Lecture 7: Spring and Damping Element

Dr.-Ing. Sudchai Boonto, Assistant Professor

Department of Control System and Instrument Engineering, KMUTT

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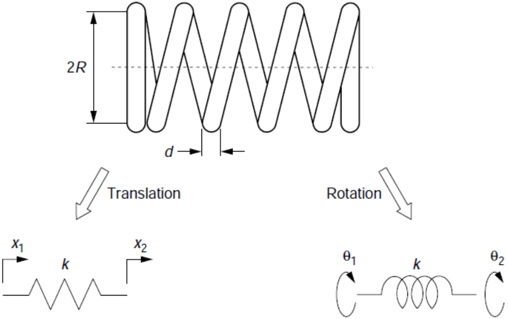
Spring and Damping Elements

Spring Elements

- Springs serving as elastic supports for translatory and rotary motion are studied in this section in relation to their lumped **stiffness** (or **spring constant**), denoted by k.
- Springs are mechanical elements that generate elastic forces in translatory motion and elastic torques in rotary motion that oppose the spring deformation; this elastic reactions are proportional to the spring deformation (linear or angular displacement).



Spring Elements

- The translatory stiffness of the helical spring is

$$k_t = \frac{Gd^4}{64nR^3},$$

where d is the wire diameter, R is the radius of the coil, and n is the number of coils. The **shear modulus of elasticity** G is a property of the wire material.
- the rotary stiffness is

$$k_r = \frac{Ed^4\left(1 + 2\frac{G}{E}\right)}{64nR},$$

where E is the modulus of elasticity (Young's modulus).
- For a spring whose end points undergo the displacements x_1 and x_2, the elastic force developed in the spring is proportional to the spring deformation, which is the difference between the two end point displacements as

$$f_e(t) = k\Delta x(t) = k\left(x_1(t) - x_2(t)\right)$$

Spring Elements

- Similarly, an elastic torque is generated by a spring in rotation whose end points undergo the rotations θ_1 and θ_2, the elastic torque is

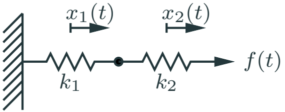
$$T_e(t) = k\Delta\theta(t) = k\left(\theta_1(t) - \theta_2(t)\right)$$

- All equation in this section, the springs are assumed to be linear; therefore, the stiffness is constant.
- For a translatory spring, the elastic energy stored corresponding to a deformation Δx is

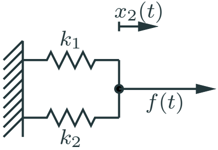
$$U_e = \frac{1}{2}k\left(\Delta x(t)\right)^2$$

- For a rotary spring, the elastic energy relative to an angular deformation $\Delta\theta$ is

$$U_e = \frac{1}{2}k\left(\Delta\theta(t)\right)^2$$



(a) Series



(b) Parallel

a) For the series (end-to-end) connection in Fig. (a), both springs have the same force but their deflection f/k_1 and f/k_2 will not be the same unless their spring constants are equal. The total deflecton x of the system is obtained from

$$x = \frac{f}{k_1} + \frac{f}{k_2} = \left(\frac{1}{k_1} + \frac{1}{k_2}\right) f$$
$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2}$$

This formula can be extended to the case of n springs connected end-to-end as follows:

b) For the parallel (side-by-side) connection in Fig. (b), both springs have the same deflection x but different forces f_1 and f_2. Then

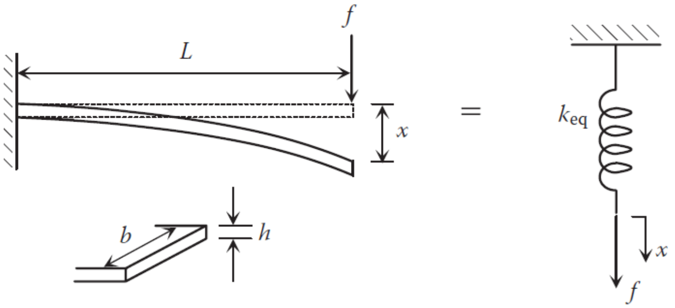
$$x = \frac{f_1}{k_1} = \frac{f_2}{k_2} \qquad f = f_1 + f_2$$

$$f = k_1 x + k_2 x = (k_1 + k_2)x = k_e x$$

This can be extended to the case of n springs connected in parallel as follows:

$$k_e = \sum_{i=1}^{n} k_i$$

Spring Elements: A beam in bending under a transverse force

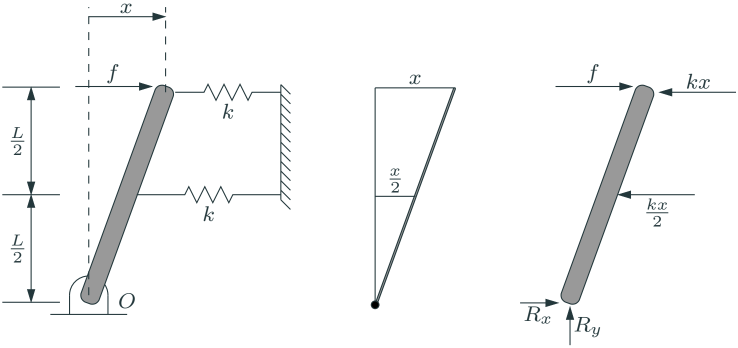


$$k_{eq} = \frac{Ebh^3}{4L^3},$$

where E is the modulus of elasticity of beam material.

Spring Elements: Lever-spring system

Figure below shows a horizontal force f acting on a lever that is attached to two springs. Assume that the resulting motion is small enough to be only horizontal and determine the expression fro the equivalent spring constant that relates the applied force f to the resulting displacement x.



Spring Elements: Lever-spring system

From the triangles shown in the Figure, for small angles θ , the upper spring deflection is x and the deflection of the lower sprig is $x/2$. thus the free body diagram is as shown in the right most of the Figure. For static equilibrium, the net moment about point O must be zero. This gives

$$\Sigma M = 0$$

$$fL - kxL - k\frac{x}{2}\frac{L}{2} = 0$$

Therefore

$$f = k\left(x + \frac{x}{4}\right) = \frac{5}{4}kx$$

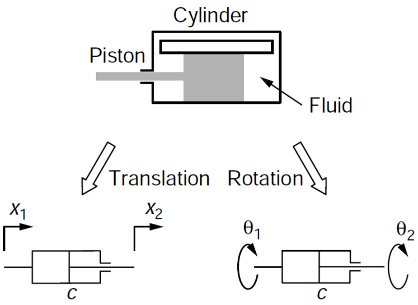
and the equivalent spring constant is $k_e = 5k/4$.

Damping Elements

Figure 1: Door Closer

Figure 2: Shock Absorbers

Damping Elements



The linear model for the damping force f as a function of the relative velocity v is

$$f = cv = c\dot{x},$$

where c is the **damping coefficient.** The units of c are force/velocity.

Damping Elements

- The damping coefficient of a piston-type damper with a single hole is

$$c = 8\pi\mu L \left[\left(\frac{D}{d} \right)^2 - 1 \right]^2,$$

  where μ is the viscosity of the fluid, L is the length of the hole through the piston, d is the diameter of the hole, and D is the diameter of the piston.
- The damping with two end points with the velocity v_1 and v_2, the net force is

$$f_e(t) = c\Delta v(t) = c\left(v_1(t) - v_2(t)\right)$$

- the linear model of a torsional damper is

$$T_e(t) = c\Delta\omega(t) = c\left(\omega_1(t) - \omega_2(t)\right)$$

Damping Elements

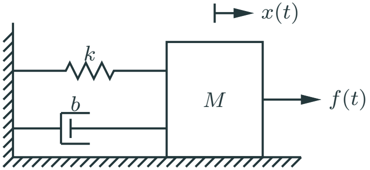
- The energy dissipated through viscous damping is equal to the work done by the damping force in translation and the damping torque in rotation:

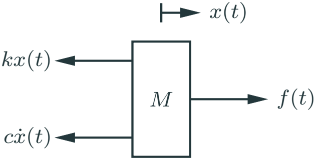
$$U_d = c\int v dx = c\int \dot{x}\frac{dx}{dt}dt = c\int \dot{x}^2 dt$$

$$U_d = c\int \omega d\theta = c\int \dot{\theta}\frac{d\theta}{dt}dt = c\int \dot{\theta}^2 dt$$

Spring-Mass-Damping

Assuming there are no friction between a mass and ground.





The equation of the motion is

$$M\ddot{x} = f - kx - c\dot{x}$$

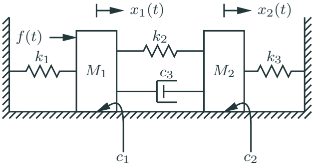
$$M\ddot{x} + c\dot{x} + kx = f$$

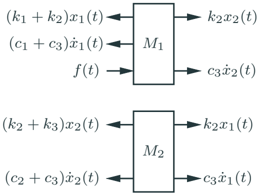
Giving $x = x_1$, $\dot{x} = x_2$, $f = u$ and $y = x_1$, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{c}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Spring-Mass-Damping

Derive the equations of motion of the two-mass system shown in Figure below.



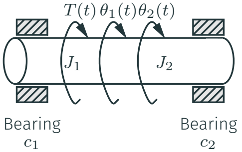


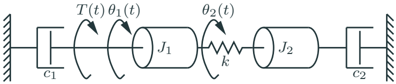
$$M_1\ddot{x}_1(t) + (c_1 + c_3)\dot{x}_1(t) - c_3\dot{x}_2(t) + (k_1 + k_2)x_1(t) - k_2 x_2 = f(t)$$
$$M_2\ddot{x}_2(t) + (c_2 + c_3)\dot{x}_2(t) - c_3\dot{x}_1(t) + (k_2 + k_3)x_2(t) - k_2 x_1 = 0$$

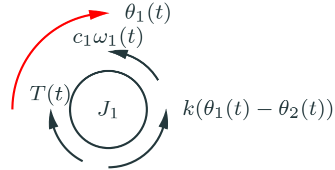
Giving $\dot{x}_1 = x_3$, $\dot{x}_2 = x_4$, and $f(t) = u$ the state-space form is

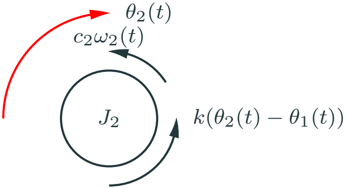
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{M_1} & \frac{k_2}{M_1} & -\frac{c_1+c_2}{M_1} & \frac{c_3}{M_1} \\ \frac{k_2}{M_2} & -\frac{k_2+k_3}{M_2} & \frac{c_3}{M_2} & -\frac{c_2+c_3}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix} u$$

Rotational Mechanical System: Example









The equations of motion are

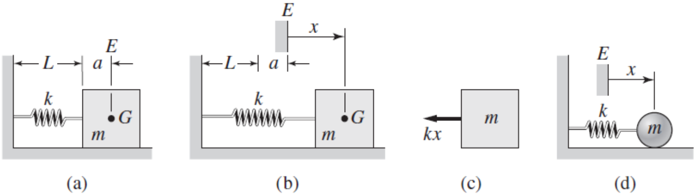
$$T(t) - c_1\dot{\theta}_1(t) - k(\theta_1(t) - \theta_2(t)) = J_1\ddot{\theta}_1$$
$$-k(\theta_2(t) - \theta_1(t)) - c_2\dot{\theta}_2(t) = J_2\ddot{\theta}_2$$

Rearranging , we have (omit the (t))

$$J_1\ddot{\theta}_1 + c_1\dot{\theta}_1 + k\theta_1 - k\theta_2 = T$$
$$J_2\ddot{\theta}_2 + c_2\dot{\theta}_2 - k\theta_1 + k\theta_2 = 0$$

Try to show the state-space from by yourself.

Effect of Spring Free Length and Object Geometry

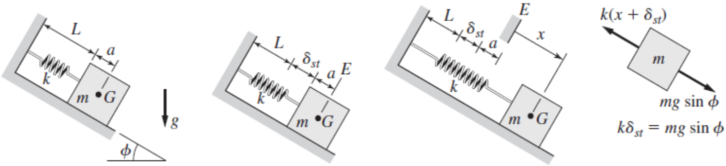


- (a) the horizontal surface is frictionless. The mass is homogeneous its center of mass is at the geometric center G of the cube. The free length of the spring is L and the mass m is in equilibrium when the spring is at its free length. The equilibrium location of G is the point marked E.
- (b) The mass displaced a distance x from the equilibrium position. The spring has been stretched a distance x from its free length, and thus its force is kx.
- (c) We have

$$m\ddot{x} = -kx$$

Note that neither the free length L nor the cube dimension a appears in the equation of motion. These two parameters need to be known only to locate the equilibrium position E of the mass center. We could consider the object as a point mass.

Effect of Gravity



(a) (b) (c) (d)

- (b) At the equilibrium, the spring stretches a distance δ_{st}, which is called the *static spring deflection*. Since the mass is in equilibrium, the sum of the forces acting on it must be zero

$$mg\sin\phi - k\delta_{st} = 0$$

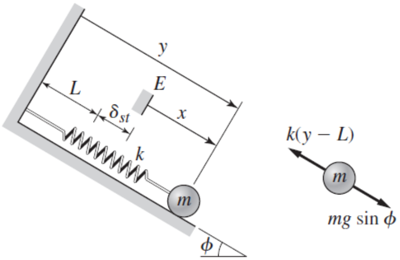
- (c) The object displaced a distance x from the equilibrium position. The spring has been stretched a distance $x + \delta_{st}$ from its free length, and thus its force is $k(x + \delta_{st})$
- (d) From the free body diagram,

$$m\ddot{x} = -k(x + \delta_{st}) + mg\sin\phi = -kx + (mg\sin\phi - k\delta_{st}) = -kx$$

Choosing the equilibrium position as coordinate reference

In the previous examples,

- a mass connected to a linear spring element, the force due to gravity is canceled out of the equation of motion by the force in the spring due to its static deflection, as log the displacement of the mass is measured form the equilibrium position.
- the force caused by its static deflection is called *static force* and the force caused by the variable displacement x as the *dynamic* spring force.



We need not choose the equilibrium location as the coordinate reference. For example, if we freely choose the coordinate y , the corresponding free body diagram is shown in the left Figure.

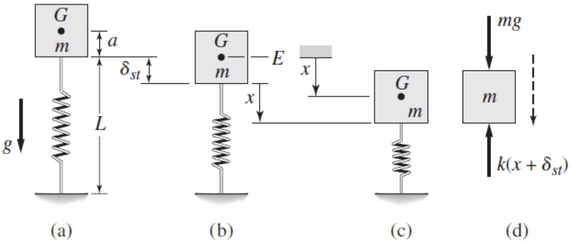
$$
\begin{aligned}
m\ddot{y} &= -k(y-L) + mg\sin\phi \\
&= -ky + kL + mg\sin\phi
\end{aligned}
$$

Here, $kL \neq mg\sin\phi$. The problem is more complicated. The static term do not cancel out of the equation.

Choosing the equilibrium position as coordinate reference

The advantages of choosing the equilibruim position as the coordinate origin are

1. we need not specify the geometric dimensions of the mass
2. this choice simplifies the equation of motion by elimainating the static forces.

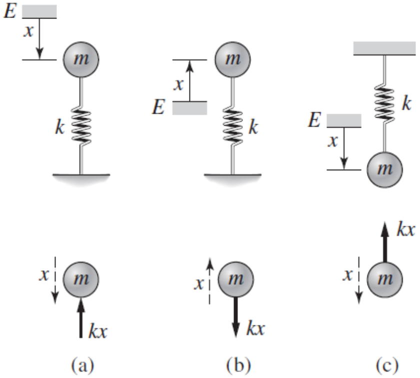


$$
\begin{aligned}
m\ddot{x} &= -k(m + \delta_{st}) + mg \\
&= -kx + (mg - k\delta_{st})
\end{aligned}
$$

Since $mg = k\delta_{st}$, then the equation of motion reduces to

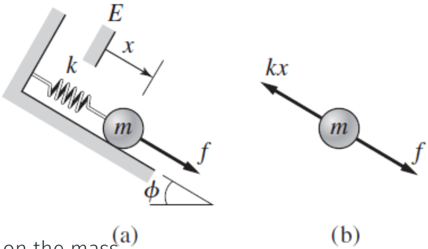
$$m\ddot{x} = -kx$$

Choosing the equilibrium position as coordinate reference



- The three situations, and the corresponding free body diagrams, have the same equation of motion, $m\ddot{x} = -kx$.
- Any forces acting on the mass, other than gravity and the spring force, are not to be included when determining the location of the equilibrium position.

Choosing the equilibrium position as coordinate reference



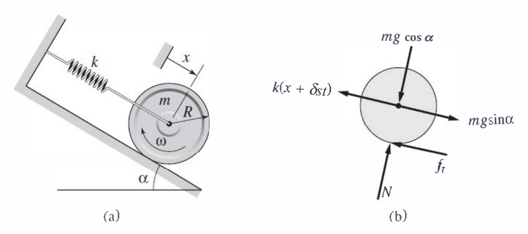
- a force f acts on the mass.
- The equilibrium position E is the location of the mass at which $k\delta_{st} = mg\sin\phi$ when $f = 0$
- From the free body diagram, the equation of motion is $m\ddot{x} = f - kx$.
- The equation of motions normally have the form

$$m\ddot{x} + kx = f \qquad \text{or} \qquad J\ddot{\theta} + k_T\theta = T$$

The natural frequencies are

$$w_n = \sqrt{\frac{k}{m}} \qquad \text{or} \qquad \omega_n = \sqrt{\frac{k_T}{J}}$$

Cylinder on an Incline



Taking $x = 0$ to be the equilibrium position, f_t to be the tangential force acting on the cylinder, and δ_{st} to be the static deflection. We have

$$
\begin{aligned}
m\ddot{x} &= mg\sin\alpha - k(x + \delta_{st}) - f_t \\
J\dot{\omega} &= Rf_t
\end{aligned}
$$

Since the cylinder does not slip, then $R\theta = x$. We find that

$$f_t = \frac{J}{R}\dot{\omega} = \frac{J}{R^2}\ddot{x}$$

Cylinder on an Incline

Using the fact from statics that

$$mg \sin \alpha = k\delta_{st},$$

hence

$$m\ddot{x} = -kx - \frac{J}{R^2}\ddot{x}$$

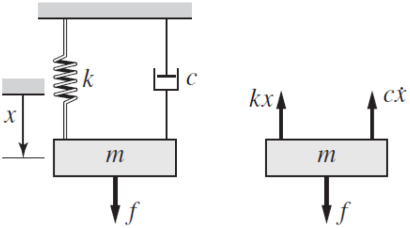
or

$$\left(m + \frac{J}{R^2}\right)\ddot{x} + kx = 0$$

The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m + J/R^2}}$$

A Generic Mass-Spring-Damper System



(a) (b)

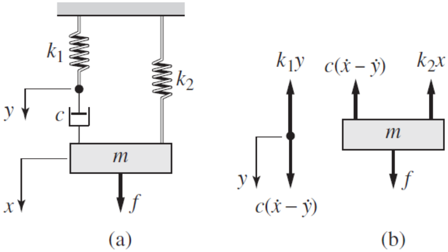
The equation of motion is

$$m\ddot{x} = -c\dot{x} - k(x + \delta_{st}) + mg + f = -c\dot{x} - kx + f$$

because $k\delta_{st} = mg$. The equation can be rearranged as

$$m\ddot{x} + c\dot{x} + kx = f$$

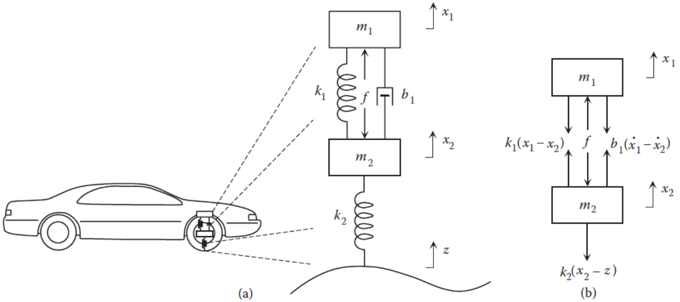
Coupled Spring and Damper



Let us assume that $x > 0, y > 0$, and $\dot{x} > \dot{y}$. If so, then the damper force pulls up on the mass, and me obtain the free body diagrams shown in part (b) of the figure.

$$
\begin{aligned}
m\ddot{x} &= f - k_2 x - c(\dot{x} - \dot{y}) \\
c(\dot{x} - \dot{y}) &= k_1 y
\end{aligned}
$$

Two-Degree-of-Freedom Quarter-Car Model



Derive the differential equations of motion.
Solution: The static equilibrium positions of m_1 and m_2 are set as the coordinate origins. Assume

$$x_1 > x_2 > z > 0,$$

which implies that th esprings are in tension and

$$\dot{x}_1 > \dot{x}_2 > \dot{z} > 0.$$

Two-Degree-of-Freedom Quarter-Car Model

At the equilibrium, the static forces in the springs cancel the weights of the masses. Note that the dampers have no effect in equilibrium and thus do not determine the location of the equilibrium position. Therefore the free body diagrams showing the dynamic forces, and not eh static forces.

Applying Newton's second law to the masses m_1 and m_2, respectively, gives

$$\begin{aligned} f - k_1(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2) &= m_1\ddot{x}_1, \\ -f + k_1(x_1 - x_2) + b_1(\dot{x}_1 - \dot{x}_2) - k_2(x_2 - z) &= m_2\ddot{x}_2. \end{aligned}$$

Rearranging the equations into the standard input-output form,

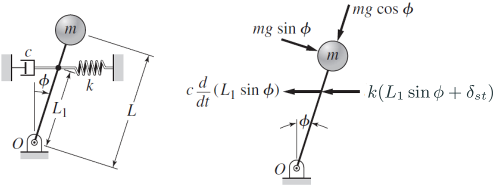
$$\begin{aligned} m_1\ddot{x}_1 + b_1\dot{x}_1 - b_1\dot{x}_2 + k_1x_1 - k_1x_2 &= f, \\ m_2\ddot{x}_2 - b_1\dot{x}_1 + b_1\dot{x}_2 - k_1x_1 + (k_1 + k_2)x_2 &= -f + k_2z, \end{aligned}$$

which can be expressed in second-order matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} b_1 & -b_1 \\ -b_1 & b_1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & k_2 \end{bmatrix} \begin{bmatrix} f \\ z \end{bmatrix}$$

Stability of an Inverted Pendulum

Determine the dynamic equation of the system below.



For the small value of ϕ the motion of the attachment point of the spring and damper is approximately horizontal; its displacement is $L_1 \sin\phi \approx L_1\phi$ and its velocity is $d(L_1 \sin\phi)/dt = L_1 \cos\phi\dot{\phi} \approx L_1\dot{\phi}$. (Note: $\cos\phi \approx 1$ if $\phi << 1$)

The equation of motion is

$J_0\ddot{\phi} = M_0$ or

$$mL^2\ddot{\phi} = mgL\sin\phi - L_1(cL_1\cos\phi\dot{\phi}) - L_1(k(L_1\sin\phi + \delta_{st}))$$

$$mL^2\ddot{\phi} = mgL\phi - cL_1^2\dot{\phi} - kL_1^2\phi - kL_1\delta_{st}$$

where $J_0 = mL^2$.

Stability of an Inverted Pendulum

At the equilibrium, we have ($\sin\phi \approx \phi$)

$$\Sigma M = 0 \quad \text{or} \quad mgL\phi = kL_1\delta_{st}$$

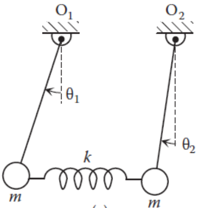
Then

$$\begin{aligned} mL^2\ddot{\phi} + cL_1^2\dot{\phi} + kL_1^2\phi &= 0 \\ \ddot{\phi} + a\dot{\phi} + b\phi &= 0, \end{aligned}$$

where

$$a = \frac{cL_1^2}{mL^2} \qquad b = \frac{kL_1^2}{mL^2}$$

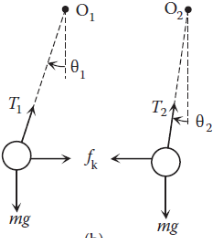
Coupled Pendulum System



(a)

Two simple pendulums are connected by a translational spring of stiffness k. Each pendulum consists of a point mass m concentrated at the tip of a massless rope of length L. When $\theta_1 = \theta_2 = 0$, the spring is at its free length. Derive the equations of motion, assuming small angles.

We choose the angular displacements θ_1 and θ_2 as the generalized coordinates. Assume $\theta_1 > \theta_2 > 0$, which implies that the spring is in tension. The free-body diagrams are shown below:



(b)

About fixed points O_1 and O_2, gives

$$\Sigma F = ma$$
$$-mgL\sin\theta_1 - f_k L\cos\theta_1 = mL^2\ddot{\theta}_1,$$
$$-mgL\sin\theta_2 + f_k L\cos\theta_2 = mL^2\ddot{\theta}_2.$$

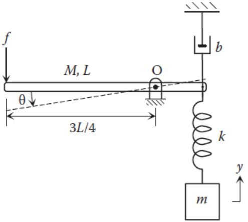
Coupled Pendulum System

The spring force f_k is in the horizontal direction due to the small-angle assumption, and its magnitude can be approximated as $k(L\sin\theta_1 - L\sin\theta_2)$. We have

$$\begin{aligned} \Sigma M &= J\ddot{\theta}, \\ mL^2\ddot{\theta}_1 + kL^2\cos\theta_1(\sin\theta_1 - \sin\theta_2) + mgL\sin\theta_1 &= 0, \\ mL^2\ddot{\theta}_2 - kL^2\cos\theta_2(\sin\theta_1 - \sin\theta_2) + mgL\sin\theta_2 &= 0. \end{aligned}$$

For small angles, we have

$$\begin{aligned} mL^2\ddot{\theta}_1 + kL^2(\theta_1 - \theta_2) + mgL\theta_1 &= 0, \\ mL^2\ddot{\theta}_2 - kL^2(\theta_1 - \theta_2) + mgL\theta_2 &= 0. \end{aligned}$$



A lever arm has a force applied on one side and a spring-damper combination on the other side with a suspended mass. When $\theta = 0$ and $f = 0$, the system is at static equilibrium. Draw the free-body diagram of the lever arm and the suspended mass. Derive the differential equations of motion for small angles θ.

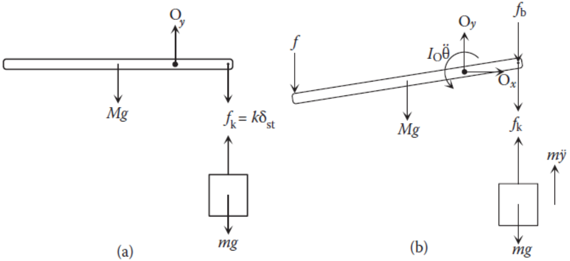
The static equilibrium, for the block, we have

$$\Sigma F_y = 0 \qquad \Rightarrow \qquad k\delta_{st} - mg = 0, \qquad \Rightarrow \qquad k\delta_{st} = mg,$$

and for the arm

$$\Sigma M_o = 0 \qquad \Rightarrow \qquad \frac{L}{4}Mg - \frac{L}{4}k\delta_{st} = 0, \qquad \Rightarrow \qquad Mg = k\delta_{st},$$

where δ_{st} is the static deformation of the spring.



When a force f is applied on one side of the rod, the deformation of the spring caused by the rotation of the rod can be approximated as $L\theta/4$ for small angels θ. Assuming that the block and the rod are displaced in their positive directions and $L\theta/4 > y > 0$, the spring is in tension and the magnitude of the spring force is $f_k = k(L\theta/4 - y + \delta_{st})$. The magnitude of the damping force is $f_b = b\frac{d}{dt}\left(\frac{L}{4}\sin\theta\right)$.

Lever mechanism

$$\Sigma F_y = ma_{cy} \qquad \Rightarrow \qquad k\left(\frac{L}{4}\theta - y + \delta_{st}\right) - mg = m\ddot{y}$$

For the rod (rotation only), applying the moment equation $\Sigma M = J\ddot{\theta}$ about the fixed point O gives ($\cos\theta \approx 1 \Rightarrow b\frac{d}{dt}\left(\frac{L}{4}\sin\theta\right) = b\frac{L}{4}\dot{\theta}$)

$$f\frac{3L}{4}\cos\theta + Mg\frac{L}{4}\cos\theta - k\left(\frac{L}{4}\theta - y + \delta_{st}\right)\frac{L}{4}\cos\theta - b\frac{L}{4}\dot{\theta}\frac{L}{4}\cos\theta = J_O\ddot{\theta},$$

where J_O can be obtained using parallel-axis theorem,

$$J_O = \frac{1}{12}ML^2 + M\left(\frac{L}{4}\right)^2 = \frac{7}{48}ML^2$$

For small angular motions, $\cos\theta \approx 1$, we have

$$m\ddot{y} + ky - \frac{kL}{4}\theta = 0,$$

$$\frac{7}{48}ML^2\ddot{\theta} + \frac{bL^2}{16}\dot{\theta} + \frac{kL^2}{16}\theta - \frac{kL}{4}y = \frac{3fL}{4}$$

Reference

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