Lecture 8: Transfer Function of Mechanical Systems

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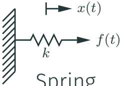
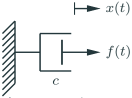
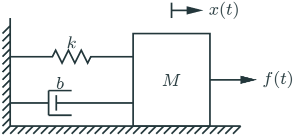
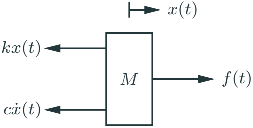
Component	Force-velocity	Force-disp	Impedance $Z_M(s) = \frac{F(s)}{X(s)}$
 Spring	$f(t) = k\int_0^t v(\tau)d\tau$	$f(t) = kx(t)$	k
 Viscous damper	$f(t) = cv(t)$	$f(t) = c\dot{x}(t)$	cs
 (a) Mass	$f(t) = M\dot{v}(t)$	$f(t) = M\ddot{x}$	Ms^2

Transfer function: Translational Mechanical System

Assuming that there are no friction between a mass and ground.





From the Newton's law $\Sigma F = ma$, we have

$$M\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$$

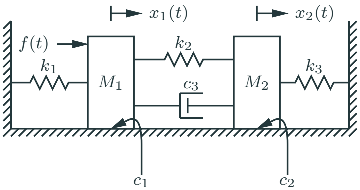
Taking the Laplace transform of above equation and setting all initial conditions to be zero, we obtain

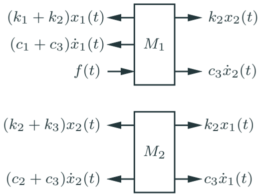
$$Ms^2X(s) + csX(s) + kX(s) = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + cs + k}$$

Transfer function: Translational Mechanical System

The point of motion in a system can still move if all other points of motion are held still. The name for the number of the linearly independent motions i sthe number of the degrees of freedom.





$$M_1\ddot{x}_1(t) + (c_1+c_3)\dot{x}_1(t) - c_3\dot{x}_2(t) + (k_1+k_2)x_1(t) - k_2x_2 = f(t)$$
$$M_2\ddot{x}_2(t) + (c_2+c_3)\dot{x}_2(t) - c_3\dot{x}_1(t) + (k_2+k_3)x_2(t) - k_2x_1 = 0$$

Transfer function: Translational Mechanical System

Taking the Laplace transform of both equations, we get

$$\left(M_1 s^2 + (c_1 + c_3)s + (k_1 + k_2)\right) X_1(s) - (c_3 s + k_2)X_2(s) = F(s)$$
$$-(c_3 s + k_2)X_1(s) + \left(M_2 s^2 + (c_2 + c_3)s + (k_2 + k_3)\right) X_2(s) = 0$$

Rearranging the equations into matrix form:

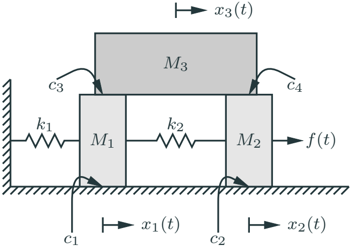
$$\begin{bmatrix} M_1 s^2 + (c_1 + c_3)s + (k_1 + k_2) & -(c_3 s + k_2) \\ -(c_3 s + k_2) & M_2 s^2 + (c_2 + c_3)s + (k_2 + k_3) \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

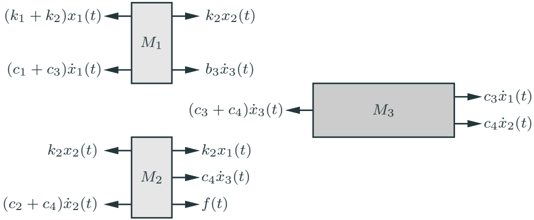
$$\frac{X_2(s)}{F(s)} = \frac{c_3 s + k_2}{\Delta}$$

where

$$\Delta = \begin{vmatrix} M_1 s^2 + (c_1 + c_3)s + (k_1 + k_2) & -(c_3 s + k_2) \\ -(c_3 s + k_2) & M_2 s^2 + (c_2 + c_3)s + (k_2 + k_3) \end{vmatrix}$$

Transfer function: Translational Mechanical System





The equations of motion are

$$
\begin{aligned}
M_1\ddot{x}_1(t) + (c_1 + c_3)\dot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) - c_3\dot{x}_3(t) &= 0 \\
M_2\ddot{x}_2(t) + (c_2 + c_4)\dot{x}_2(t) + k_2x_2(t) - k_2x_1(t) - c_4\dot{x}_3(t) &= f(t) \\
M_3\ddot{x}_3(t) + (c_3 + c_4)\dot{x}_3(t) - c_3\dot{x}_1(t) - c_4\dot{x}_2(t) &= 0
\end{aligned}
$$

Transfer function: Translational Mechanical System

Taking the Laplace transform to all equations, we have

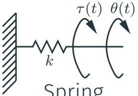
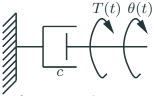
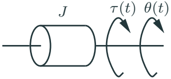
$$\left(M_1s^2 + (c_1 + c_3)s + (k_1 + k_2)\right) X_1(s) - k_2X_2(s) - c_3sX_3(s) = 0$$
$$-k_2X_1(s) + \left(M_2s^2 + (c_2 + c_4)s + k_2\right) X_2(s) - c_4sX_3(s) = F(s)$$
$$-c_3sX_1(s) - c_4sX_2(s) + \left(M_3s^2 + (c_3 + c_4)s\right) X_3(s) = 0$$

and in matrix from

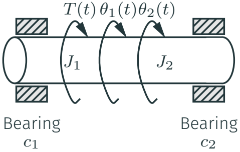
$$\begin{bmatrix} M_1s^2 + (c_1 + c_3)s + (k_1 + k_2) & -k_2 & -c_3s \\ -k_2 & M_2s^2 + (c_2 + c_4)s + k_2 & -c_4s \\ -c_3s & -c_4s & M_3s^2 + (c_3 + c_4)s \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \begin{bmatrix} 0 \\ F(s) \\ 0 \end{bmatrix}$$

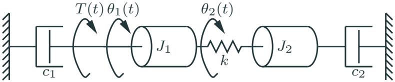
Note: the matrix is symmetry.

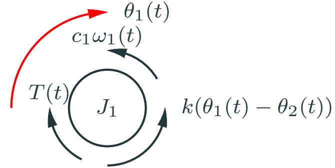
Transfer function: Rotational Mechanical System

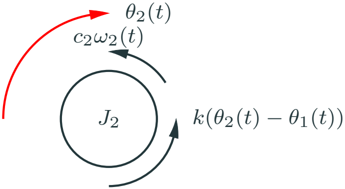
Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = \hat{T}(s)/\Theta(s)$
 Spring	$T(t) = k \int_0^{t_1} \omega(t) dt$	$T(t) = k\theta(t)$	k
 Viscous damper	$T(t) = b\omega(t)$	$T(t) = c\dot{\theta}(t)$	cs
 Inertia	$T(t) = J\dot{\omega}(t)$	$T(t) = J\ddot{\theta}$	Js^2

Transfer function: Rotational Mechanical System









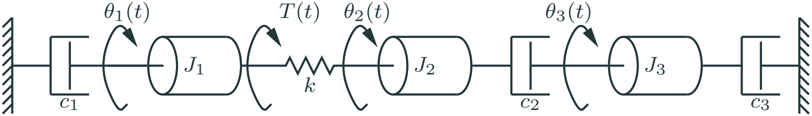
The equations of motion are

$$
\begin{aligned}
T(t) - c_1\dot{\theta}_1(t) - k(\theta_1(t) - \theta_2(t)) &= J_1\ddot{\theta}_1 \\
-k(\theta_2(t) - \theta_1(t)) - c_2\dot{\theta}_2(t) &= J_2\ddot{\theta}_2
\end{aligned}
$$

Taking the Laplace transform, we have

$$
\begin{aligned}
\left(J_1 s^2 + c_1 s + k\right)\Theta_1(s) - k\Theta_2(s) &= T(s) \\
\left(J_2 s^2 + c_2 s + k\right)\Theta_2(s) - k\Theta_1(s) &= 0
\end{aligned}
$$

Transfer function: Rotational Mechanical System



The equations of motion are

$$
\begin{aligned}
T(t) - c_1\dot{\theta}_1(t) - k\left(\theta_1(t) - \theta_2(t)\right) &= J_1\ddot{\theta}_1 \\
-k\left(\theta_2(t) - \theta_1(t)\right) - c_2\left(\dot{\theta}_2 - \dot{\theta}_3\right) &= J_2\ddot{\theta}_2 \\
-c_2\left(\dot{\theta}_3 - \dot{\theta}_2\right) - c_3\dot{\theta}_3 &= J_3\ddot{\theta}_3
\end{aligned}
$$

Taking the Laplace transform, we have

$$
\begin{aligned}
\left(J_1 s^2 + c_1 s + k\right)\Theta_1(s) - k\Theta_2(s) &= T(s) \\
-k\Theta_1(s) + \left(J_2 s^2 + c_2 s + k\right)\Theta_2(s) - c_2 s\Theta_3(s) &= 0 \\
-c_2 s\Theta_2(s) + \left(J_3 s^2 + (c_2 + c_3)s\right)\Theta_3(s) &= 0
\end{aligned}
$$

Transfer function: Rotational Mechanical System

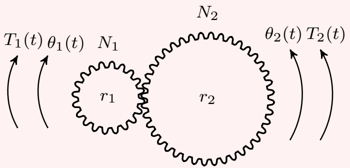
In matrix from

$$\begin{bmatrix} \left(J_1 s^2 + c_1 s + k\right) & -k & 0 \\ -k & \left(J_2 s^2 + c_2 s + k\right) & -c_2 s \\ 0 & -c_2 s & \left(J_3 s^2 + (c_2 + c_3) s\right) \end{bmatrix} \begin{bmatrix} \Theta_1(s) \\ \Theta_2(s) \\ \Theta_3(s) \end{bmatrix} = \begin{bmatrix} T(s) \\ 0 \\ 0 \end{bmatrix}$$

Transfer Functions for Systems with Gears

- Gears provide mechanical advantage to rotational system, e.g. a bicycle with gears.
- Gears are nonlinear. They exhibit *backlash*, which occurs from the loose fit between two meshed gears.
- In this course, we consider only the linearized version of gears.

A gear system



- a small gear has radius r_1 and N_1 teeth is rotated through angle $\theta_1(t)$ due to a torque, $T_1(t)$.
- a big gear have radius r_2 and N_2 teeth responds by rotating through angle $\theta_2(t)$ and delivering a torque, $T_2(t)$.

Transfer Functions for Systems with Gears

- The gears turn, the distance traveled along each gear's circumference is the same. Thus

$$r_1\theta_1 = r_2\theta_2$$

or

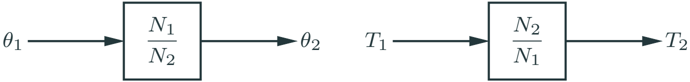
$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

- If the gears are lossless, that is they do not absorb or store energy, the energy into Gear 1 equals the energy out of Gear 2. Since the translational energy of force times displacement becomes the rotational energy of torque time angular displacement.

$$T_1\theta_1 = T_2\theta_2$$

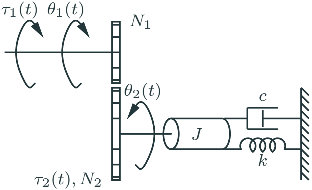
or

$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$



(a) (b)

- (a) is a transfer functions for angular displacement in lossless gears
- (b) is a transfer functions for torque in lossless gears.

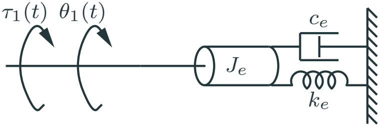


$$\tau_2 - k\theta_2 - c\dot{\theta}_2 = J\ddot{\theta}_2$$
$$J\ddot{\theta}_2 + c\dot{\theta}_2 + k\theta_2 = \tau_2$$

The first gear (lossless) generates torque (T_1) to drive the second gear by T_2, then

$$\left(Js^2 + cs + k\right)\Theta_2(s) = T_2(s)$$
$$\frac{T_2}{T_1} = \frac{N_2}{N_1} = \frac{\theta_1}{\theta_2}$$
$$\left(Js^2 + cs + k\right)\frac{N_1}{N_2}\Theta_1(s) = \frac{N_2}{N_1}T_1(s)$$
$$\left(J\left(\frac{N_1}{N_2}\right)^2 s^2 + c\left(\frac{N_1}{N_2}\right)^2 s + k\left(\frac{N_1}{N_2}\right)^2\right)\Theta_1(s) = T_1(s)$$

Transfer Functions for Systems with Gears

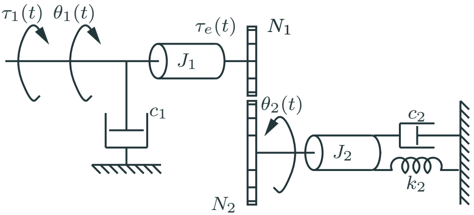


$$J_e = J\left(\frac{N_1}{N_2}\right)^2$$

$$c_e = c\left(\frac{N_1}{N_2}\right)^2$$

$$k_e = k\left(\frac{N_1}{N_2}\right)^2$$

Transfer Functions for Systems with Gears



Find the transfer function, $\Theta_2(s)/T_1(s)$. Assuming that $T_e(t)$ is the torque generates at the first gear by the torque $T_1(t)$, the we have

$$\left(J_1 s^2 + c_1 s\right)\Theta_1(s) = T_1(s) - T_e(s)$$

$$\left(J_1 s^2 + c_1 s\right)\frac{N_2}{N_1}\Theta_2(s) + T_e(s) = T_1(s)$$

Transfer Functions for Systems with Gears

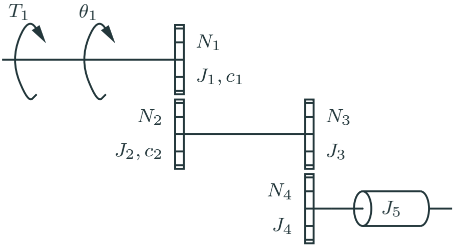
At the second gear, we have

$$
\begin{aligned}
\left(J_2 s^2 + c_2 s + k_2\right)\Theta_2(s) &= T_2(s) \\
\left(J_2 s^2 + c_2 s + k_2\right)\Theta_2(s) &= \frac{N_2}{N_1} T_e(s) \\
\left(J_2 s^2 + c_2 s + k_2\right)\frac{N_1}{N_2}\Theta_2(s) &= T_e(s)
\end{aligned}
$$

Combining the equations of both gears, we have

$$
\begin{aligned}
\left[J_1\left(\frac{N_2}{N_1}\right)s^2 + c_1\left(\frac{N_2}{N_1}\right)s + J_2\left(\frac{N_1}{N_2}\right)s^2 + c_2\left(\frac{N_1}{N_2}\right)s + k_2\left(\frac{N_1}{N_2}\right)\right]\Theta_2(s) &= T_1(s) \\
\left[\left(J_1\left(\frac{N_2}{N_1}\right)^2 + J_2\right)s^2 + \left(c_1\left(\frac{N_2}{N_1}\right)^2 + c_2\right)s + k_2\right]\frac{N_1}{N_2}\Theta_2(s) &= T_1(s) \\
\frac{\Theta_2(s)}{T_1(s)} &= \frac{N_2/N_1}{\left[\left(J_1\left(\frac{N_2}{N_1}\right)^2 + J_2\right)s^2 + \left(c_1\left(\frac{N_2}{N_1}\right)^2 + c_2\right)s + k_2\right]}
\end{aligned}
$$

Transfer Functions - Gears with Loss



Starting from the left most of the gear, we can find the transfer function $\Theta_1(s)/T_1(s)$ as follow:

$$\left(J_1 s^2 + c_1 s\right)\Theta_1(s) + T_{e1}(s) = T_1(s)$$
$$\left[(J_2 + J_3)\, s^2 + c_2 s\right]\Theta_2(s) + T_{e2}(s) = T_2(s)$$
$$\left[(J_4 + J_5)\, s^2\right]\Theta_4(s) = T_4(s)$$

Transform all torques and angle to be in terms of $T_1(s)$ and $\Theta_1(s)$ respectively.

Transfer Functions - Gears with Loss

$$\left[(J_4 + J_5)\, s^2\right] \frac{N_3}{N_4} \Theta_2(s) = \frac{N_4}{N_3} T_{e2}(s)$$

Substituting $T_{e2}(s)$ to one above equation, we have

$$\left[(J_2 + J_3)\, s^2 + (J_4 + J_5) \left(\frac{N_3}{N_4}\right)^2 s^2 + c_2 s\right] \Theta_2(s) = T_2(s)$$

$$\left[(J_2 + J_3)\, s^2 + (J_4 + J_5) \left(\frac{N_3}{N_4}\right)^2 s^2 + c_2 s\right] \frac{N_1}{N_2} \Theta_1(s) = \frac{N_2}{N_1} T_{e1}(s)$$

Substituting $T_{e1}(s)$ to one above equation, we have

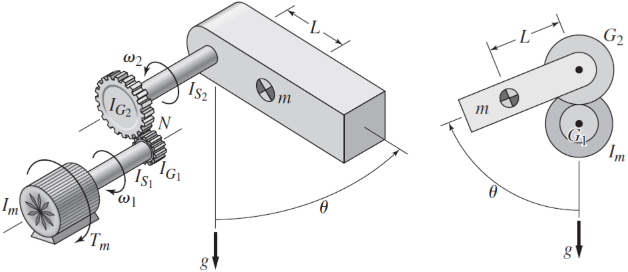
$$\left[\left(J_1 + (J_2 + J_3) \left(\frac{N_1}{N_2}\right)^2 + (J_4 + J_5) \left(\frac{N_1}{N_2}\frac{N_3}{N_4}\right)^2\right) s^2 + \left(c_1 + c_2 \left(\frac{N_1}{N_2}\right)^2\right) s\right] \Theta_1(s) = T_1(s)$$

$$\frac{\Theta_1(s)}{T_1(s)} = \frac{1}{\left[\left(J_1 + (J_2 + J_3) \left(\frac{N_1}{N_2}\right)^2 + (J_4 + J_5) \left(\frac{N_1}{N_2}\frac{N_3}{N_4}\right)^2\right) s^2 + \left(c_1 + c_2 \left(\frac{N_1}{N_2}\right)^2\right) s\right]}$$

Electromechanical System Transfer Functions

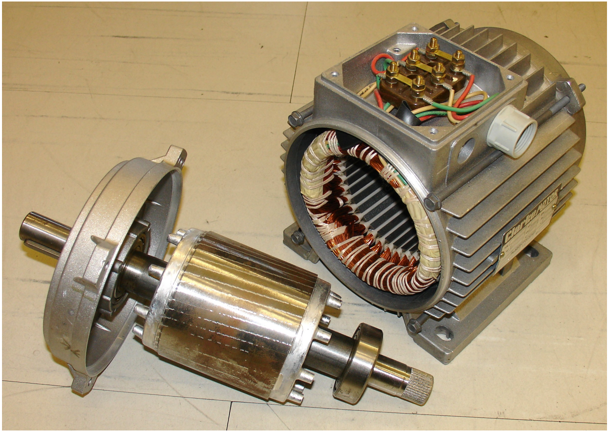
An *electromechanical systems* is a hybrid system of electrical and mechanical variables. This system has a lot of application for examples

- an antenna azimuth position control system
- robot and robot arm controls
- sun and star trackers
- disk-drive position controls



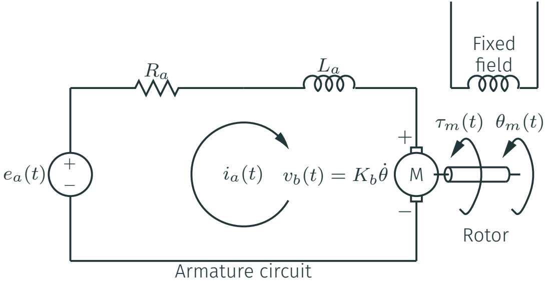
Industrial robot arm

Electromechanical System Transfer Functions: DC Motor



Cutaway view of a permannet magnet motor.

Electromechanical System Transfer Functions: DC Motor



- $v_b(t) = K_b \dfrac{d\theta_m(t)}{dt} = K_b\dot{\theta}_m$, where $v_b(t)$ is the *back electromotive force (back emf)*; K_b is a constant of proportionality called the back emf constant.
- The relationship between the armature current, $i_a(t)$, the applied armature voltage, $e_a(t)$, and the back emf, $v_b(t)$ is

$$R_a i_a(t) + L_a \frac{di_a(t)}{dt} + K_b\dot{\theta}_m(t) = e_a(t)$$

$$(R_a + L_a s)\, I_a(s) + K_b s\Theta_m(s) = E_a(s)$$

Electromechanical System Transfer Functions: DC Motor

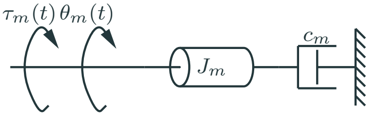
- The torque developed by the motor is proportional to the armature current; thus

$$\tau_m(t) = K_t i_a(t) \;\Rightarrow\; T_m(s) = K_t I_a(s)$$

  where $T_m(t)$ is the torque developed by the motor, and K_t is a constant for proportionality, called the motor torque constant.

- Taking the Laplace transform of both relationship and substituting $I_a(s)$ into the mesh equation, we have $(\Omega_m(s) = s\Theta_m(s))$

$$\frac{(R_a + L_a s)T_m(s)}{K_t} + K_b s\Theta_m(s) = E_a(s)$$



The figure shows a typical equivalent mechanical loading on a motor. We have

$$\tau_m(t) = J_m\ddot{\theta}(t) + c_m\dot{\theta}(t)$$
$$T_m(s) = \left(J_m s^2 + c_m s\right)\Theta_m(s)$$

Electromechanical System Transfer Functions: DC Motor

Substituting $T_m(s)$ into the armature equation yields

$$\frac{(R_a + L_a s)\left(J_m s^2 + c_m s\right)\Theta_m(s)}{K_t} + K_b s\Theta_m(s) = E_a(s)$$

The transfer function from $e_a(t)$ to $\theta_m(t)$ is

$$\left[\frac{R_a + L_a s}{K_t}\left(J_m s + c_m\right) + K_b\right] s\Theta_m(s) = E_a(s)$$

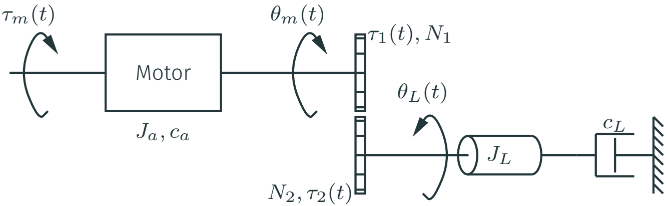
If we assume that the armature inductance L_a is small compared to the armature resistance, R_a, the equation become

$$\left[\frac{R_a}{K_t}\left(J_m s + c_m\right) + K_b\right] s\Theta_m(s) = E_a(s)$$

After simplification, the desired transfer function is

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K_t}{R_a(J_m s + c_m) + K_b K_t} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(c_m + \frac{K_t K_b}{R_a}\right)\right]}$$

Electromechanical System Transfer Functions



A motor with inertia J_a and damping c_a at the armature driving a load consisting of inertia J_L and damping c_L. Assuming that J_a, J_L, c_a, and c_L are known. Then, we have

$$\left(J_a s^2 + c_a s\right) \Theta_m(s) = T_m(s) - T_1(s)$$

At the load side, we obtain

$$\left(J_L s^2 + c_L s\right) \Theta_L(s) = T_2(s)$$
$$\left(J_L s^2 + c_L s\right) \frac{N_1}{N_2} \Theta_m(s) = \frac{N_2}{N_1} T_1(s)$$

Electromechanical System Transfer Functions

Substituting the $T_1(s)$ back to the motor side equation, the equivalent equation is

$$\left[\left(J_a + J_L\left(\frac{N_1}{N_2}\right)^2\right)s^2 + \left(c_a + c_L\left(\frac{N_1}{N_2}\right)^2\right)s\right]\Theta_m(s) = T_m(s)$$

Or the equivalent inertial, J_m, and the equivalent damping, c_m, at the armature are

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2; \qquad c_m = c_a + c_L\left(\frac{N_1}{N_2}\right)^2$$

Next step, we going to find the electrical constants by using a *dynamometer* test of motor. This can be done by measuring the torque and speed of a motor under the condition of a constant applied voltage. Substituting $V_b(s) = K_b s\Theta_m(s)$ and $T_m(s) = K_t I_a(s)$ in to the Laplace transformed armature circuit, with $L_a = 0$, yields

$$\frac{R_a}{K_t}T_m(s) + K_b s\Theta_m(s) = E_a(s)$$

Electromechanical System Transfer Functions

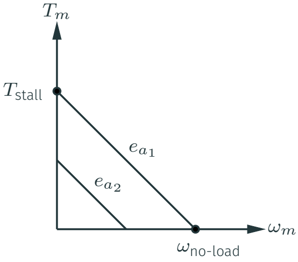
Taking the inverse Laplace transform, we get

$$\frac{R_a}{K_t}T_m(t) + K_b\omega_m(t) = e_a(t)$$

If $e_a(t)$ is a DC voltage, at the steady state, the motor should turn a a constant speed, ω_m, with a constant torque, T_m. With this, we have

$$\frac{R_a}{K_t}T_m + K_b\omega_m = e_a \qquad \Rightarrow \qquad T_m = -\frac{K_tK_b}{R_a}\omega_m + \frac{K_t}{R_a}e_a$$

Torque-peed curve



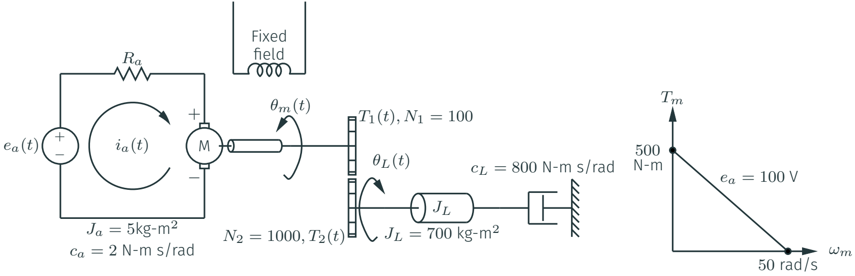
- $\omega_m = 0$, the value of torque is called the *stall torque*, T_{stall}. Thus

$$T_{\text{stall}} = \frac{K_t}{R_a}e_a \Rightarrow \frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a}$$

- $T_m = 0$, the angular velocity becomes *no-load speed*, $\omega_{\text{no-load}}$. Thus

$$\omega_{\text{no-load}} = \frac{e_a}{K_b} \Rightarrow K_b = \frac{e_a}{\omega_{\text{no-load}}}$$

Transfer Function–DC Motor and Load



Find $\Theta_L(s)/E_a(s)$ from the given system and torque-speed curve. The total inertia and the total damping at the armature of the motor are

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12$$

$$c_m = c_a + c_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10$$

Transfer Function–DC Motor and Load

Next, we find the electrical constants K_t/R_a and K_b from the torque-speed curve. Hence,

$$\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a} = \frac{500}{100} = 5$$

and

$$K_b = \frac{e_a}{\omega_{\text{no-load}}} = \frac{100}{50} = 2$$

We have

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(c_m + \frac{K_t K_b}{R_a}\right)\right]} = \frac{0.417}{s(s+1.667)}.$$

Using the gear ratio, $N_1/N_2 = 0.1$

$$\frac{\Theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)}$$

Reference

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