Constrained Optimization I: Introduction

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Objective

At the end of this chapter you should be able to:

- Describe and implement the constrained optimization problems
- Understand the concept of **Lagrange multipliers**
- Understand the **Karush-Kuhn-Tucker** conditions

Notation and Basic Assumptions

Definition: Constrained Optimization Problem

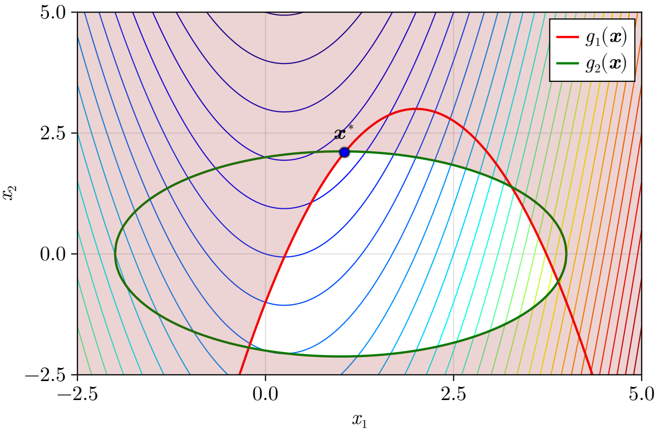
$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) \\
\text{subject to} \quad & h_i(\mathbf{x}) = 0 \quad \text{for } i = 1, 2, \ldots, p \\
& g_j(\mathbf{x}) \leq 0 \quad \text{for } j = 1, 2, \ldots, q
\end{aligned}
$$

where $h_i(\mathbf{x})$ is a **equality constraint**, and $g_j(\mathbf{x})$ is the vector of **inequality constraint**.

Consider a two-variable problem

$$
\begin{aligned}
\underset{x_1, x_2}{\text{minimize}} \quad & f(x_1, x_2) = x_1^2 - \frac{1}{2}x_1 - x_2 - 2 \\
\text{subject to} \quad & g_1(x_1, x_2) = x_1^2 - 4x_1 + x_2 + 1 \leq 0 \\
& g_2(x_1, x_2) = \frac{1}{2}x_1^2 + x_2^2 - x_1 - 4 \leq 0
\end{aligned}
$$

Notation and Basic Assumptions



A graphical method can be used to solve simple problems. However, it is difficult or impossible to use such a method for more constrained functions and high-dimensional systems.

Equality Constraints: Direct Substitution

Consider a problem with the next level of complexity: Optimization with equality constraints

$$\begin{aligned} \underset{\mathbf{y}}{\text{minimize}} \quad & f(\mathbf{y}) \\ \text{subject to} \quad & \mathbf{h}(\mathbf{y}) = 0 \end{aligned}$$

- To simplify the notation, let be partitioned into a decision vector and a state vector, such that

$$\mathbf{y} = \begin{bmatrix} \mathbf{x}^\top & \mathbf{u}^\top \end{bmatrix}^\top \in \mathbb{R}^p,\ \mathbf{x} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m,\ p = m + n$$

where is implicitly defined by the constraints that relate it to the decision variables. The problem now becomes:

$$\begin{aligned} \underset{\mathbf{u}}{\text{minimize}} \quad & f(\mathbf{x}, \mathbf{u}) \\ \text{subject to} \quad & \mathbf{h}(\mathbf{x}, \mathbf{u}) = 0 \end{aligned}$$

Equality Constraints: Direct Substitution

- p must be greater than n otherwise the problem is completely specified by the constraints (or over specified or not depend of f).
- One solution approach to solve the problem is direct substitution, which involves
  - Solving for $\mathbf{x}$ in terms of $\mathbf{u}$ using $\mathbf{h}(\mathbf{x}, \mathbf{u})$
  - Substituting this expression into $f(\mathbf{x}, \mathbf{u})$ and solving for $\mathbf{u}$ using an unconstrained optimization.
  - The method is good if $f(\mathbf{x}, \mathbf{u})$ is linear (assumption is that not both of f and $\mathbf{h}$ are linear.)

Equality Constraints: Direct Substitution

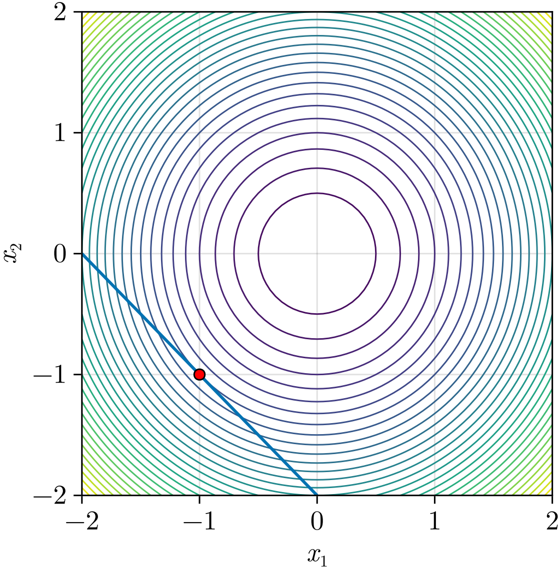
$$\underset{\mathbf{x}}{\text{minimize}} \quad f(\mathbf{x}) = x_1^2 + x_2^2$$
$$\text{subject to} \quad x_1 + x_2 + 2 = 0$$

- Clearly the unconstrained minimum is at $x_1 = x_2 = 0$
- From the constrain, $x_1 = -2 - x_2$ or $x_2 = -2 - x_1$, we have equivalent problems:

$$\underset{x_1}{\text{minimize}} \quad f_1(x_1) = x_1^2 + (-2 - x_1)^2 \text{ or } \underset{x_2}{\text{minimize}} \quad f_2(x_2) = (-2 - x_2)^2 + x_2^2$$

- the solution ($\partial f_i / \partial x_i = 0$ is $x_1 = x_2 = -1$
- The substitution method works well for linear constraints, but it is hard to generalize for larger systems/ nonlinear constraints.

Equality Constraints: Direct Substitution



Notation and Basic Assumptions

- For unconstrained gradient-based optimization, we only require the gradient of the objective, $\nabla f(\mathbf{x})$. To solve a constrained problem, we also require the gradients of all the constraints. Because the constraints are vectors, their derivatives yield a **Jacobian** matrix. For the equality constraints, we have

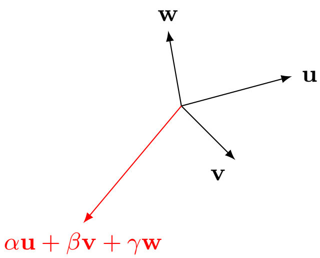
$$\mathbf{J_h} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}} = \underbrace{\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix}}_{p \times n} = \begin{bmatrix} \nabla h_1^\top \\ \vdots \\ \nabla h_p^\top \end{bmatrix}$$

- Similarly, the Jacobian of the inequality constraints is an $(q \times n)$ matrix.

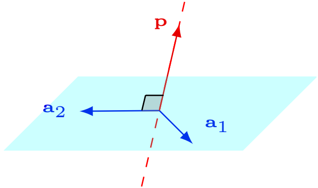
n-dimension space

There are several essential linear algebra concepts for constrained optimization.

- The **span** of a set of vectors is the space formed by all points that can be obtained by a linear combination of those vectors.
- The **null space** of a matrix $\mathbf{A}$ is the set of all n-dimensional vector $\mathbf{p}$ such that $\mathbf{A}\mathbf{p} = 0$.

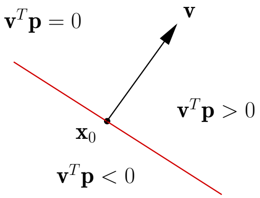


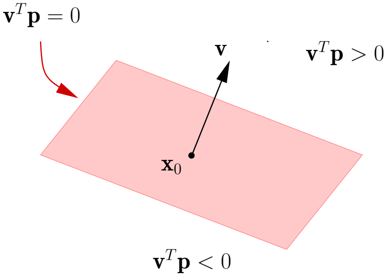
Span in three-dimensional space.



Nullspace of a 2×3 matrix A of rank 2, where a_1 and a_2 are the row vectors of A.

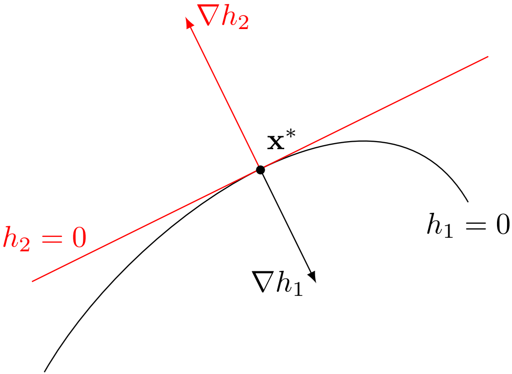
Hyperplanes and Half-space





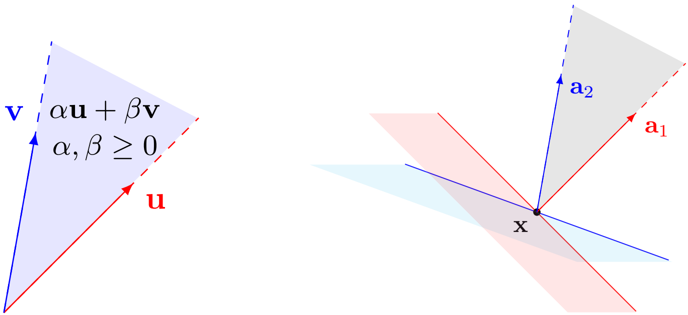
- In n dimensions, a hyperplane of $n-1$ dimensions divides the space into two **half-spaces**: in one of these, $\mathbf{v}^\top\mathbf{p} > 0$, and in the other, $\mathbf{v}^\top\mathbf{p} < 0$.
- Each half-space is closed if it includes the hyperplane ($\mathbf{v}^\top\mathbf{p} = 0$) and open otherwise.

Hyperplanes and Half-space



- The function gradient at the point on the isosurface is locally perpendicular to the isosurface. The gradient vector defines the **tangent hyperplane** and the point.
- The set of points such that $\nabla f^\top p = 0$.

Hyperplanes and Half-space



- The intersection of multiple half-spaces yields a **polyhedral cone.**
- A polyhedral cone is the set of all the points that can be obtained by the linear combination of a given set of vectors using nonnegative coefficients.

Equality Constraints

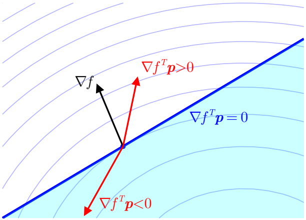
For the unconstrained case, by taken a first-order Taylor series expansion of the objective function with some step $\mathbf{p}$ that is small enough by neglecting the second-order term:

$$f(\mathbf{x}+\mathbf{p}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x})^\top \mathbf{p}$$

At the minimum point $\mathbf{x}^*$, we should have

$$f(\mathbf{x}^* + \mathbf{p}) \geq f(\mathbf{x}^*) \qquad \Rightarrow \qquad \nabla f(\mathbf{x}^*)^\top \mathbf{p} \geq 0$$

For unconstraint problem, $\nabla f^\top \mathbf{p} \geq 0$ is satisfied if $\nabla f(\mathbf{x}^*) = 0$



The gradient $f(\mathbf{x})$, which is the direction of steepest function increase, splits the design space into two halves. All $\mathbf{p}$ direction that make the function decrease always make $\nabla f^\top \mathbf{p} < 0$ except when $\nabla f^\top \mathbf{p} = 0$.

Equality Constraints

- For constrained problem, the function increase condition still applies, but $\mathbf{p}$ must also be a **feasible** direction. To find the feasible directions, we use a first-order Taylor series expansion for each equality constraint function as

$$h_j(\mathbf{x}+\mathbf{p}) \approx h_j(\mathbf{x}) + \nabla h_j(\mathbf{x})^\top \mathbf{p}, \qquad j = 1, \ldots, p$$

- $\mathbf{x}$ is a feasible point, then $h_j(\mathbf{x}) = 0$ for all constraints j, then

$$\nabla h_j(\mathbf{x})^\top \mathbf{p} = 0, \qquad \text{for all } j = 1, \ldots, p$$

- The **direction $\mathbf{p}$ is feasible when it is orthogonal to all equality constraint gradients.** Or,

$$\mathbf{J}_h(\mathbf{x})\mathbf{p} = 0$$

- Any feasible direction has to lie in the nullspace of the Jacobian of the constraints, $\mathbf{J}_h$.

Equality Constraints

- For constrained optimality, we need to satisfy both $\nabla f(\mathbf{x}^*)^\top \mathbf{p} \geq 0$ and $\mathbf{J}_h(\mathbf{x})\mathbf{p} = 0$
- For equality constraints, if a direction $\mathbf{p}$ is feasible, then $-\mathbf{p}$ must also be feasible (from Taylor series), Therefore, the only way to satisfy $\nabla f(\mathbf{x}^*)^\top \mathbf{p} \geq 0$ is if $\nabla f(\mathbf{x})^\top \mathbf{p} = 0$.

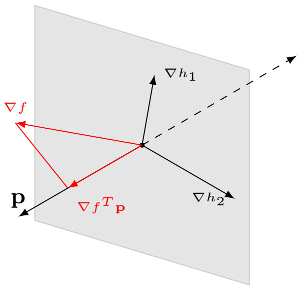
Theorem: 1^{st} order condition

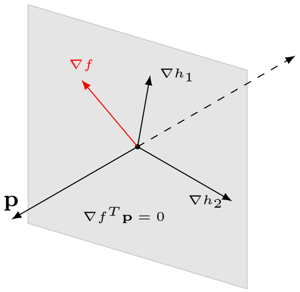
For $\mathbf{x}^*$ to be constrained optimum, we require

$$\nabla f(\mathbf{x}^*)^\top \mathbf{p} = 0 \quad \text{for all } \mathbf{p} \text{ such that} \quad \mathbf{J}_h(\mathbf{x}^*)\mathbf{p} = 0$$

- On other words, **the projection of the objective function gradient onto the feasible space must vanish.**

Equality Constraints





- The objective function gradient must be a linear combination of the gradients of the constraints. (left) we still have decent direction. (right) $\mathbf{x}$ is optimal.
- We can write

$$\nabla f(\mathbf{x}^*) = -\sum_{j=1}^{p} \lambda_j \nabla h_j(\mathbf{x}^*)$$

- λ_j are called the **Lagrange multipliers**. For equality constraints, the sign of Lagrange multipliers is arbitrary.

Equality Constraints

It is more convenient to use the **Lagrangian function:**

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^\top \mathbf{h}(\mathbf{x})$$

$$\nabla_\mathbf{x} \mathcal{L} = \nabla f(\mathbf{x}) + \boldsymbol{\lambda}^\top \mathbf{J}_h(\mathbf{x}) = 0, \qquad \nabla_{\boldsymbol{\lambda}} \mathcal{L} = \mathbf{h}(\mathbf{x}) = 0$$

With the Lagrangian function, we have transformed a constrained problem into an unconstrained problem by introducing new variables, $\boldsymbol{\lambda}$.

Theorem: 1^{st}-order optimality conditions

The optimality conditions for the equality-constrained case are

$$\begin{aligned} \nabla f(\mathbf{x}^*) + (\boldsymbol{\lambda}^*)^\top \mathbf{J}_h(\mathbf{x}^*) &= 0 \\ \mathbf{h}(\mathbf{x}^*) &= 0 \end{aligned}$$

These conditions assume that the gradients of the constraints are linearly independent; that is, $\mathbf{J}_h$ has full row rank.

Equality Constraints

The set of equality constraints

$$h_1(\mathbf{x}) = 0, h_2(\mathbf{x}) = 0, \cdots, h_p(\mathbf{x}) = 0$$

$$\mathbf{h}(\mathbf{x}) = \begin{bmatrix} h_1(\mathbf{x}) & h_2(\mathbf{x}) & \cdots & h_p(\mathbf{x}) \end{bmatrix}^\top, \mathbf{h}(\mathbf{x}) = 0$$

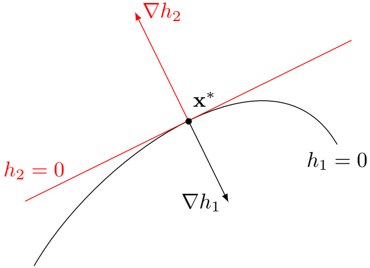
Definition: Regular point

A point $\mathbf{x}$ is called a **regular point** of the constraints $\mathbf{h}(\mathbf{x})$ if $\mathbf{x}$ satisfies $\mathbf{h}(\mathbf{x}) = 0$ and column vectors $\nabla h_1(\mathbf{x})$, $\nabla h_2(\mathbf{x})$, $\cdots$, $\nabla h_p(\mathbf{x})$ are linearly independent.

- The definition states that $\mathbf{x}$ is a regular point of the constraints if it is a solution of $\mathbf{h}(\mathbf{x}) = 0$ and the Jacobian $\mathbf{J}_h = \begin{bmatrix} \nabla h_1(\mathbf{x}) & \nabla h_2(\mathbf{x}) & \cdots & \nabla h_p(\mathbf{x}) \end{bmatrix}^\top$
- It is impossible for $\mathbf{x}$ to be a regular point of the constraints if $p > n$. It is the upper bound for the number of independent equality constraints, i.e., $p \leq n$.

Equality Constraints

- For $p > n$, the constraint qualification condition does not hold in this case because the gradients of the two constraints not linearly independent.



- The optimality conditions using first-oder conditions is a necessary but not sufficient. We need the Hessian of the objective function to be positive definite.

$$\mathbf{H}_{\mathcal{L}} = \mathbf{H}_f + \sum_{j=1}^{p} \lambda_j \mathbf{H}_{h_j}$$

Equality Constraints

Theorem: 2^{st}-order optimality conditions

The second-order sufficient conditions are as follows:

$$\mathbf{p}^\top \mathbf{H}_{\mathcal{L}} \mathbf{p} > 0 \qquad \text{for all } \mathbf{p} \text{ such that } \mathbf{J}_{\mathbf{h}} \mathbf{p} = 0$$

This conditions assumes that the gradients of the constraints are linearly independent; that is, $\mathbf{J}_{\mathbf{h}}$ has full row rank.

Equality Constraints

Discuss and sketch the feasible region described by the equality constraints

$$-x_1 + x_3 - 1 = 0$$
$$x_1^2 + x_2^2 - 2x_1 = 0$$

The Jacobian of the constraints is given by

$$\mathbf{J}_h(\mathbf{x}) = \begin{bmatrix} -1 & 0 & 1 \\ 2x_1 - 2 & 2x_2 & 0 \end{bmatrix}$$

which has rank 2 by giving any values of x_2.

- The $\mathbf{J}_h(\mathbf{x})$ has rank less than 2 when $\mathbf{x} = \begin{bmatrix} 1 & 0 & x_3 \end{bmatrix}^\top$.
- Sine $\mathbf{x} = \begin{bmatrix} 1 & 0 & x_3 \end{bmatrix}^\top$ does not satisfy the circle constrain, any point $\mathbf{x}$ satisfying both constraints is regular. (make $\mathbf{J}_h$ has full row rank.)

Equality Constraints: Example I

Consider a constrained problem with a linear objective function and a quadratic equality constraint:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = x_1 + 2x_2 \\ \text{subject to} \quad & h(\mathbf{x}) = \frac{1}{4}x_1^2 + x_2^2 - 1 = 0 \end{aligned}$$

The Lagrangian is

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 + 2x_2 + \lambda\left(\frac{1}{4}x_1^2 + x_2^2 - 1\right)$$

Then,

$$\begin{aligned} \nabla\mathcal{L}_{\mathbf{x}} &= \begin{bmatrix} 1 + \frac{1}{2}\lambda x_1 \\ 2 + 2\lambda x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \nabla\mathcal{L}_{\lambda} &= \frac{1}{4}x_1^2 + x_2^2 - 1 = 0 \end{aligned}$$

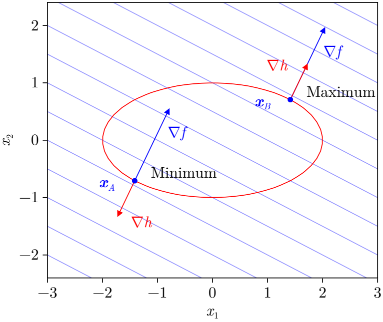
We have $x_1 = -2/\lambda$, and $x_2 = -1/\lambda$, then $\lambda = \pm\sqrt{2}$.

Equality Constraints: Example I

For each $\lambda_A = \sqrt{2}$ and $\lambda_B = -\sqrt{2}$, we obtain two possible solutions:

$$\mathbf{x}_A = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}, \quad \lambda_A = \sqrt{2}$$

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \lambda_B = -\sqrt{2}$$



- The Hessian of the Lagrangian is

$$\mathbf{H}_{\mathcal{L}} = \begin{bmatrix} \frac{1}{2}\lambda & 0 \\ 0 & 2\lambda \end{bmatrix}$$

- It is clear that $\mathbf{H}$ is positive for $\mathbf{x}_A$, and negative for $\mathbf{x}_B$. Then $\mathbf{x}_A$ is a minimum point, and $\mathbf{x}_B$ is a maximum point.

Equality Constraints: Example II

Consider the following problem:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = x_1^2 + 3(x_2 - 2)^2 \\ \text{subject to} \quad & h(\mathbf{x}) = \beta x_1^2 - x_2 = 0, \end{aligned}$$

where β is a parameter that we will vary to change the characteristics of the constraint. The Lagrangian for this problem is

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \lambda) &= x_1^2 + 3(x_2 - 2)^2 + \lambda \left(\beta x_1^2 - x_2\right) \\ \nabla_{\mathbf{x}} \mathcal{L} &= \begin{bmatrix} 2x_1(1 + \lambda\beta) \\ 6(x_2 - 2) - \lambda \end{bmatrix} = 0 \\ \nabla_{\lambda} \mathcal{L} &= \beta x_1^2 - x_2 = 0 \end{aligned}$$

Form $2x_1(1 + \lambda\beta) = 0$ we get $x_1 = 0$, then the solution is $\begin{bmatrix} x_1 & x_2 & \lambda \end{bmatrix}$ $= \begin{bmatrix} 0 & 0 & -12 \end{bmatrix}$, which is independent of β.

Equality Constraints: Example II

To determine if this is a minimum, we must check the second-order conditions by evaluating the Hessian of the Lagrangian,

$$\mathbf{H}_{\mathcal{L}} = \begin{bmatrix} 2(1 - 12\beta) & 0 \\ 0 & 6 \end{bmatrix}$$

- The feasible directions are all $\mathbf{p}$ such that $\mathbf{J}_h^\top \mathbf{p} = 0$. Here $\mathbf{J}_h^\top = \begin{bmatrix} 2\beta x_1 & -1 \end{bmatrix}$, yielding $\mathbf{J}_h(\mathbf{x}^*) = \begin{bmatrix} 0 & -1 \end{bmatrix}^\top$
- The feasible directions at the solution can be represented as $\mathbf{p} = \begin{bmatrix} \alpha & 0 \end{bmatrix}^\top$, where α is any number.
- For positive curvature in the feasible directions, we require that

$$\mathbf{p}^\top \mathbf{H}_{\mathcal{L}} \mathbf{p} = 2\alpha^2(1 - 12\beta) > 0$$
$$\beta < \frac{1}{12}$$

Equality Constraints: Example III

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = x_1^2 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 + 2 = 0 \end{aligned}$$

- Form the Lagrangian

$$\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda h(\mathbf{x}) = x_1^2 + x_2^2 + \lambda(x_1 + x_2 + 2)$$

$$\nabla_{\mathbf{x}} \mathcal{L} = \begin{bmatrix} 2x_1 + \lambda \\ 2x_2 + \lambda \end{bmatrix} = 0$$

$$\nabla_{\lambda} \mathcal{L} = x_1 + x_2 + 2 = 0$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \quad x_1^* = x_2^* = -1, \lambda^* = 2$$

Inequality Constraints

We can use some of the concepts from the equality constrained optimality conditions for inequality constrained problems.

- An inequality constraint j is feasible when $g_j(\mathbf{x}^*) \leq 0$ and it is said to be **active** if $g_j(\mathbf{x}^*) = 0$ and **inactive** if $g_i(\mathbf{x}^*) < 0$.
- Based on the Taylor series, for any small enough feasible step $\mathbf{p}$, we get the condition

$$f(\mathbf{x}^* + \mathbf{p}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^\top \mathbf{p}$$
$$\nabla f(\mathbf{x}^*)^\top \mathbf{p} \geq 0, \text{ since } \mathbf{x} \text{ is the optimal point.}$$

- The decent directions, if it is feasible, is in the open half-space defined by the hyperplane tangent to the gradient of the objective.
- Consider the Taylor series of the inequality constraints

$$g_j(\mathbf{x} + \mathbf{p}) \approx g_j(\mathbf{x}) + \nabla g_j(\mathbf{x})^\top \mathbf{p} \leq 0, \qquad j = 1, \ldots, q$$

Inequality Constraints

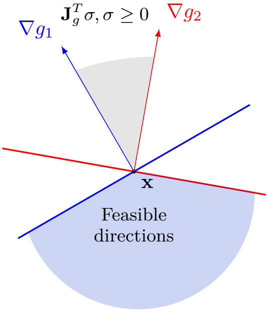
There are two possibilities to consider for each inequality constraint: inactive $g_j(\mathbf{x}) < 0$ or active $g_j(\mathbf{x}) = 0$.

- If the constraint is inactive we can take a step $\mathbf{p}$ in any direction and remain feasible as long as the step is small enough.
- Inequality constraints do not need the nullspace of the Jacobian matrix. From

$$g_j(\mathbf{x} + \mathbf{p}) \approx g_j(\mathbf{x}) + \nabla g_j(\mathbf{x})^\top \mathbf{p} \leq 0, \qquad j = 1, \ldots, q$$

  if constraint j is active ($g_j(\mathbf{x}) = 0$), then the nearby point $g_j(\mathbf{x} + \mathbf{p})$ is only feasible if $\nabla g_j(\mathbf{x})^\top \mathbf{p} \leq 0$ for all constraints j that are active. In matrix form, we can write $J_g(\mathbf{x})\mathbf{p} \leq 0$, where the Jacobian matrix includes only the gradients of the active constraints.

Inequality Constraints



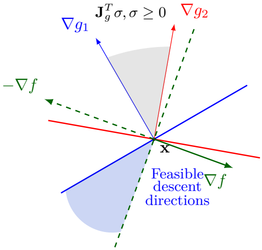
- The set of feasible directions that satisies all active constraints is the intersection of all the closed half-spaces defined by the inequality constraints, that is all $\mathbf{p}$ such that $\mathbf{J}_g(\mathbf{x})\mathbf{p} \leq 0$.
- The intersection of the feasible directions forms a polyhedral cone.
- To find the cone of feasible directions, first consider the cone formed by the active inequality constraint gradients (shown in gray).

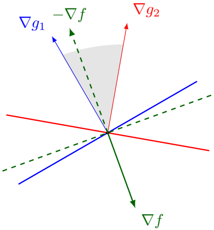
The cone is defined by all vectors $\mathbf{d}$ such that (linear combination of ∇g_j)

$$\mathbf{d} = \mathbf{J}_g^\top \sigma = \sum_{j=1}^{q} \sigma_j \nabla g_j, \quad \text{where } \sigma_j \geq 0$$

A direction $\mathbf{p}$ is feasible if $\mathbf{p}^\top \mathbf{d} \leq 0$ for all $\mathbf{d}$ in the cone. The set of all feasible directions forms the **polar cone** of the cone defined above and is shown in blue.

Inequality Constraints: Farkas' lemma





We need to establish under which condition there is no feasible descent direction or when is there no intersection between the cone of feasible directions and the open half-space of descent direction?

- There exists a $\mathbf{p}$ (dashed line) such that $\mathbf{J}_g\mathbf{p} \leq 0$ and $\nabla f^\top \mathbf{p} < 0$ (a descent direction is feasible. (above))
- There exists a σ such that $\mathbf{J}_g^\top \sigma = -\nabla f$ with $\sigma \geq 0$ (This corresponds to optimality. There is no feasible direction.(below))
- The optimality criterion for inequality constraints:

$$\nabla f + \sigma^\top \mathbf{J}_g(\mathbf{x}) = 0, \text{ with } \sigma \geq 0$$

Inequality Constraints: Farkas' lemma

- The criteria of the inequality constraints is similar to the equality constraints. However, σ corresponds to the Lagrange multipliers for the inequality constraints and carries the additional restriction that $\sigma \geq 0$ (nonnegative)
- If equality constraints are present, the conditions for the inequality constraints apply only in the subspace of the directions feasible with respect to the equality constraints.
- We can add all inequality constraints (we don't know which one we should use.) to the Lagrangian by replacing them with the equality constraint as

  $$g_j + s_j^2 = 0, \qquad j = 1, \ldots, q$$

  where s_j is a new unknown associated with each inequality constraint called a **slack variable**. This variable must be positive.
- If $s_j = 0$, the corresponding inequality constraint is active ($g_j = 0$), and when $s_j \neq 0$, the corresponding constraint is inactive.

The Lagrangian

The Lagrangian including both equality and inequality constraints is

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\sigma}, \mathbf{s}) = \mathbf{f}(\mathbf{x}) + \boldsymbol{\lambda}^\top \mathbf{h}(\mathbf{x}) + \boldsymbol{\sigma}^\top \left(\mathbf{g}(\mathbf{x}) + \mathbf{s} \odot \mathbf{s}\right),$$

where σ represents the Lagrange multipliers associated with the inequality constraints. The $\odot$ is represented the element-wise multiplication of $\mathbf{s}$.
At the stationary point

$$\begin{aligned}
\nabla_{\mathbf{x}} \mathcal{L} = 0 \quad &\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{l=1}^{p} \lambda_l \frac{\partial h_l}{\partial x_i} + \sum_{j=1}^{q} \sigma_j \frac{\partial g_j}{\partial x_i} = 0, i = 1, \ldots, n \\
\nabla_{\boldsymbol{\lambda}} \mathcal{L} = 0 \quad &\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \lambda_l} = h_l = 0, \quad l = 1, \ldots, p \\
\nabla_{\boldsymbol{\sigma}} \mathcal{L} = 0 \quad &\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \sigma_j} = g_j + s_j^2 = 0, \quad j = 1, \ldots, q \\
\nabla_{\mathbf{s}} \mathcal{L} = 0 \quad &\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial s_j} = 2\sigma_j s_j = 0, \quad j = 1, \ldots, q
\end{aligned}$$

The last one is call **complementary slackness condition**. It can help us to distinguish the active constraints from the inactive constraint.

Karush-Kuhn-Tucker (KKT) condition

Theorem: KKT 1st-order condition

$$
\begin{aligned}
\nabla \mathbf{f} + \mathbf{J}_{\mathbf{h}}^{\top} \boldsymbol{\lambda} + \mathbf{J}_{\mathbf{g}}^{\top} \boldsymbol{\sigma} &= 0 \\
\mathbf{h} &= 0 \\
\mathbf{g} + \mathbf{s} \odot \mathbf{s} &= 0 \\
\boldsymbol{\sigma} \odot \mathbf{s} &= 0 \\
\boldsymbol{\sigma} &\geq 0
\end{aligned}
$$

Theorem: 2nd-order condition

$$
\begin{aligned}
\mathbf{p}^{\top} \mathbf{H}_{\mathcal{L}} \mathbf{p} &> 0 && \text{for all } \mathbf{p} \text{ such that:} \\
\mathbf{J}_{\mathbf{h}} \mathbf{p} &= 0 \\
\mathbf{J}_{\mathbf{g}} \mathbf{p} &\leq 0 && \text{for the active constraints.}
\end{aligned}
$$

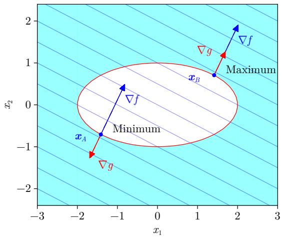
Problem with one inequality constraint

Consider a problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = x_1 + 2x_2 \\ \text{subject to} \quad & g(\mathbf{x}) = \frac{1}{4}x_1^2 + x_2^2 - 1 \leq 0 \end{aligned}$$

The Lagrangian for this problem is

$$\mathcal{L}(x_1, x_2, \sigma, s) = x_1 + 2x_2 + \sigma\left(\frac{1}{4}x_1^2 + x_2^2 - 1 + s^2\right)$$



- Inequality constrained problem with linear objective.
- Feasible space within a ellipse.

Problem with one inequality constraint

Differentiating the Lagrangian with respect to all the variables, we get the first-order optimality conditions

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 + \frac{1}{2}\sigma x_1 = 0, \quad \frac{\partial \mathcal{L}}{\partial x_2} = 2 + 2\sigma x_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \sigma} = \frac{1}{4}x_1^2 + x_2^2 - 1 = 0, \quad \frac{\partial \mathcal{L}}{\partial s} = 2\sigma s = 0$$

The last equation, we can set $s = 0$ (meaning the constraint is active) and $\sigma = 0$ (meaing the constraint is inactive). However, σ cannot be zero because the first two equation will not yield a solution. Setting that $s = 0$ and $\sigma \neq 0$, we can solve the equations to obtain:

$$\mathbf{x}_A = \begin{bmatrix} x_1 \\ x_2 \\ \sigma \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ -\frac{\sqrt{2}}{2} \\ \sqrt{2} \end{bmatrix}, \quad \mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ \sigma \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \frac{\sqrt{2}}{2} \\ -\sqrt{2} \end{bmatrix}$$

According to the KKT conditions, the Lagrange multiplier σ must be nonnegative. Point $\mathbf{x}_A$ satisfies this condition. There is no feasible descent direction a $\mathbf{x}_A$.

Problem with two inequality constraint

Consider

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = x_1 + 2x_2 \\ \text{subject to} \quad & g_1(\mathbf{x}) = \frac{1}{4}x_1^2 + x_2^2 - 1 \le 0 \\ & g_2(\mathbf{x}) = -x_2 \le 0. \end{aligned}$$

The Lagrangian for this problem is

$$\mathcal{L}(x, \sigma, s) = x_1 + 2x_2 + \sigma_1 \left(\frac{1}{4}x_1^2 + x_2^2 - 1 + s_1^2 \right) + \sigma_2 \left(-x_2 + s_2^2 \right)$$

Differentiating the Lagrangian with respect to all the variables, we get the first-order optimality conditions

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= 1 + \frac{1}{2}\sigma_1 x_1 = 0, \quad \frac{\partial \mathcal{L}}{\partial x_2} = 2 + 2\sigma_1 x_2 - \sigma_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial \sigma_1} &= \frac{1}{4}x_1^2 + x_2^2 - 1 + s_1^2 = 0, \quad \frac{\partial \mathcal{L}}{\partial \sigma_2} = -x_2 + s_2^2 = 0 \\ \frac{\partial \mathcal{L}}{\partial s_1} &= 2\sigma_1 s_1 = 0, \quad \frac{\partial \mathcal{L}}{\partial s_2} = 2\sigma_2 s_2 = 0 \end{aligned}$$

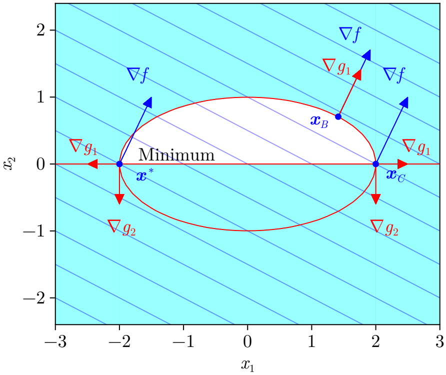
Problem with two inequality constraint

We have two complementary slackness conditions, which yield the four potential combinations listed below:

Assumption	Meaning	x_1	x_2	σ_1	σ_2	s_1	s_2	Point
$s_1 = 0$	g_1 is active	-2	0	1	2	0	0	$\mathbf{x}^*$
$s_2 = 0$	g_2 is active	2	0	-1	2	0	0	$\mathbf{x}_C$
$\sigma_1 = 0$ $\sigma_2 = 0$	g_1 is inactive g_2 is inactive	-	-	-	-	-	-	
$s_1 = 0$ $\sigma_2 = 0$	g_1 is active g_2 is inactive	$\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	0	0	$2^{-\frac{1}{4}}$	$\mathbf{x}_B$
$\sigma_1 = 0$ $s_2 = 0$	g_1 is inactive g_2 is active	-	-	-	-	-	-	

Assuming that both constraints are active yields two possible solutions ($\mathbf{x}^*$ and $\mathbf{x}_C$) cooresponding to two different Lagrange multipliers. According to the KKT conditions, the Lagrange multipliers for all active inequality constraints have to be positive, so only the solution with $\sigma_1 = 1$, then $\mathbf{x}^*$ is a candidate for a minimum.

The feasible region is the top half of the ellipse, as show below



Simple Constrained Example

Consider:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = x_1^2 + x_1 x_2 + x_2^2 \\ \text{subject to} \quad & x_2 \geq 1 \\ & x_1 + x_2 \geq 3 \end{aligned}$$

Form the Lagrangian

$$\mathcal{L}(\mathbf{x}, \sigma) = x_1^2 + x_1 x_2 + x_2^2 + \sigma_1(1 - x_2) + \sigma_2(x_1 + x_2 - 3)$$

Form necessary conditions:

$$\nabla_{\mathbf{x}} \mathcal{L} = \begin{bmatrix} 2x_1 + x_2 + \sigma_2 \\ x_1 + 2x_2 - \sigma_1 + \sigma_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\nabla_{\sigma} \mathcal{L} = \begin{bmatrix} 1 - x_2 \\ x_1 + x_2 - 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Simple Constrained Example

Consider the various options:

- $\sigma_1 = \sigma_2 = 0$, both constraints are inactive

$$\nabla_{\mathbf{x}}\mathcal{L} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;\Rightarrow\; x_1 = x_2 = 0$$

- $\sigma_1 = 0$ (inactive), $\sigma_2 \geq 0$ (active)

$$\nabla_{\mathbf{x}}\mathcal{L} = \begin{bmatrix} 2x_1 + x_2 + \sigma_2 \\ x_1 + 2x_2 + \sigma_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\nabla_{\sigma_2}\mathcal{L} = x_1 + x_2 - 3 = 0$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \;\Rightarrow\; x_1 = x_2 = \frac{3}{2}, \sigma_2 = -\frac{9}{2}$$

Satisfy the constraints but σ_2 is negative, and $f(\mathbf{x}) = 6.75$ is not minimum.

Simple Constrained Example

- $\sigma_1 \geq 0$ (active), $\sigma_2 = 0$ (inactive)

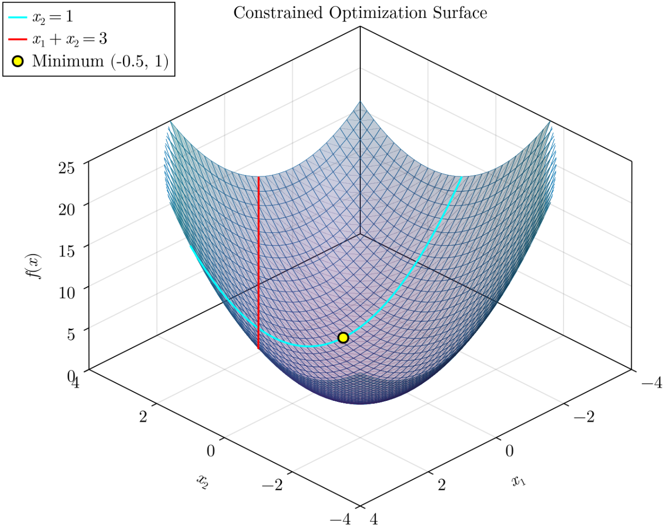
$$\nabla_{\mathbf{x}} \mathcal{L} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 - \sigma_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\nabla_{\sigma_1} \mathcal{L} = 1 - x_2 = 0$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \sigma_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow x_1 = -0.5, x_2 = 1, \sigma_1 = \frac{3}{2}$$

Satisfy the constraints, σ_1 is positive, and $f(\mathbf{x}) = 0.75$ is minimum.

Simple Constrained Example



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