# Lecture 9: Fluid Systems

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# Fluid Systems

- The fluid systems have a lot of applications: actuators and processes that involve mixing, heating, and cooling of fluids.
- Active vehicle suspensions use hydraulic and pneumatic actuators to provide forces that supplement the passive spring and damping elements.
- water supply, waste treatment, and other chemical processing applications are examples of a general category of fluid systems called *liquid-level systems*, because they involve regulating the volumes, and therefore the levels of liquids in containers such as tanks.

# Conservation of Mass

For incompressible fluids, conservation of mass is equivalent to conservation of volume, because the **fluid density** is constant. That is

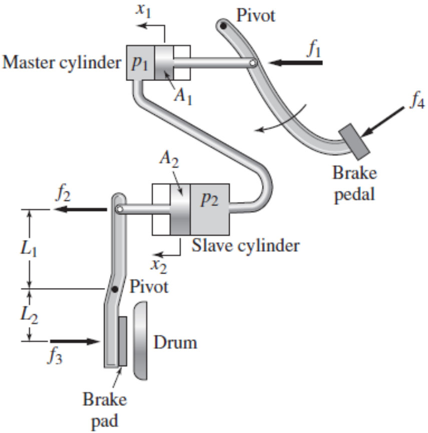
$$q_m = \rho q_v,$$

where $q_m$ and $q_v$ are the mass and volume flow rates. (In SI system, we use kg/s and m$^2$/s as units of $q_m$ and $q_v$, respectively. )

Density and Pressure

- **weight density:** whose common symbol is $\gamma$. Its units is N/m$^3$, and it is related to the mass density as $\gamma = \rho g$, where $g$ is the acceleration due to gravity. The mass density of fresh water near room temperature is 1000 kg/m$^3$.
- **Pressure** is the force per unit area that is exerted by the fluid (F/A). The SI unit of pressure is the Pascal (1 Pa = 1 N/m$^2$). At sea level near room temperature, atmospheric pressure, usually abbreviated $p_a$, is 1.0133 $\times$ 10$^5$ Pa.
- **Gage pressure** is the pressure difference between the **absolute pressure** and **atmospheric pressure**.
- **Hydrostatic pressure** is the pressure that exists in a fluid at rest. It is caused by the weight of the fluid. For example, the hydrostatic pressure at the bottom of a column of fluid of height $h$ is $\rho g h$.

# A Hydraulic Brake System




- $f_1 = p_1 A_1$ and $f_2 = p_2 A_2$
- The point 1 is higher than point 2 , then $p_1 = p_2 + \rho g h$. If $h$ is small the pressure $\rho g h$ can be negligible compared to $p_2$, then $p_1 = f_1/A_1 = p_2 = f_2/A_2$.
- The forces are therefore related as $f_2 = f_1 A_2/A_1$.
- The force $f_3$ can be obtained from the lever relation $f_3 = f_2 L_1/L_2$, assuming static equilibrium or negligible lever inertia.

# Conservation of Mass

The conservation of mass can be stated as follows:

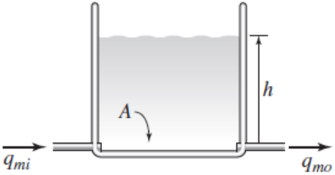
- For a container holding a mass of fluid $m$, the time rate of change $\dot{m}$ of mass in the container must equal the total mass inflow rate minus the total mass outflow rate.
- $\dot{m} = q_{mi} - q_{mo}$, where $q_{mi}$ is the mass inflow rate and $q_{mo}$ is the mass outflow rate.
- The fluid mass $m$ is related to the container volume $V$ by $m = \rho V$. For the incompressible fluid, $\rho$ is constant, and thus $\dot{m} = \rho \dot{V}$. Let $q_{vi}$ and $q_{vo}$ be the total volume inflow and outflow rates. Thus, $q_{mi} = \rho q_{vi}$, and $q_{mo} = \rho q_{vo}$. Substituting thee relationship into above equation gives

$$\rho \dot{V} = \rho q_{vi} - \rho q_{vo}$$

- Cancel $\rho$ to obtain

$$\dot{V} = q_{vi} - q_{vo}$$

# Conservation of Mass



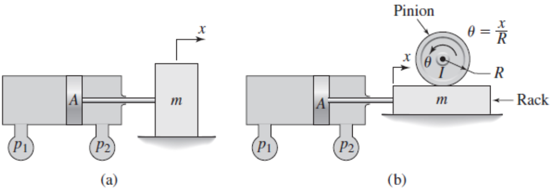

We have

$$\dot{m} = \frac{d}{dt}\rho A h = q_{mi}(t) - q_{mo}(t)$$
$$\rho A \frac{dh}{dt} = q_{mi}(t) - q_{mo}(t)$$

which can be integrated as follows:

$$h(t) = h(0) + \frac{1}{\rho A}\int_0^t (q_{mi}(\tau) - q_{mo}(\tau))d\tau$$

# A Hydraulic Cylinder




a) Develop a model of the motion of the displacement $x$ of the mass in part (a) of the figure.

b) Develop a model of the displacement $x$ in part (b) of the figure.

(a) Assuming that $p_1(t) > p_2(t)$. The net force acting on the piston and mass $m$ is $(p_1 - p_2)A$, and thus from Newton's law,

$$F = (p_1(t) - p_2(t))A = m\ddot{x}$$

$$\dot{x} = \dot{x}(0) + \frac{A}{m}\int_0^t (p_1(\tau) - p_2(\tau))d\tau$$

(b) Because we want an expression for the displacement $x$, we obtain an expression for the equivalent mass of the rack, pinion, and load. The kinetic energy of the system is

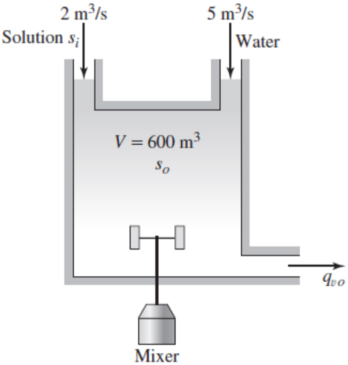
$$\text{KE} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{x}^2$$

because $R\dot{\theta} = \dot{x}$.
Thus the equivalent mass is

$$m_e = m + \frac{I}{R^2}$$

The required model can now be obtained by replacing $m$ with $m_e$ in the model developed in part (a).

# A Mixing Process




- Pure water flows into the tank of volume $V = 600$ m$^3$ at the constant volume rate of 5 m$^3$/s.
- A solution with a salt concentration of $s_i$ kg/m$^3$ flows into the tank at a constant volume rate of 2 m$^3$/s.
- Assume that the solution is perfect mixed. The salt concentration $s_o$ kg/m$^3$ in the outflow is the same as the concentration in the tank.
- The input is the concentration $s_i(t)$, whose value may change during the process, thus changing the value of $s_o$.

Obtain a dynamic model of the concentration $s_o$.

Two mass species are conserved here: water mass and salt mass. The tank is always full, so that the mass of water $m_w$ in the tank is constant and thus conservation of water mass gives ($\rho_w$ is the mass density of fresh water.)

$$\frac{dm_w}{dt} = \underbrace{5\rho_w + 2\rho_w}_{\text{inlet flow}} - \underbrace{\rho_w q_{vo}}_{\text{outlet flow}} = 0 \implies q_{vo} = 7 \text{ m}^3/\text{s}$$

# A Mixing Process

The salt mass in the tank is $s_o V$ (the concentration times volume), and the concentration of salt mass gives

$$\frac{d}{dt}(s_o V) = \underbrace{0(5)}_{\text{water}} + \underbrace{2s_i}_{\text{solution}} - s_o q_{vo} = 2s_i - 7s_o$$

$$600\frac{ds_o}{dt} = 2s_i - 7s_o$$

The last equation is the model.

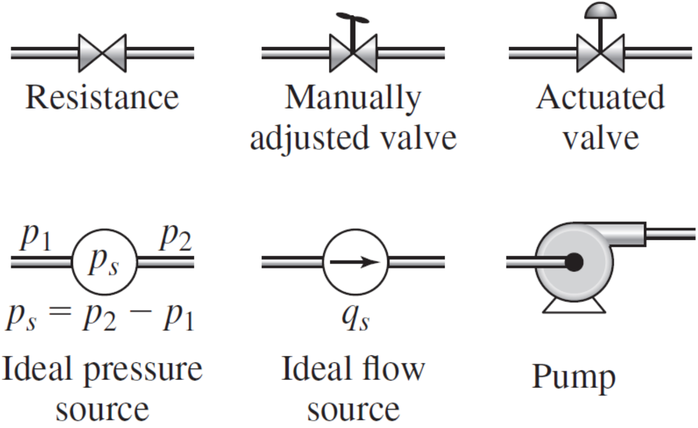
# Fluid Capacitance

It is very useful to think of fluid systems in terms of electrical circuits as follow:

| Fluid quantity | Electrical quantity |
|---|---|
| Fluid mass, $m$ | Charge, $Q$ |
| Mass flow rate, $q_m$ | Current , $i$ |
| Pressure, $p$ | Voltage, $v$ |
| Fluid linear resistance, $R$ | Electrical resistance, $R$ |
| $R = p/q_m$ | $R = v/i$ |
| Fluid capacitance, $C$ | Electrical capacitance, $C$ |
| $C = m/p$ | $C = Q/v$ |
| Fluid inertance, $I$ | Electrical inductance, $L$ |
| $I = p/(dq_m/dt)$ | $L = v/(di/dt)$ |

The compatibility law is analogous to Kirchhoff's voltage law, which states that the sum of signaed voltage differences around a closed loop must be zero.

# Fluid Symbols and Sources



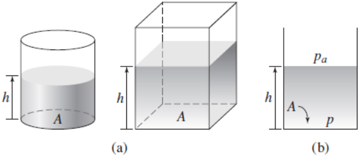

Fluid system symbols

## Fluid capacitance

The fluid capacitance is the relation between stored fluid mass and the resulting pressure caused by the stored mass. At a particular reference point $(p_r, m_r)$ the slope is $C$, where

$$C = \left.\frac{dm}{dp}\right|_{p=p_r} \text{ or } C_v = \left.\frac{dV}{dp}\right|_{p=p_r}$$

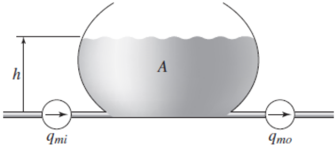
Thus, fluid capacitance $C$ is the ratio of the change and $C = \rho C_v$.




We have $m = \rho V = \rho A h$ and $p = \rho g h$, then $h = p/\rho g$. The pressure can be expressed as a function of the mass $m$ stored in the tank as $p = mg/A$. Thus

$$m = \frac{pA}{g} \text{ and } C = \frac{dm}{dp} = \frac{A}{g}$$

# Capacitance Relations




The fluid mass stored in the container is

$$m = \rho V = \rho \int_0^h A(x)dx$$

$$\frac{dm}{dh} = \rho A$$

For such a container, conservation of mass give

$$\frac{dm}{dt} = q_{mi} - q_{mo} \implies \frac{dm}{dt} = \frac{dm}{dp}\frac{dp}{dt} = C\frac{dp}{dt}$$

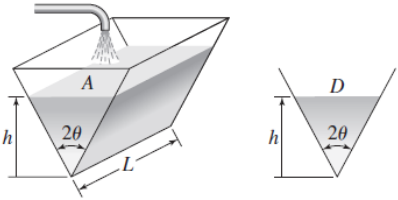
Thus

$$C\frac{dp}{dt} = \frac{dm}{dt} = q_{mi} - q_{mo} \implies \frac{dm}{dt} = \frac{dm}{dh}\frac{dh}{dt} = \rho A\frac{dh}{dt}$$

so

$$\rho A\frac{dh}{dt} = q_{mi} - q_{mo}$$

# Capacitance of a V-Shaped Trough



(a) (b)

We have $D = 2h\tan\theta$, and the vertical cross-sectional area of the liquid is $hD/2$. The the fluid mass is given by

$$m = \rho V = \rho\left(\frac{1}{2}hD\right)L = (\rho L\tan\theta)h^2, \text{ and } p = \rho g h$$

$$m = (\rho L\tan\theta)\left(\frac{p}{\rho g}\right)^2 = \left(\frac{L\tan\theta}{\rho g^2}\right)p^2$$

# Capacitance of a V-Shaped Trough

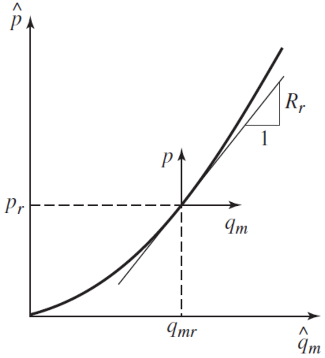
Since $q_{mo} = 0$ then $C\dot{p} = q_{mi}$, or

$$\left( \frac{2L \tan \theta}{\rho g^2} \right) p \frac{dp}{dt} = q_{mi}$$

which is a nonlinear equation because of the product $p\dot{p}$. We can obtain the model for the height by substituting $h = p/\rho g$. The result is

$$(2\rho L \tan \theta) h \frac{dh}{dt} = q_{mi}$$

# Fluid Resistance




General fluid resistance relation

## fluid resistance

We define the *fluid resistance* $R$ as

$$R = \left.\frac{dp}{dq_m}\right|_{q=q_{mr}}$$

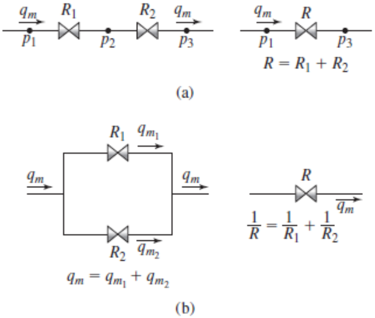
The resistance is the slope of the $p$ versus $q_m$ curve at some reference flow rate $q_{mr}$ and reference pressure $p_r$.

# Laminar Pipe Resistance

If the pipe flow is laminar, the laminar resistance for a level pipe of diameter $D$ and length $L$ is given by

$$R = \frac{128\mu L}{\pi\rho D^4},$$

where $\mu$ is the fluid viscosity.




Combination of series and parallel resistance

# Dynamic Models of Hydraulic Systems

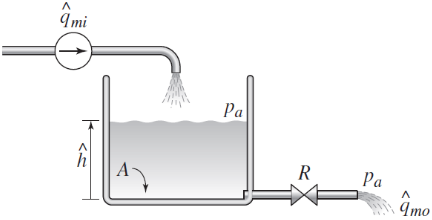
# Liquid Level Systems

In liquid-level systems energy is stored in two ways:

- as potential energy in the mass of liquid in the tank.
- as kinetic energy in the mass of liquid flowing in the pipe.

In many systems, the mass of the liquid in the pipes is small compared to the liquid mass in the tanks. If the mass of liquid in the pipe is small enough or is flowing at a small enough velocity, the kinetic energy contained in it will be negligible compared to the potential energy stored in the liquid in the tank.

# Liquid-Level System with a Flow Source




From conservation of mass ($\hat{h} = h + \bar{h}$ and the reference condition are $\bar{h}$ and $\bar{q}_{mi}$)

$$\begin{aligned}\frac{dm}{dt} &= \rho A \frac{d\hat{h}}{dt} = \hat{q}_{mi} - \hat{q}_{mo} \\ &= \rho A \frac{d(h + \bar{h})}{dt} = (q_{mi} + \bar{q}_{mi}) - (q_{mo} + \bar{q}_{mo})\end{aligned}$$

At the equilibrium point, the level is constant at $\bar{h}$ and the inflow $\bar{q}_{mi}$ and outflow $\bar{q}_{mo}$ are equal. The model becomes

$$\rho A \frac{dh}{dt} = q_{mi} - q_{mo}$$

# Liquid-Level System with a Flow Source

Because $R$ is a linearized resistance, then for small changes $h$ in the height,

$$q_{mo} = \frac{1}{R}\left[(\rho g h + p_a) - p_a\right] = \frac{1}{R}\rho g h$$

Then we have

$$\rho A \frac{dh}{dt} = q_{mi} - \frac{1}{R}\rho g h,$$

which can be rearranged as

$$\frac{dh}{dt} + \frac{1}{R}\frac{g}{A}h = \frac{1}{\rho A}q_{mi}$$

Noting that when $q_{mi} = 0$ the inflow rate remains constant a $\bar{q}_{mi}$.

**Second method:** Since

$$\frac{dm}{dt} = \frac{dm}{dp}\frac{dp}{dh}\frac{dh}{dt} = C(\rho g)\frac{dh}{dt}$$

# Liquid-Level System with a Flow Source

We have

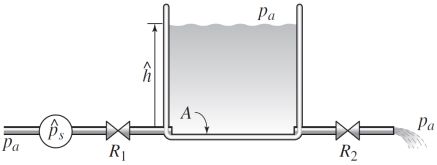
$$\begin{aligned}\rho g C \frac{dh}{dt} &= q_{mi} - \frac{1}{R}\rho g h \\ RC\frac{dh}{dt} + h &= \frac{R}{g} q_{vi} \\ \frac{dh}{dt} + \frac{1}{RC} h &= \frac{1}{gC} q_{vi}\end{aligned}$$

Take the Laplace transform of both sides and let all initial conditions to be zero, we have

$$\begin{aligned}\left(s + \frac{1}{RC}\right) H(s) &= \frac{1}{gC} Q_{vi}(s) \\ T_{hq_{vi}}(s) = \frac{H(s)}{Q_{vi}(s)} &= \frac{1/gC}{s + 1/RC} \\ &= \frac{R/g}{RCs + 1}\end{aligned}$$

**Note** We can also use $C = A/g$ to link both methods.

# Liquid Level Systems with Pressure Source




The tank shown in cross section in Figure has a bottom area $A$. A pressure source $\hat{p}_s = p_s(t) + \bar{p}_s$ is connected through a resistance to the bottom of the tank, where $p_s(t)$ is a given function of time. The resistances $R_1$ and $R_2$ are linearized resistances about the reference condition ($p_{sr}$), Develop a model of $h$, the deviation of the liquid height from the constant reference height $\bar{h}$, where $\hat{h} = h + \bar{h}$.

**Solution** The total mass in the tank in $m = \rho A\hat{h} = \rho A(h + \bar{h})$, and from conservation of mass

$$\frac{dm}{dt} = \rho A \frac{d(h + \bar{h})}{dt} = \rho A \frac{dh}{dt} = \hat{q}_{mi} - \hat{q}_{mo}$$

$$\rho A \frac{dh}{dt} = (q_{mi} + \bar{q}_{mi}) - (q_{mo} + \bar{q}_{mo}) = (q_{mi} - q_{mo}) + (\bar{q}_{mi} - \bar{q}_{mo}).$$

# Liquid Level Systems with Pressure Source

Because at the reference equilibrium, the outflow rate equals the inflow rate, $\bar{q}_{mi} - \bar{q}_{mo} = 0$, and we have

$$\rho A \frac{dh}{dt} = q_{mi} - q_{mo}$$

This is a linearized model that is valid for small changes around the equilibrium state. At the outlet flow we have

$$q_{mo} = \frac{1}{R_2}\left[(\rho g h + p_a) - p_a\right] = \frac{\rho g h}{R_2}.$$

Similarly for the mass inflow rate, we have

$$q_{mi} = \frac{1}{R_1}\left[(p_s + p_a) - (\rho g h + p_a)\right] = \frac{1}{R_1}(p_s - \rho g h)$$

The level model is

$$\frac{dh}{dt} + \frac{g}{A}\left(\frac{R_1 + R_2}{R_1 R_2}\right) h = \frac{1}{\rho A R_1} p_s$$

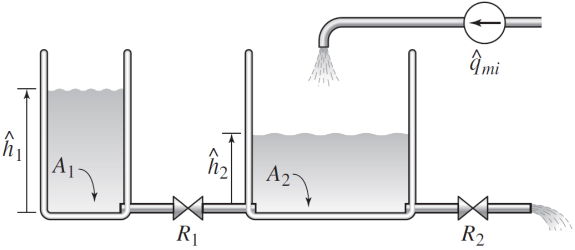
# Liquid Level Systems with Pressure Source

Using $C = A/g$, we have

$$\frac{dh}{dt} + \frac{1}{C}\left(\frac{R_1 + R_2}{R_1 R_2}\right) h = \frac{1}{\rho g R_1 C} p_s$$
$$\frac{dh}{dt} + \frac{1}{R_e C} h = \frac{1}{\rho g R_1 C} p_s,$$

where $R_e = (R_1 + R_2)/(R_1 R_2)$. Taking the Laplace transform of both sides, we obtain

$$\left(s + \frac{1}{R_e C}\right) H(s) = \frac{1}{\rho R_1 C} P_s(s)$$
$$\frac{H(s)}{P_s(s)} = \frac{R_e/(\rho g R_1)}{R_e C s + 1}$$




The cylindrical tanks shown in Figure have bottom areas $A_1$ and $A_2$. The total mass inflow rate from the flow source is $\hat{q}_{mi}(t)$, a given function of time. The resistances are linearized resistances about the reference condition $\bar{h}_1$, $\bar{h}_2$, $\bar{q}_{mi}$. (a) Develop a model of the liquid heights $h_1$ and $h_2$. (b) Suppose the resistances are equal: $R_1 = R_2 = R$, and the areas are $A_1 = A$ and $A_2 = 3A$. Obtain the transfer function $H_1(s)/Q_{mi}(s)$. (c) Use the transfer function to solve for the steady-state response for $h_1$ if the inflow rate $q_{mi}$ is a unit-step function, and estimate how long it will take to reach steady state. Is it possible for liquid heights to oscillate in the step response?

# Two Connected Tanks

a. Note that $\hat{h}_1 = \bar{h}_1 + h_1$, $\hat{h}_2 = \bar{h}_2 + h_2$, and $\hat{q}_{mi} = \bar{q}_{mi} + q_{mi}$. Assume that $h_1 > h_2$. From the conservation of mass applied to tank 1, we obtain

$$\rho A_1 \frac{d\hat{h}_1}{dt} = \rho A_1 \frac{d(h_1 + \bar{h}_1)}{dt} = \rho A_1 \frac{dh_1}{dt} = -\hat{q}_{1mo} = -(q_{1mo} + \bar{q}_{1mo})$$

From physical reasoning we can see that the two heights must be equal at equilibrium, and thus $\bar{q}_{1mo} = 0$. Therefore

$$\rho A_1 \frac{dh_1}{dt} = -q_{1mo}$$

Because $R_1$ is a linearized resistance,

$$q_{1mo} = \frac{\rho g}{R_1}(h_1 - h_2) \qquad \text{at the equilibrium point } \bar{h}_1 = \bar{h}_2$$

Finally, we have

$$\frac{dh_1}{dt} = -\frac{g}{R_1 A_1}(h_1 - h_2)$$

# Two Connected Tanks

Similarly for tank 2,

$$\frac{d(\rho A_2 \hat{h}_2)}{dt} = \frac{d\left[\rho A_2 (h_2 + \bar{h}_2)\right]}{dt} = \rho A_2 \frac{dh_2}{dt}$$

conservation of mass gives

$$\rho A_2 \frac{dh_2}{dt} = \hat{q}_{mi} + \hat{q}_{1mo} - \hat{q}_{2mo} = (q_{mi} + \bar{q}_{mi}) + (q_{1mo} + \bar{q}_{1mo}) - (q_{2mo} + \bar{q}_{2mo})$$

Recalling that $\bar{q}_{1mo} = 0$, we note that this implies that $\bar{q}_{mi} = \bar{q}_{2mo}$, and thus

$$\rho A_2 \frac{dh_2}{dt} = q_{mi} + q_{1mo} - q_{2mo}$$

Because the resistances are linearized, we have

$$\rho A_2 \frac{dh_2}{dt} = q_{mi} + \frac{\rho g}{R_1}(h_1 - h_2) - \frac{\rho g}{R_2} h_2 \implies A_2 \frac{dh_2}{dt} = \frac{1}{\rho} q_{mi} + \frac{g}{R_1}(h_1 - h_2) - \frac{g}{R_2} h_2$$

The model consists of equations of tank 1 and tank 2.

Using $C_1 = A_1/g$ and $C_2 = A_2/g$, we have

$$\frac{dh_1}{dt} = -\frac{1}{R_1 C_1}(h_1 - h_2)$$
$$\frac{dh_2}{dt} = \frac{1}{\rho g C_2} q_{mi} + \frac{1}{R_1 C_2}(h_1 - h_2) - \frac{1}{R_2 C_2} h_2$$

b. Substituting $R_1 = R_2 = R$, $A_1 = A$, and $A_2 = 3A$ into the differential equations and dividing by $A$, and letting $B = 1/RC$ $\left(\frac{1}{R_1 C1} = \frac{1}{RC} \text{ and } \frac{1}{R_2 C_2} = \frac{1}{3RC}\right)$, we obtain

$$\frac{dh_1}{dt} = -B(h_1 - h_2)$$
$$\frac{dh_2}{dt} = \frac{1}{3\rho g C} q_{mi} + \frac{B}{3}(h_1 - h_2) - \frac{B}{3} h_2$$

# Two Connected Tanks

Taking the Laplace transform and assuming zero initial conditions, we have

$$
\begin{aligned}
H_1(s) &= \frac{B}{(s+B)} H_2(s) = \frac{1}{\frac{1}{B}s+1} H_2(s) \\
H_2(s) &= \frac{B/3}{(s+2B/3)} H_1(s) + \frac{1/(3\rho g C)}{(s+2B/3)} Q_{mi}(s) \\
&= \frac{1/2}{\frac{3}{2B}s+1} H_1(s) + \frac{R/(2\rho g)}{\frac{3}{2B}s+1} Q_{mi}(s)
\end{aligned}
$$

Then

$$
\begin{aligned}
H_1(s) &= \frac{1}{\frac{1}{B}s+1} \left( \frac{1/2}{\frac{3}{2B}s+1} H_1(s) + \frac{R/(2\rho g)}{\frac{3}{2B}s+1} Q_{mi}(s) \right) \\
\frac{H_1(s)}{Q_{mi}(s)} &= \frac{RB^2/3\rho g}{s^2 + \frac{5B}{3}s + \frac{B^2}{3}}
\end{aligned}
$$

c. The characteristic equation is $s^2 + \frac{5}{3}Bs + \frac{1}{3}B^2 = 0$ and has the two real roots

$$s = \frac{-5/3 \pm \sqrt{13/9}}{2}B = -1.43B, -0.232B$$

Thus the system is stable, and there will be a constant steady-state response to a step input. The step response cannot oscillate because both roots are real. The steady-state height can be obtained by applying the final value theorem as below:

$$h_{1ss} = \lim_{s \to 0} sH_1(s) = \lim_{s \to 0} s \frac{RB^2/3\rho g}{s^2 + \frac{5}{3}Bs + \frac{1}{3}B^2} \frac{1}{s} = \frac{R}{\rho g}$$

# Reference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2. Norman S. Nise, "*Control Systems Engineering*,$6^{th}$ edition, Wiley, 2011
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