Lecture 9: Thermal Systems

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May 18, 2022

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Thermal system

A thermal system

- is one in which energy is stored and transferred as thermal energy commonly called heat.
- The thermal systems include heating and cooling systems in buildings and mixing processes where heat must be added or removed to maintain an optimal reaction temperature.
- Thermal systems operate because of temperature differences, as heat energy flows from an object with the higher temperature to an object with the lower temperature.
- Conservation of heat energy forms the basis of thermal system models, along with the concepts of thermal resistance and thermal capacitance.
- For a system with well-defined boundaries, the law of conservation of energy states

$$\Delta E = Q - W,$$

where ΔE is the change in energy of the system, Q is the heat flow into or out of the system, and W is the work done by or on the system.

Thermal system

- Q is positive if heat is supplied to the system and negative if heat is given off by the system. W is positive if work is done by the system and negative if work is done to the system. We have

$$\Delta E = (Q_{\text{in}} - Q_{\text{out}}) - (W_{\text{out}} - W_{\text{in}}),$$

where $Q_{\text{in}}, Q_{\text{out}}, W_{\text{in}}$, and W_{out} are all positive quantities.

- The net amount of energy added to the system is equal to the net increase in the energy stored internally in the system and any change in the mechanical energy of the system's center of mass,

$$\Delta E = \Delta U + \Delta ME_C,$$

where U is the internal energy (or internal thermal energy), which is the energy stored at the molecular level. It includes the kinetic energy due to the motion of molecules and the potential energy that holds the atoms together. ME_C stands for the mechanical energy, which includes the kinetic energy and the potential energy associated with the system's mass center.

- For systems with negligible change in mechanical energy,

$$\Delta U = Q - W = (Q_{\text{in}} - Q_{\text{out}}) - (W_{\text{out}} - W_{\text{in}})$$

Thermal system

- Heat Q is the energy transfer at the molecular level. Work W is the energy transfer that is capable of producing macroscopic mechanical motion of tye system's mass center. For the thermal systems with pure heat transfer and no work involved, that is, $W_{\text{in}} = W_{\text{out}} = 0$, the law of conservation of energy presented the previous equatin can be rewritten as

$$\Delta U = Q = Q_{\text{in}} - Q_{\text{out}}$$

or

$$\frac{dU}{dt} = q_{\text{hi}} - q_{\text{ho}},$$

where $q_{\text{h}} = dQ/dt$ is the heat flow rate having units of J/s, which is a watt of ft·lb/s.

Thermal Capacitance

Thermal capacitance relates an object's temperature to the amount of heat energy stored. It is defined as the ratio of the change in heat flow to the change in the object's temperature,

$$C = \frac{dQ}{dT},$$

where Q is the stored heat energy, C has units of J/K, J/°C.

For a constant-volume process, no work is involved and all the heat goes into the internal energy of the substance,

$$Q = \Delta U = mc_v \Delta T,$$

where m is the mass of the substance, c_v is the constant-volume specific heat capacity of the substance in units of J/K, J/°C, and ΔT is the change in the temperature of the substance.
For the constant-pressure process,

$$Q = \Delta H = mc_p \Delta T,$$

where H is the enthalpy and c_p is the constant-pressure specific heat capacity.

Thermal Capacitance

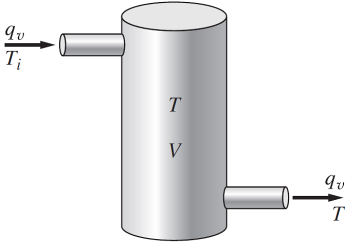
We have

$$C = mc_v \qquad \text{and} \qquad mc_p = \rho V c_p,$$

where ρ and V are the density and the volume of the mass m.

Concept of thermal capacitance applies to fluids as well as solids. For example, at room temperature and atmospheric pressure, the ratio of the specific heat of water to that of air is 4.16. Thus for the same mass of air and water, to raise the water temperature by 1° requires 4.16 times more energy than for air.

Temperature Dynamics of a Mixing Process



Liquid at a temperature T_i is pumped into a mixing tank at a constant volume flow rate q_v. The container walls are perfectly insulated so that no heat escapes through them. The container volume is V, and the liquid within is well mixed so that its temperature throughout is T. The liquid's specific heat and mass density are c_p and ρ. Develop a model for the temperature T as a function of time, with T_i as the input.

The amount of heat energy in the tank liquid is $Q = \rho c_p V(T - T_r)$, where T_r is an arbitrarily selected reference temperature. From conservation of energy,

$$\frac{d\left[\rho c_p V(T - T_r)\right]}{dt} = \text{heat rate in - heat rate out}$$

The mass is flowing into the tank at the rate $\dot{m} = \rho q_v$. Thus heat energy is flowing into the tank at the rate

$$\text{heat rate in} = \dot{m} c_p (T_i - T_r) = \rho q_v c_p (T_i - T_r)$$

Temperature Dynamics of a Mixing Process

Similarly,

$$\text{heat rate out} = \rho q_v c_p (T - T_r)$$

Therefore, since ρ, c_p, V, and T_r are constants,

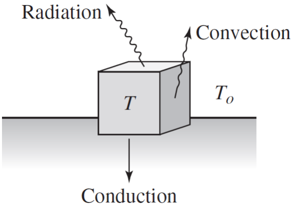
$$\begin{aligned}
\rho c_p V \frac{dT}{dt} &= \rho q_v c_p (T_i - T_r) - \rho q_v c_p (T - T_r) = \rho q_v c_p (T_i - T) \\
V \frac{dT}{dt} &= q_v (T_i - T) \\
\frac{dT}{dt} + \frac{q_v}{V} T &= \frac{q_v}{V} T_i
\end{aligned}$$

Note that T_r, ρ, and c_p do not appear in the final model form, so their specific numerical values are irrelevant to the problem. The time constant is V/q_v, and thus the liquid temperature T changes slowly if the tank volume V is large or if the inflow rate q_v is small.

Thermal Resistance

The heat energy is conserved:

- the heat in thermal system analysis plays the same role as charge in electrical systems. The flow of heat, called **heat transfer**, causes a change in an object's temperature.
- Heat transfer between two objects is caused by a difference in their temperatures.
- Thus temperature difference in thermal systems plays the same role as voltage difference in electrical systems.
- We utilize the concept of **thermal resistance** in a manner similar to electrical resistance.
- Heat transfer can occur by one or more modes: **conduction**, **convection**, and **radiation**.



Newton's Law of Cooling

Newton's law of cooling is a linear model for heat flow rate as a function of temperature difference. The law, which is used for both convection and conduction models, is expressed as

$$q_h = \frac{1}{R}\Delta T,$$

where q_h is the heat flow rate, R is the thermal resistance, and ΔT is the temperature difference. In SI, q_h has the units of J/s, which is a watt (W).

For **conduction** through material of thickness L, and approximate formula for the conductive resistance is

$$R = \frac{L}{kA} \implies q_h = \frac{kA}{L}\Delta T,$$

where k is the **thermal** conductivity of the material and A is the surface area.

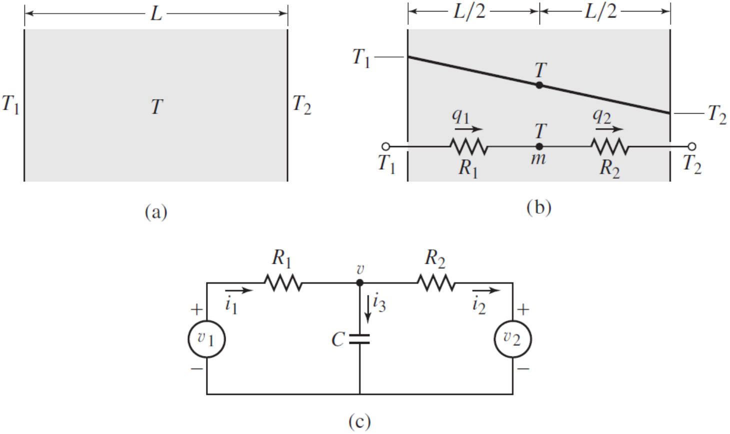
Newton's Law of Cooling

For the convection, we might need to analyze the system as a fluid as well as a thermal system. The thermal resistance for convection occurring at the boundary of a fluid and a solid is given by

$$R = \frac{1}{hA} \implies q_h = hA\Delta T,$$

where h is the so-called film coefficient or convection coefficient of the fluid-solid interface and A is the involved surface area. The film coefficient might be a complicated function of the fluid flow characteristics.

Heat Transfer Through a Plate



If $T_1 > T_2$, heat will flow from the left side to the right side. The temperatures T_1 and T_2 of the adjacent objects will remain constant if the objects are large enough. Under the transient conditions the temperature profile is not linear and must be obtained by solving a partial differential equation.

Heat Transfer Through a Plate

Under **steady-state** conditions, the average temperature is at the center, and we can select as the center temperature to be the representative temperature for the transient calculations. The mass m of the plate is assumed to be lumped at the plate centerline, and consider conductive heat transfer to occur over a path of length $L/2$ between temperature T_1 and temperature T and $L/2$ between temperature T and temperature T_2. Thus, the thermal resistance for each path is

$$R_1 = \frac{L/2}{kA}, \qquad R_2 = \frac{L/2}{kA}$$

We can derive the thermal model by applying conservation of heat energy. Assuming that $T_1 > T > T_2$, we obtain

$$mc_p \frac{dT}{dt} = q_1 - q_2 = \frac{1}{R_1}(T_1 - T) - \frac{1}{R_2}(T - T_2)$$

The thermal capacitance is $C = mc_p$.

This system is analogous to the circuit in Figure (c), where the voltages v, v_1, and v_2 are analogous to the temperatures T, T_1 and T_2.

Series Thermal Resistances

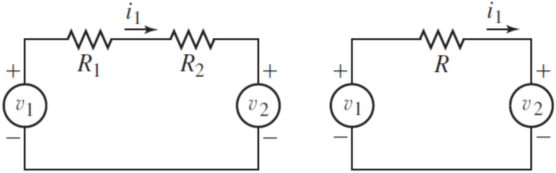
If the plate mass m is very small, its thermal capacitance $C = mc_p$ is also very small. In this case, the mass absorbs a negligible amount of heat energy, so the heat flow rate q_1 through the left-hand conductive path must equal the rate q_2 though the right-hand path. That is, if $C = 0$,

$$q_1 = \frac{1}{R_1}(T_1 - T) = q_2 = \frac{1}{R_2}(T - T_2)$$

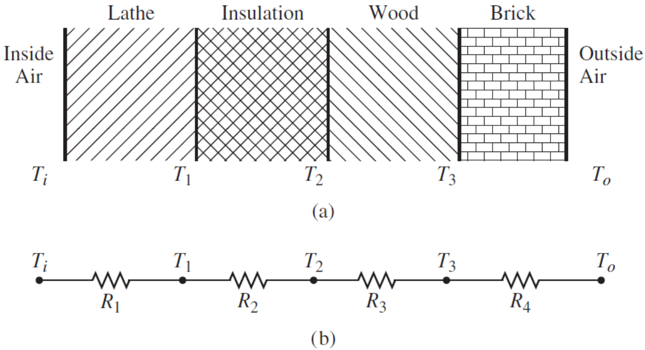
$$T = \frac{R_2 T_1 + R_1 T_2}{R_1 + R_2}$$

$$q_1 = q_2 = \frac{T_1 - T_2}{R_1 + R_2} = \frac{T_1 - T_2}{R} \implies R = R_1 + R_2$$

Thus thermal resistances are in **series** if they pass the same heat flow rate.



Thermal Resistance of Building Wall



The resistances for a wall area of 1 m^2 are $R_1 = 0.036$, $R_2 = 4.01$, $R_3 = 0.408$, and $R_4 = 0.038$ °C/W. Suppose that $T_i = 20°$C and $T_0 = -10°$C. (a) Compute the total wall resistance for 1 m^2 of wall area, and compute the heat loss rate if the wall's area is 3 m by 5 m. (b) Find the temperatures T_1 , T_2 and T_3 assuming steady-state conditions.

Thermal Resistance of Building Wall

a. The series resistance law gives

$$R = R_1 + R_2 + R_3 + R_4 = 0.036 + 4.01 + 0.408 + 0.038 = 4.492^\circ\text{C/W}$$

which is the total resistance for 1 m^2 of wall area. The wall area is $3(5) = 15\text{m}^2$, and thus the total heat loss is

$$q_h = 15\frac{1}{R}(T_i - T_0) = 15\frac{1}{4.492}(20 + 10) = 100.2\text{W}$$

This is the heat rate that must be supplied by the building's heating system to maintain the inside temperature at 20°C, if the outside temperature is -10°C.

b. Applying conservation of energy gives the following equations:

$$q_h = \frac{1}{R_1}(T_i - T_1) = \frac{1}{R_2}(T_1 - T_2) = \frac{1}{R_3}(T_2 - T_3) = \frac{1}{R_4}(T_3 - T_0)$$

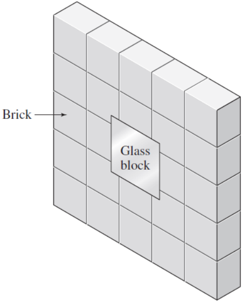
Then,

$$
\begin{aligned}
(R_1 + R_2)T_1 - R_1 T_2 &= R_2 T_i \\
R_3 T_1 - (R_2 + R_3)T_2 + R_2 T_3 &= 0 \\
-R_4 T_2 + (R_3 + R_4)T_3 &= R_3 T_0
\end{aligned}
$$

For the given values of T_i and T_0, the solution to these equations is $T_1 = 19.7596$, $T_2 = -7.0214$, and $T_3 = -9.7462°$C.

Parallel Resistances

A certain wall section is composed of a 15 cm by 15 cm glass block 8 cm thick. Surrounding the block is a 50 cm brick section, which is also 8 cm thick. The thermal conductivity of the glass is $k = 0.81$ W/m·°C. For the brick, $k = 0.45$ W/m·°C. (a) Determine the thermal resistance of the wall section. (b) Compute the heat flow rate through (1) the glass (2) the brick, and (3) the wall if the temperature difference across the wall is 30°C.



The resistance

$$R = \frac{L}{kA}$$

$$R_1 = \frac{0.08}{0.81(0.15)^2} = 4.39$$

$$R_2 = \frac{0.08}{0.45[(0.5)^2 - (0.15)^2]} = 0.781$$

The resistances are in parallel,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = 0.228 + 1.28 = 1.51$$

or $R = 0.633$°C/W.

Parallel Resistances

The heat flow through the glass is

$$q_1 = \frac{1}{R_1}\Delta T = \frac{1}{4.39}30 = 6.83 \text{ W}$$

The heat flow through the brick is

$$q_2 = \frac{1}{R_2}\Delta T = \frac{1}{0.781}30 = 38.4 \text{ W}$$

Thus the total heat flow through the wall section is

$$q_h = q_1 + q_2 = 45.2 \text{ W}$$

This rate could also have been calculated from the total resistance as followed

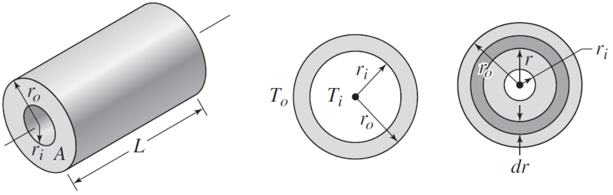
$$q_h = \frac{1}{R}\Delta T = \frac{1}{0.663}30 = 45.2 \text{ W}L$$

Radial Conductive Resistances

Consider a cylindrical tube whose inner and outer radii are r_i and r_o. Heat flow in the tube wall can occur in the axial direction along the length of the tube and in the radial direction. If the tube surface is insulated, there will be no radial heat flow, and the heat flow in the axial direction is given by

$$q_h = \frac{kA}{L}\Delta T,$$

where L is the length of the tube, ΔT is the temperature difference between the ends a distance L apart, and A is area of the solid cross section. If only the ends of the tube are insulated, then the heat flow will be entirely radial. Derive and expression for the conductive resistance in the radial direction.



(a) (b) (c)

Radial Conductive Resistances

From the figure (b) the inner and outer temperatures are T_i and T_o, and are assumed to be constant along the length L of the tube. From *the Fourier's law, the heat flow rate per unit area through an element of thickness dr is proportional to the negative of the temperature gradient dT/dr.*

$$\frac{q_h}{2\pi rL} = -k\frac{dT}{dr} \Rightarrow q_h = -2\pi kLr\frac{dT}{dr}$$

$$\int_{r_i}^{r_o} \frac{q_h}{r}\,dr = -2\pi Lk \int_{T_i}^{T_o} dT$$

Because q_h is constant, the integration yields

$$q_h \ln\frac{r_o}{r_i} = -2\pi Lk(T_o - T_i) \text{ or } q_h = \frac{2\pi Lk}{\ln(r_o/r_i)}(T_i - T_o)$$

The radial resistance is thus given by

$$R = \frac{\ln(r_o/r_i)}{2\pi Lk}$$

Dynamic Models of Thermal Systems

Heat Transfer Systems

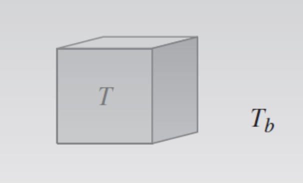
Heat transfer occurs between two objects. To obtain an ordinary differential equation model of the temperature dynamics of an object, we must be able to assign a single temperature that is representative of the object.

The BIOT Criterion For solid bodies immersed in a fluid, a useful criterion for determiing the validity of the uniform-temperature assumption is based on the *Biot number*, defined as

$$N_B = \frac{hL}{k}$$

where L is a representative dimension of the object, which is usually taken to be the ratio of the volume to the surface area of the body. For example a cube $L = d^3/6d^2$. If $N_B < 0.1$, the temperature is taken to be uniform (The object can treat as a lumped-parameter system with a single uniform temperature, denoted T).

Quenching with Constant Bath Temperature



Consider a lead cube with a side length of $d = 20$ mm. The cube is immersed in an oil bath for which $h = 200$ W/(m^2·°C), The oil temperature is T_b. Develop a model of the cube's temperature as a functin of the liquid temperature T_b, which is assumed to be known. (k for lead is 34 W/m·°C and the density of lead is 1.134×10^4 kg/m^3)

Here $L = d^3/6d^2 = 0.02/6$, and $N_B = 200(0.02)/34(6) = 0.02$. We can treat the cube as a lumped-parameter system with a single uniform temperature, denoted T. If $T > T_b$, from the conservation of energy we obtain

$$C\frac{dT}{dt} = -\frac{1}{R}(T - T_b)$$

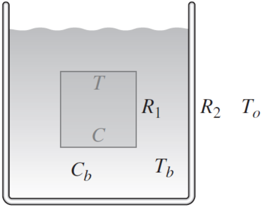
$$C = mc_p = \rho V c_p = 1.134 \times 10^4(0.02)^3(129) = 11.7 \text{ J}/^\circ C$$

The thermal resistance R is due to convection $R = 1/hA = 1/200(0.02)^2 = 2.08^\circ$·s/J, and is

$$11.7\frac{dT}{dt} = -\frac{1}{2.08}(T - T_b) \Rightarrow 24.4\frac{dT}{dt} + T = T_b$$

Note The time constant is $\tau = RC = 24.4$ s.

Quenching with Variable Bath Temperature



The temperature outside the bath is T_o, which is assumed to be known. The convective resistance between the cube and the bath is R_1 , and the combined convective/conductive resistance of the container wall and the liquid surface is R_2. The capacitances of the cube and the liquid bath are C and C_b, respectively.

1. Derive a model of the cube temperature and the bath temperature assuming that the bath loses no heat to the surroundings (that is, $R_2 = \infty$).
2. Obtain the model's characteristic roots and the form of the response.

Quenching with Variable Bath Temperature

(1) Assume that $T > t_b$. Then the heat flow is out of the cube and into the bath. From conservation of energy for the cube.

$$C\frac{dT}{dt} = -\frac{1}{R_1}(T - T_b)$$

and for the bath

$$C_b\frac{dT_b}{dt} = \frac{1}{R_1}(T - T_b)$$

The heat flow rate in the lower equation must have a sign opposite to that in the above equation because the heat flow out of the cube must be the same as the heat flow into the bath.
(2) Appliying th eLaplace transform to equations with zero initial conditions, we obtain

$$(R_1Cs + 1)T(s) - T_b(s) = 0$$
$$(R_1C_bs + 1)T_b(s) - T(s) = 0$$

Quenching with Variable Bath Temperature

We have

$$[(R_1C_bs+1)(R_1Cs+1)-1]\,T(s)=0$$

from which we obtain

$$R_1^2C_bCs^2+R_1(C+C_b)s=0$$
$$s(R_1^2C_bCs+R_1(C+C_b))=0$$

The form of the response is $T(t)=A_1e^{-\lambda_1 t}+B_1e^{-\lambda_2 t}$, then

$$T(t)=A_1+B_1e^{-t/\tau} \text{ and } T_b(t)=A_2+B_2e^{-t/\tau}$$

and

$$\tau=\frac{1}{\lambda_2}=\frac{R_1CC_b}{C+C_b}$$

where the constants A_1, A_2, B_1 depend on the initial conditions. The two temperatures become constant after approximately 4τ. Note that $T(t)\to A_1$ and $T_b(t)\to A_2$ as $t\to\infty$. From physical insight we know that T and T_b will become equal as $t\to\infty$. Therefore, $A_2=A_1$.

Reference

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