerical solution method.
- **Mathematical programming** is used as an alternative to numerical optimization. The term *programming* referred to the solution of *planning problems.*

Formulation of Optimization Problems

The general formulation of an optimization problem consists of:

- the **variables** (also called decision variables, degrees of freedom, parameters, ...)
- An **objective function**
- A **mathematical model** for the description of the system to be optimized
- **Additional restrictions** on the optimal solution, including bounds of the variables.

The mathematical model of the system under consideration and the additional restrictions are also referred to as **constraints.**
The objective function can either be **minimized** or **maximized**.

Formulation of Optimization Problems

- The **objective function** describes an economical measure (operating costs, investment costs, profit, etc.) or technological, or ...
- The mathematical modeling of the system results in models to be added to the optimization problem as **equality constraints.**
- The **additional constraints** (mostly linear inequalities)(result, for instance, from:)
    - plant- or equipment-specific limitations (capacity, pressure, etc.)
    - material limitations (explosion limit, boiling point, corrosivity, etc.)
    - product requirements (quality, etc.)
    - resources (availability, quality, etc.)

Solution of Optimization Problems

What defines the solution of an optimization problem?

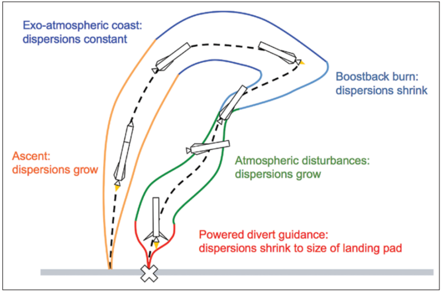
- Those values of the influencing variables (decision variables or degrees of freedom) are sought, which maximize of minimize the objective function.
- The values of the degrees of freedom must satisfy the mathematical model and all additional constraints like, for instance, physical or resource limitations at the optimum.
- The solution is, typically, a compromise between opposing effects. In process design, for instance, the investment costs can be reduced while increasing the operating costs (and vice versa).

Applications of Optimization

- Adjusts its speed over time
- Jobs must be scheduled. Each job has: arrival time, deadline, total work required.
- Budget

- **Business decisions** (determination of product portfolio, choice of location of production sites, analysis fo completing investments, etc.)
- **Design decisions: Process, plant and equipment** (structure of a process or energy conversion plant, favorable operating point, selection and dimensions of major equipment, modes of process operation, etc.)

Applications



- Autonomous Precision Landing of Space Rockets of F9R return-to-launch-site mission.
- The high speed on board Convex Optimization.

- **Operational decisions**(adjustment of the operating point to changing environmental conditions, production planning, control for disturbance mitigation and set-point tracking, etc.)
- **Model Identification** (parameter estimation, design of experiments, model structure discrimination, etc.)

Example: Optimal Motion Planning of Robots

Task:

- Transportation and accurate positioning of a part, e.g., during the assembly of an automobile windscreen.

Aims:

- Short cycle time for production, e.g., minimization of transportation time through optimal motion planning
- Correct positioning of the part during assembly
- no collisions during movement

Example: Wind Turbines Problem

https://commons.wikimedia.org/wiki/File:Wind_farm_near_North_Sea_coast.jpg

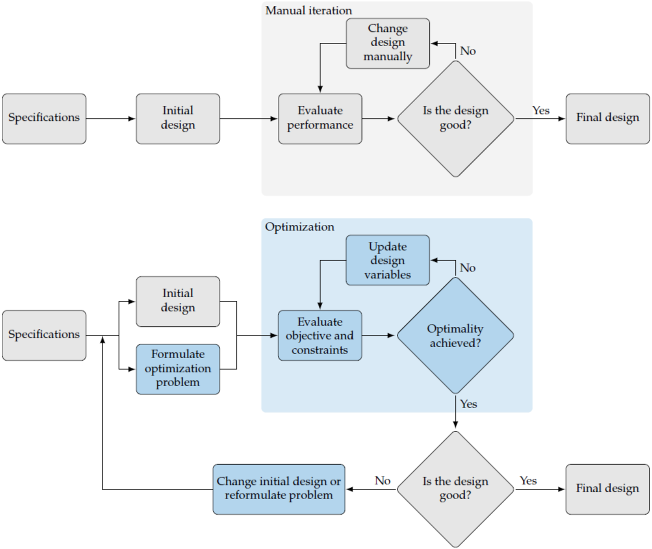
Task:

- Wind turbines are built in groups (=wind farms) to produce more electricity in a given limited area
- Wind farm layout: Where to position turbines within farm limits? How many turbines?
- Maximize annual electricity production. Minimize levelized cost of electricity

Solution:

- Fixed cells, Continuous positions, Patterns

Conventional vs Design optimization process [1]



General Form

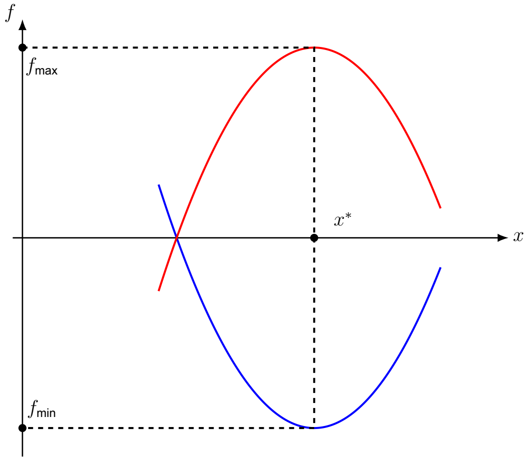
The General Form of Optimization Problem

$$
\begin{aligned}
\text{minimize} \quad & f(\mathbf{x}) \\
\text{subject to} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\
& h_j(\mathbf{x}) = 0, \quad j = 1, \ldots, l \\
& \mathbf{x}^L \leq \mathbf{x} \leq \mathbf{x}^U
\end{aligned}
$$

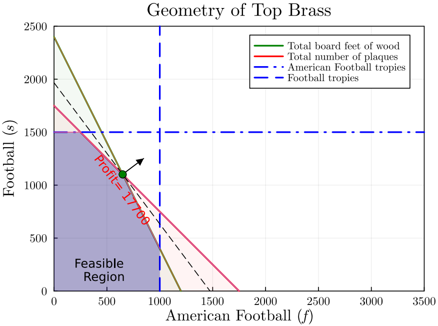
- $\mathbf{x} \in \mathbb{R}^n$ is the optimization variable
- $f(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ is the objective or cost function
- $g_i : \mathbb{R}^n \mapsto \mathbb{R}, i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbb{R}^n \mapsto \mathbb{R}, i = 1, \ldots, l$, are the equality constraint functions.

Maximization vs. Minimization

Maximization of $f(x)$ is equivalent to minimization of $-f(x)$.



Feasible Region



We can express the general form as

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{x} \in \Omega, \end{aligned}$$

where $\Omega = \{\mathbf{x} : g(\mathbf{x}) \leq 0, h(\mathbf{x}) = 0, \mathbf{x}^L \leq \mathbf{x} \leq \mathbf{x}^U\}$ is a subset of $\mathbb{R}^n$ Ω is called the *feasible region*.

Type of Variables and Problems

Additions restrictions may be imposed on a variables x_j as:

- x_j is continuous (default)
- x_j is binary (equals 0 or 1)
- x_j is integer (equals 1 or 2 or 3, $\ldots$ or N)
- x_j is discrete (e.g., takes values 10 mm or 20 mm or 30 mm, etc.)

Specialized name are given to the standard form such as:

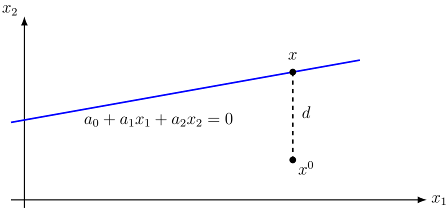
- Linear Programming (LP): When all functions (objective and constraints) are linear (in x)
- Integer Programming (IP): An LP when all variables are required to be integers.
- Mixed Integer Programming (MIP): An IP where some variables are required to be integers, others are continuous.
- MINP: an MIP with nonlinear functions.
- Quadratic Programming (QP): When objective function is a quadratic function in x and all constraints are linear.

Type of Variables and Problems

- **Convex Programming (CP):** When objective is convex (for minimization) or concave (for maximization), and the feasible region Ω is a convex set. Here , an local minimum is also a global minimum. Powerful solution techniques that can handle large number of variables exist for this category. Convexity of Ω is guaranteed when all inequality constraints g_i are convex functions and all equality constraints h_j are linear.
- **Combinatorial Problems:** These generally involve determining an optimum permutation of a set of integers, or equivalently, and optimum choice among a set of discrete choices. Some combinatorial problems can be posed as LP problems (which are much easier to solve). Heuristic algorithms (containing *thumb rules*) play a crucial role in solving large scale combinatorial problems where the aim is to obtain near-optimal solutions rather than the exact optimum.

Determine the shortest distance d between a given point $\mathbf{x}^0 = \begin{bmatrix} x_1^0 & x_2^0 \end{bmatrix}$ and a given lines

$$a_0 + a_1 x_1 + a_2 x_2 = 0$$



$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 \\ \text{subject to} \quad & a_0 + a_1 x_1 + a_2 x_2 = 0 \end{aligned}$$

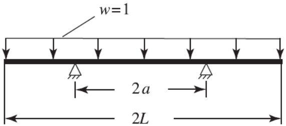
Matrix Form:

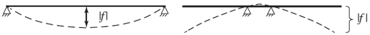
$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & (\mathbf{x} - \mathbf{x}^0)^T (\mathbf{x} - \mathbf{x}^0) \\ \text{subject to} \quad & \mathbf{a}^T \mathbf{x} - b = 0, \end{aligned}$$

where $b = -a_0$.

Optimization Problem Modeling

Consider a uniformly loaded beam on two supports as shown in Fig. The beam length is $2L$ units,and the spacing between supports is $2a$ units. We wish to determine the halfspacing a/L so as to minimize the maximum deflection that occurs in the beam.





- When the support spacing is too large wherein the maximum deflection occurs at the center.
- When the spacing is too small wherein the maximum deflection occurs at the ends.
- The graph of deflection vs. spacing is convex or cup-shaped with a well-defined minimum.

We may state that the the maximum deflection δ at any locations in the beam, which is a function of both position and support spacing, can be reduced to checking the maximum at just two locations.

Optimization Problem Modeling

The objective function f which is to be minimized is given by

$$f(a) = \max_{a \leq x \leq 1} \delta(x, a) = \max\{\delta(0, a), \delta(1, a)\}$$

We can determine the optimum spacing a by solving the optimization problem

$$\begin{aligned} \underset{a}{\text{minimize}} \quad & f(a) \\ s.t. \quad & 0 \leq a \leq 1 \end{aligned}$$

Optimization Problem Modeling

- VLSI is a process used to build electronic components such as microprocessors and memory ships comprising millions of transistors.
- The first stage is to produce a set of indivisible rectangular blocks called *cells*.
- The second stage, interconnection information is used to determine the relative placements of these cells.
- The third state, implementation are selected for the various cells with the goal of optimizing the total area, which is related to cost of the chips.
- floor plan optimization is the third stage.

We have three rectangular cell. Dimensions of C1 is 5×10, C2 is one of $(3 \times 8), (2 \times 12)$ or (6×4), and C3 is chosen from (5×8) or (8×5). Relative ordering of the cells must satisfy the following vertical and horizaontal ordering:

- **Vertical:** C2 must lie above C3
- **Horizontal:** C1 must lie to the left of C2 and C3.

We can solve this problem by using a mixed-integer nonlinear program.

- Let $(x_i,\ h_i)$, $i = 1, 2, 3$ denote the width and height of cell i.
- (x_i, y_i) denote the coordinates of the left bottom corner of cell i
- Let W, H represent the width and height, respectively.

We have the constraints:

$$y_1 \geq 0,\ y_1 + h_1 \leq H,\ y_3 \geq 0,\ y_3 + h_3 \leq y_1, y_2 + h_2 \leq H$$
$$x_1 \geq 0,\ x_1 + w_1 \leq x_2,\ x_1 + w_1 \leq x_3,\ x_2 + w_2 \leq W,\ x_3 + w_3 \leq W$$

Optimization Problem Modeling

Introducing the binary or $0/1$ variables δ_{ij} to implement discrete selection.

$$
\begin{aligned}
w_2 &= 8\delta_{21} + 12\delta_{22} + 4\delta_{23}, & h_2 &= 3\delta_{21} + 2\delta_{22} + 6\delta_{23} \\
w_3 &= 5\delta_{31} + 8\delta_{31}, & h_3 &= 8\delta_{31} + 5\delta_{32} \\
&\delta_{21} + \delta_{22} + \delta_{23} = 1 & &\delta_{31} + \delta_{32} = 1 \\
&\delta_{ij} = 0 \text{ or } 1
\end{aligned}
$$

There are 13 variables inthe problem are $(x_i, y_i), \delta_{ij}, W, H$, and the objective function is

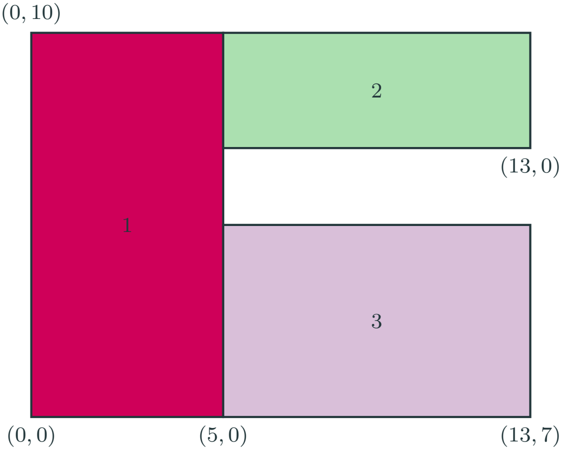
$$f(W, H) = WH \text{ is to be minimized}$$

The solution of this problem (using Branch and Bound procedure) is

$$\left[x_1, y_1, w_1, h_1, x_2, y_2, w_2, h_2, x_3, y_3, w_3, h_3\right] = \left[0, 0, 5, 10, 5, 7, 8, 3, 5, 0, 8, 5\right]$$

The associated optimum objective is $f = \text{Area} = 130$ units.

Optimization Problem Modeling



Optimization Problem Modeling

The problem is as follow:

- Assume you have \$ 100 and you want to invest it in four stocks– how much will you invest in each?
- You have to balace between profit and risk, as some stocks may yield high return with high risk, others may yield lower return with lower risk.

Mathematical modeling of this problem is as follows:

- Let $c_i, i = 1, \ldots, n$, represent the average return (for example two months period) of stock i , with n = total number of stocks, σ_i^2 represent the variance of each stock, where σ_i is the standard deviation, and $x_i =$ money invested in stock i, expressed as a percentage of total investment.

$$\text{Expected (average) return on total investment } = \sum_{i=1}^{n} c_i x_i$$

$$\text{Variance of total investment } = \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j,$$

where $\sigma_{ij} =$ correlation coefficient between a pair of stocks i and j.

Optimization Problem Modeling

The inventor's objective is to maximize return with limited risk (as indicated by variance). The balance is achieved by a penalty function as

$$\text{Objective function } f = \sum_{i=1}^{n} c_i x_i - \frac{1}{\lambda} \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j \text{ to be maximized,}$$

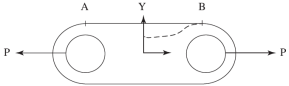
where $\lambda =$ a penalty parameter.
The optimization problem may be summarized in matrix form as

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T \mathbf{x} - \frac{1}{\lambda} \mathbf{x}^T \mathbf{R} \mathbf{x} \\ \text{subject to} \quad & \mathbf{x}^T \mathbb{1} = 1, \quad \mathbf{x} \geq 0 \end{aligned}$$

where $\mathbb{1}$ is a vector that all element are one.
This problem is a quadratic programming (QP) problem, owing to a quadratic objective function and linear constraints. $\mathbf{R}$ is a square, symmetric correlation matrix with σ along the diagonal. Solution $\mathbf{x}$ gives the percentage portfolio investment.

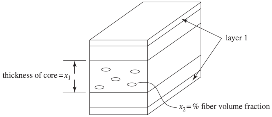
Optimization Problem Modeling

- Refueling Optimization
- Renumbering for Efficient Equation Solving in Finite Elements
- Shape Optimization



Bicycle Chain link

- Sizing Optimization



- Noise Reduction Problem

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