Lecture 4 Forward Kinematics

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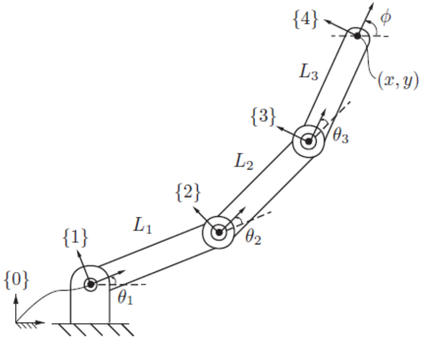
Forward Kinematics

The **forward kinematics** of a robot is the calculation of the position and orientation of its end-effector frame from its joint coordinates θ.



$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3),$$

$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3),$$

$$\phi = \theta_1 + \theta_2 + \theta_3$$

Forward Kinematics

- Using only basic trigonometry, the more general spatial chains, the more complicated.
- A more systematic method of deriving the forward kinematics is by attaching reference frames to each link
- In this case, there are three links $\{1\}$, $\{2\}$, and $\{3\}$.
- The forward kinematics can be written as a product of four homogeneous transformation matrices:

$$^0T_4 = {^0T_1}\,{^1T_2}\,{^2T_3}\,{^3T_4},$$

where

$$^0T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {^1T_2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^2T_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {^3T_4} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics

- Observe that ${}^{3}T_{4}$ is constant and that each remaining ${}^{i-1}T_{i}$ depends only on the joint variable θ_i.
- This representation is called Denavit-Hartenberg parameters (D-H parameters).

Product of Exponentials

- Let us define M to be the position and orientation of frame $\{4\}$ when all joint angles are set to zero. Then

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- If θ_1 and θ_2 are held at their zero position theta the screw axis corresponding to rototing about joint 3 can be expressed in the $\{0\}$ frame as

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

- Note $v_3 = -\omega_3 \times q_3$, where q_3 is any point on the axis of joint 3 expressed in $\{0\}$, e.g. $q_3 = (L_1 + L_2, 0, 0)$.

Product of Exponentials

- The screw axis $\mathcal{S}_3$ can be expressed in $se(3)$ matrix form as

$$[\mathcal{S}_3] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Therefore, for any θ_3, the matrix exponential representation for screw motions is

$$^0T_4 = e^{[\mathcal{S}_3]\theta} M \qquad \text{(for } \theta_1 = \theta_2 = 0)$$

- For $\theta_1 = 0$ and any fixed θ_3, rotation about joint 2 can be viewed as applying a screw motion to the rigid (link 2/ link 3) pair

$$^0T_4 = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{ for } \theta_1 = 0),$$

Product of Exponentials

- where $[\mathcal{S}_3]$ and M are defined previously, and

$$[\mathcal{S}_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

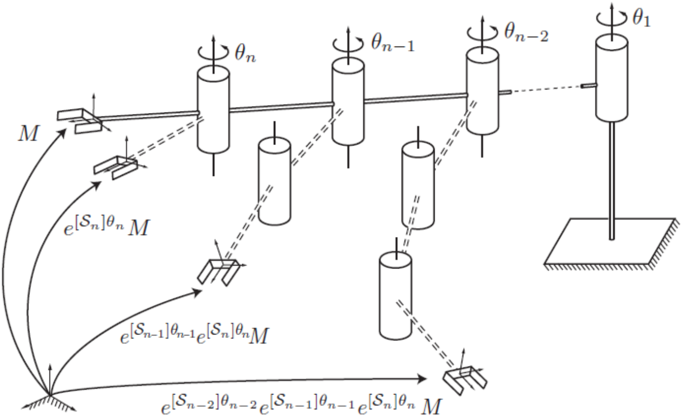
- Finally, keeping θ_2 and θ_3 fixed, rotation about joint 1 can be viewed as applying a screw motion to the entire rigid three-link assembly. We can write

$$^0T_4 = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M,$$

where

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Product of Exponentials



- Choose a fixed base frame $\{s\}$ and an end-effector frame $\{b\}$ attached to the last link.
- Let $M \in SE(3)$ denote the configuration of the end-effector frame relative to the fixed base frame when the robot is in its zero position.

Product of Exponentials

- Suppose that joint n is displaced to some joint value θ_n the end effector from M then undergoes a displacement of the form

$$T = e^{[\mathcal{S}_n]\theta_n} M,$$

where $T \in SE(3)$ is the new configuration of the end-effector frame and $\mathcal{S}_n = (\omega_n, v_n)$ is the screw axis of joint n as expressed in the fixed base frame.

- If joint n is revolute then $\omega_n \in \mathbb{R}^3$ is a unit vector in the positive direction of joint axis n; $v_n = -\omega_n \times q_n$ and θ_n is the joint angle.
- If joint n is prismatic then $\omega_n = 0$, $v_n \in \mathbb{R}^3$ is a unit vector in the direction of positive translation, and θ_n represents the prismatic extension/retraction.
- If the joint $n - 1$ is also allowed to vary then this has the effect of applying a screw motion to link $n - 1$. The en-effector frame thus undergoes a displacement of the form

$$T = e^{[\mathcal{S}_{n-1}]\theta_{n-1}} \left(e^{[\mathcal{S}_n \theta_n]} M \right)$$

Product of Exponentials

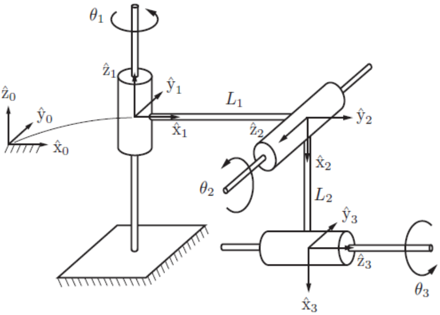
- Then

$$T = e^{[\mathcal{S}_1]\theta_1} \ldots e^{\left[\mathcal{S}_{n-1}\right]\theta_{n-1}} e^{[\mathcal{S}_n \theta_n]} M$$

Summary:

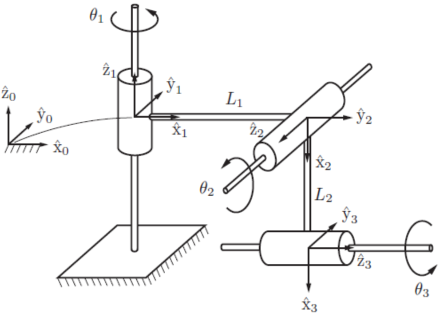
- The end-effector configuration $M \in SE(3)$ when the robot is at its home position;
- The screw axes $\mathcal{S}_1, \ldots, \mathcal{S}_n$ expressed in the fixed base frame, corresponding to the joint motions when the robot is at its home position;
- The joint variables $\theta_1, \ldots, \theta_n$.

Product of Exponentials ex1



What are M and $[\mathcal{S}_1]$, $[\mathcal{S}_2]$ and $[\mathcal{S}_3]$?

Product of Exponentials ex1



What are M and $[\mathcal{S}_1]$, $[\mathcal{S}_2]$ and $[\mathcal{S}_3]$?

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Product of Exponentials ex1

- The screw axis $\mathcal{S}_1 = (\omega_1, v_1)$ for joint axis 1 is given by $\omega_1 = (0,0,1)$ and $v_1 = -\omega_1 \times q_1 = (0,0,0)$.
- For joint 2, $\omega_2 = (0,-1,0)$ (consider $\hat{y}_0$) and $q_2 = (L_1, 0, 0)$, then $v_2 = -\omega_2 \times q_2 = (0,0,-L_1)$
- For joint 3, $\omega_3 = (1,0,0)$ and $q_3 = (0,0,-L_2)$ (distance compared with the rotating frame), then $v_3 = -\omega_3 \times q_3 = (0,-L_2,0)$

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ [\mathcal{S}_2] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ [\mathcal{S}_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,-1,0)$	$(0,0,-L_1)$
3	$(1,0,0)$	$(0,-L_2,0)$

Product of Exponentials ex1

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% Lecture 4 supplymentary
syms L1 L2

M = [ 0 0 1 L1; 0 1 0 0; -1 0 0 -L2; 0 0 0 1];
w1 = [0; 0; 1]; q1 = [0; 0; 0];
w2 = [0; -1; 0]; q2 = [L1; 0 ; 0];
w3 = [1; 0; 0]; q3 = [0; 0; -L2];
v1 = cross(-w1, q1); v2 = cross(-w2, q2); v3 = cross(-w3, q3);

Sc1 = [skew(w1), v1; zeros(1,4)]
Sc2 = [skew(w2), v2; zeros(1,4)]
Sc3 = [skew(w3), v3; zeros(1,4)]
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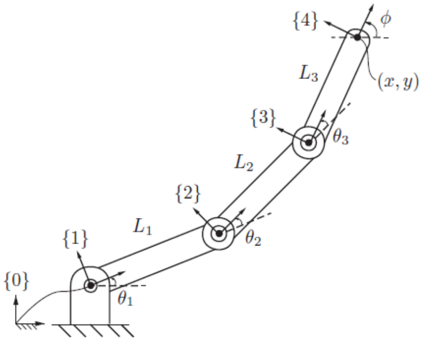
Product of Exponentials ex2



What are M and $[\mathcal{S}_1]$, $[\mathcal{S}_2]$ and $[\mathcal{S}_3]$?

Product of Exponentials ex2

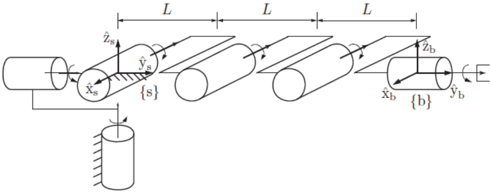
i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,0,1)$	$(0,-L_1,0)$
3	$(0,0,1)$	$(0,-(L_1+L_2),0)$

We can neglect the z-axis, in this case.

i	ω_i	v_i
1	1	$(0,0)$
2	1	$(0,-L_1)$
3	1	$(0,-(L_1+L_2))$

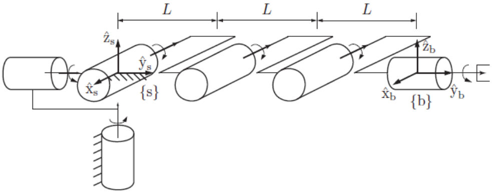
$$M = \begin{bmatrix} 1 & 0 & L_1+L_2+L_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Product of Exponentials ex3



What are M and $[\mathcal{S}_1]$, $[\mathcal{S}_2]$, $[\mathcal{S}_3]$, $[\mathcal{S}_4]$, $[\mathcal{S}_5]$ and $[\mathcal{S}_6]$?

Product of Exponentials ex3

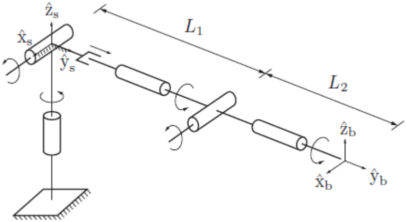


What are M and $[\mathcal{S}_1]$, $[\mathcal{S}_2]$, $[\mathcal{S}_3]$, $[\mathcal{S}_4]$, $[\mathcal{S}_5]$ and $[\mathcal{S}_6]$?

i	ω_i	p_i	v_i
1	$(0,0,1)$	$(0,0,0)$	$(0,0,0)$
2	$(0,1,0)$	$(0,0,0)$	$(0,0,0)$
3	$(-1,0,0)$	$(0,0,0)$	$(0,0,0)$
4	$(-1,0,0)$	$(0,L,0)$	$(0,0,L)$
5	$(-1,0,0)$	$(0,2L,0)$	$(0,0,2L)$
6	$(0,1,0)$	$(0,3L,0)$	$(0,0,0)$

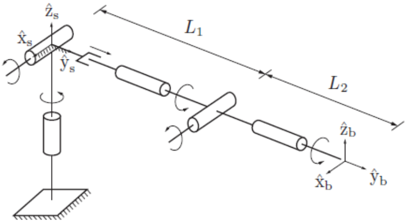
$$, \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Product of Exponentials ex1



What are M and $[\mathcal{S}_1]$, $[\mathcal{S}_2]$, $[\mathcal{S}_3]$, $[\mathcal{S}_4]$, $[\mathcal{S}_5]$ and $[\mathcal{S}_6]$?

Product of Exponentials ex1

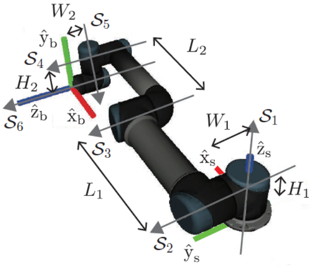


What are M and $[\mathcal{S}_1]$, $[\mathcal{S}_2]$, $[\mathcal{S}_3]$, $[\mathcal{S}_4]$, $[\mathcal{S}_5]$ and $[\mathcal{S}_6]$?

i	ω_i	p_i	v_i
1	$(0,0,1)$	$(0,0,0)$	$(0,0,0)$
2	$(1,0,0)$	$(0,0,0)$	$(0,0,0)$
3	$(0,0,0)$	$(0,1,0)$	$(0,1,0)$
4	$(0,1,0)$	$(0,0,0)$	$(0,0,0)$
5	$(1,0,0)$	$(0,L_1,0)$	$(0,0,-L_1)$
6	$(0,1,0)$	$(0,L_1+L_2,0)$	$(0,0,0)$

$$, \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1+L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Product of Exponentials ex1



What are M and $[\mathcal{S}_1]$, $[\mathcal{S}_2]$, $[\mathcal{S}_3]$, $[\mathcal{S}_4]$, $[\mathcal{S}_5]$ and $[\mathcal{S}_6]$? If we set $\theta_2 = -\pi/2$ and $\theta_5 = \pi/2$, with all other joint angles equal to zero. What is $T(\theta)$? If $W_1 = 109$ mm , $W_2 = 82$ mm, $L_1 = 425$ mm, $L_2 = 392$ mm , $H_1 = 89$ mm, and $H_2 = 95$ mm

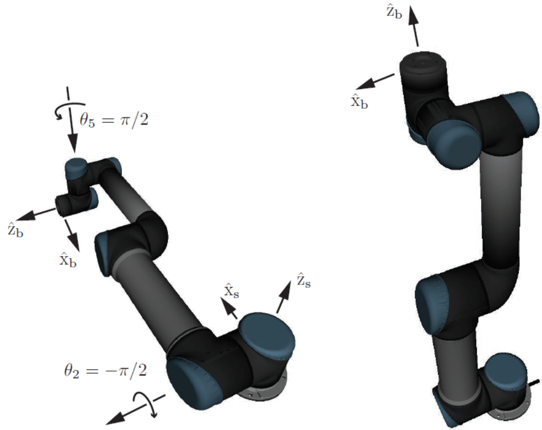
Product of Exponentials ex1

i	ω_i	p_i	v_i
1	$(0,0,1)$	$(0,0,0)$	$(0,0,0)$
2	$(0,1,0)$	$(0,W_1,H_1)$	$(-H_1,0,0)$
3	$(0,1,0)$	(L_1,W_1,H_1)	$(-H_1,0,L_1)$
4	$(0,1,0)$	$(L_1+L_2,0,H_1)$	$(-H_1,0,L_1+L_2)$
5	$(0,0,-1)$	(L_1+L_2,W_1,H_1)	$(-W_1,L_1+L_2,0)$
6	$(0,1,0)$	$(L_1+L_2,W_1+W_2,H_1-H_2)$	$(H_2-H_1,0,L_1+L_2)$

,

$$M = \begin{bmatrix} -1 & 0 & 0 & L_1+L_2 \\ 0 & 0 & 1 & W_1+W_2 \\ 0 & 1 & 0 & H_1-H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Product of Exponentials ex1



Product of Exponentials second formulation

From the fact that $A = PDP^{-1}$ for some $D \in \mathbb{R}^{n\times n}$ and invertible $P \in \mathbb{R}^{n\times n}$ then $e^{At} = Pe^{Dt}P^{-1}$. We have

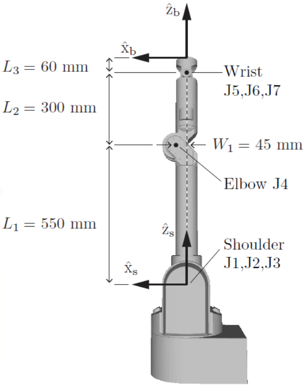
$$e^{M^{-1}PM} = M^{-1}e^{P}M \quad \Rightarrow \quad Me^{M^{-1}PM} = e^{P}M$$

We have

$$\begin{aligned}
T(\theta) &= e^{[\mathcal{S}_1]\theta_1}\dots e^{[\mathcal{S}_n]\theta_n}M \\
&= e^{[\mathcal{S}_1]\theta_1}\dots Me^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= e^{[\mathcal{S}_1]\theta_1}\dots Me^{M^{-1}[\mathcal{S}_{n-1}M\theta_{n-1}]}e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= Me^{M^{-1}[\mathcal{S}_1]M\theta_1}\dots e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}}e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= Me^{[\mathcal{B}_1]\theta_1}\dots e^{\mathcal{B}_{n-1}\theta_{n-1}}e^{[\mathcal{B}_n]\theta_n},
\end{aligned}$$

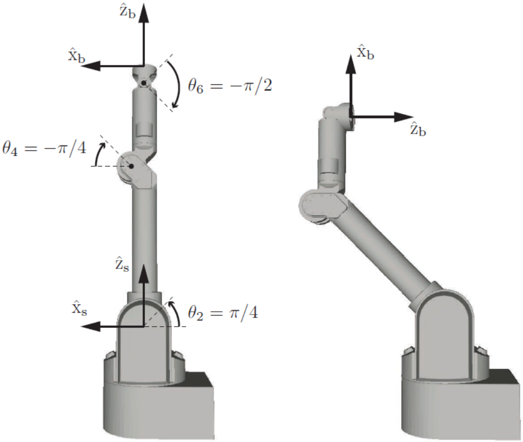
where each $[\mathcal{B}_i]$ is given by $M^{-1}\,[\mathcal{S}_i]\,M$, i.e., $\mathcal{B}_i = [\mathrm{Ad}_{M^{-1}}]\,\mathcal{S}_i, i = 1, \dots, n$. We call this form as a **body form**.

Product of Exponentials 2nd formulation ex1



What are M and $[\mathcal{S}_1]$, $[\mathcal{S}_2]$, $[\mathcal{S}_3]$, $[\mathcal{S}_4]$, $[\mathcal{S}_5]$, $[\mathcal{S}_6]$ and $[\mathcal{S}_7]$? If we set $\theta_2 = 45°$ and $\theta_4 = -45°$, $\theta_6 = -90°$ and all other joint angles equal to zero. What is $T(\theta)$?

Product of Exponentials 2nd formulation ex1



Reference

1. P. Corke, *Robotics, Vision and control: Fundamental Algorithms in MATLAB,* 2nd, Springer, 2011
2. K. M. Lynch, and F. C. Park, *Modern Robotics: Mechanics, Planning, and Control,* Cambridge U. Press, 2017