Lecture 5: Modeling of Rigid-Body Mechanical Systems

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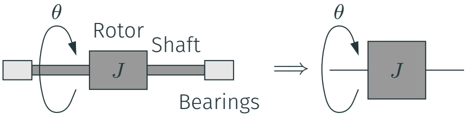
Basic Mechanical Elements: Inertia, Stiffness, Damping, and Forcing

Inertia Elements

- For **Rigid bodies**, inertia properties can be considered point like; therefore, inertia features corresponding to either translatory or rotary motion are naturally **lumped**.
- **Inertia** is represented by **mass** (usually denoted by m) is translatory motion (a) and **mechanical (mass) moment of inertia** (generally symbolized by J) is rotary motion (b).

(a)



(b)

Inertia Elements

- The inertia (mass) is defined as the change in force (torque) required to make a unit change in acceleration (angular acceleration).

$$\text{inertia (mass)} = \frac{\text{change in force}}{\text{change in acceleration}} \; \frac{\text{N}}{\text{m/s}^2} \text{ or kg}$$

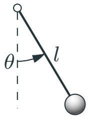
$$\text{inertia (moment of inertia)} = \frac{\text{change in torque}}{\text{change in angular acceleration}} \; \frac{\text{N-m}}{\text{rad/s}^2} \text{ or kg-m}^2$$

Mechanical moment of inertia

The mechanical moment of inertia of a body of mass m rotating about and axis is defined by

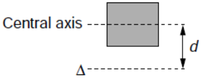
$$J = \int_m r^2 dm,$$

where r is the distance from the reference axis to the mass element dm.



$$J = ml^2$$

If the rotation axis of a homogeneous rigid body does not coincide with the body's axis of symmetry, but is parallel to it at a distance d, then the mass moment of inertia about the rotation axis is given by the **parallel-axis theorem**,

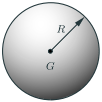


$$J_\Delta = J + md^2,$$

where J is the mechanical moment of inertia of the body about its centroid axis.

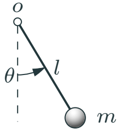
Mechanical moments of inertia of common elements

Sphere



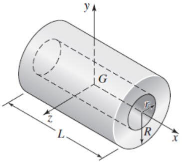
$$J_G = \frac{2}{5} m R^2$$

Mass rotating about point o



$$J_o = m l^2$$

Hollow cylinder

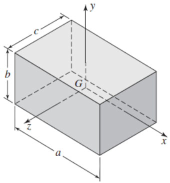


$$J_x = \frac{1}{2} m \left(R^2 + r^2\right)$$

$$J_y = J_z = \frac{1}{12} m \left(3R^2 + 3r^2 + L^2\right)$$

for the proof of J_y see
http://dynref.engr.illinois.edu/rem.html

Rectangular prism



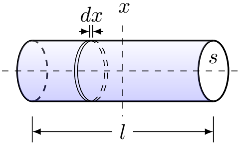
$$J_x = \frac{1}{12} m \left(b^2 + c^2\right)$$

Mechanical moments of inertia: Example

The moment of inertia of a thin rod (radian r of rod is very small compared to the length of the rod.) with constant cross-section s and density ρ and with length l about a perpendicular axis through its center of mass is determined by integration. Align the x-axis with the rod and locate the origin its center of mass at the center of the rod.

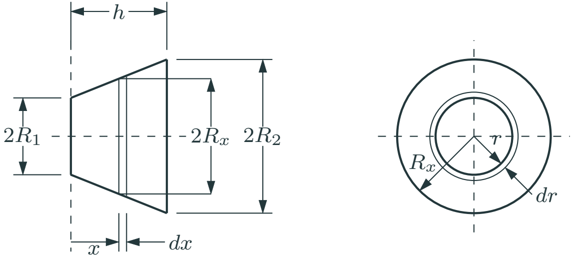
Solution Here, x is a distance from the reference axis to the dm, $dm = \rho s dx$, and $dV = sdx$. Note: $s = \pi r^2$, where $r \approx 0$.

$$
\begin{aligned}
J &= \int_m x^2 dm = \rho \int_V x^2 dV \\
&= \rho s \int_{-l/2}^{l/2} x^2 dx \\
&= \rho s \left. \frac{x^3}{3} \right|_{-l/2}^{l/2} = \frac{\rho s}{3} \left(\frac{l^3}{8} + \frac{l^3}{8} \right) = \frac{1}{12} m l^2
\end{aligned}
$$



If the rotating axis is at the one end, the moment of inertia is $J = \frac{ml^2}{3}$

Mechanical moments of inertia: Example



Calculate the mass moment of inertia about the centroidal (symmetry) axis of the right circular cone frustum in Figure above in side view and defined by R_1, R_2 and h. Use the obtained result to also calculate the mass moment of inertia of a cylinder, both about its centroidal axis and about a parallel axis that is offset at a distance $d = 2R_2$ from the centroidal axis. (the mass density is ρ)

Mechanical moments of inertia: Example

Solution:

The mechanical moment of inertia is expressed as

$$J = \int r^2 dm = \int_V r^2 \rho dV = \rho \int_0^h \left(\int_A r^2 dA \right) dx$$

From the Figure above, the area of an elementary circular strip of width dr and inner radius r is

$$dA = 2\pi r dr$$

The mass moment of inertia of the cone frustum becomes

$$J = 2\pi\rho \int_0^h \left(\int_0^{R_x} r^3 dr \right) dx = \frac{\pi\rho}{2} \int_0^h R_x^4 dx$$

The variable external radius, can be calculated as

$$R_x = R_1 + \frac{R_2 - R_1}{h} x$$

Mechanical moments of inertia: Example

Matlab code

```
syms R1 R2 h x rho r

Rx = R1 + (R2 - R1) * x / h;

J = (2*pi*rho*int(int(r^3, r, 0, Rx), x, 0, h))
```

This will return

$$J = \frac{\pi \rho h}{10} \left(R_1^4 + R_1^3 R_2 + R_1^2 R_2^2 + R_1 R_2^3 + R_2^4 \right)$$

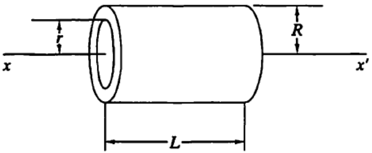
When $R_1 = R_2 = R$, the cone frustum becomes a cylinder the result simplies to

$$J = \frac{\pi \rho h}{10} \left(5R^4 \right) = \frac{1}{2} \pi \rho h R^4 = \frac{1}{2} m R^2$$

Mechanical moments of inertia: Example

The cylinder's mass moment of inertia about an axis situated at $d = 2R_2$ from its centroidal axis is found from the above equation by means of the parallel-axis theorem as

$$J = \frac{1}{2}mR^2 + m(2R)^2 = \frac{9}{2}mR^2$$



Calculate the moment of inertia about axis xx' of the hollow cylinder shown in Fig.

Solution: The moment of inertia about axis xx' of the solid cylinder of radius R is

$$J_R = \frac{1}{2} m_1 R^2, \text{ where } m_1 = \pi R^2 L \rho$$

The moment of inertia about axis xx' of the solid cylinder of radius r is

$$J_r = \frac{1}{2} m_2 r^2, \text{ where } m_2 = \pi r^2 L \rho$$

Mechanical moments of inertia: Example

Then the moment of inertia about axis xx' of the hollow cylinder shown in Fig. is

$$\begin{aligned} J &= J_R - J_r = \frac{1}{2}m_1R^2 - \frac{1}{2}m_2r^2 = \frac{1}{2}\left[(\pi R^2 L\rho)R^2 - (\pi r^2 L\rho)r^2\right] \\ &= \frac{1}{2}\pi L\rho(R^4 - r^4) = \frac{1}{2}\pi L\rho(R^2 + r^2)(R^2 - r^2) \end{aligned}$$

The mass of the hollow cylinder is

$$m = \pi(R^2 - r^2)L\rho$$

Hence,

$$\begin{aligned} J &= \frac{1}{2}(R^2 + r^2)\pi(R^2 - r^2)L\rho \\ &= \frac{1}{2}m(R^2 + r^2) \end{aligned}$$

Translational Motion

Newton's Laws

- A **Particle** is a mass of negligible dimensions. We can consider a body to be a particle if its dimensions are irrelevant for specifying its position and the forces acting on it. Ex. we don't need to know the size of an satellite to describe its orbital path.
- **Newton's first law** states that a particle originally at rest, or moving in a straight line with a constant speed, will remain that way as long as it is not acted upon by an unbalanced external force.
- **Newton's second law** states that the acceleration of a mass particle is proportional to the vector resultant force acting on it and is in the direction of this force.

$$\Sigma F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = m\ddot{x}$$

- **Newton's third law** states that the forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and collinear.

Mechanical Energy

The force f can be a function of displacement x,

$$m\frac{dv}{dt} = f(x)$$

$$m\frac{dv}{dt}vdt = f(x)vdt \qquad \Longrightarrow \qquad mvdv = f(x)\frac{dx}{dt}dt = f(x)dx$$

Integrate both sides, we have

$$\int mvdv = \frac{mv^2}{2} = \int f(x)dx + C$$

- Since work is force times displacement, the integral on the right represents the total work done on the mass by the force $f(x)$. The term $\dfrac{mv^2}{2}$ is called **kinetic energy (KE)**
- If the work done by $f(x)$ is independent of the path and depends only on the end points, the force $f(x)$ is derivable from a function $V(x)$ as follows:

$$f(x) = -\frac{dV}{dx}$$

The negative sign provides the convention that work done against a force field.

Mechanical Energy

- The force $f(x)$ is called **conservative force**. If we integrate both sides, we obtain

$$V(x) = \int dV = -\int f(x)dx$$

$$\frac{mv^2}{2} + V(x) = C$$

  This equation shows that $V(x)$ has the same units as kinetic energy. $V(x)$ is called the **potential energy (PE) function**.
- The equation states that the sum of the kinetic and potential energies must be constant, if no force other than the conservative force is applied.
- If v and x have the values v_0 and x_0 at the time t_0, then

$$\frac{mv_0^2}{2} + V(x_0) = C$$

  Hence,

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} + V(x) - V(x_0) = 0$$

Mechanical Energy

- The result is the conservative of energy as

$$\Delta\text{KE} + \Delta\text{PE} = 0,$$

  where the change in kinetic energy is ΔKE = $m(v^2 - v_0^2)/2$ and the change in potential energy is ΔPE = $V(x) - V(x_0)$.
- Gravity is an example of a conservative force, for which $f = -mg$. The gravity force is conservative because the work done lifting an object depends only on the change in height and not on the path taken. If x represents vertical displacement,

$$\begin{aligned} V(x) &= mgx \\ \frac{mv^2}{2} + mgx &= C \\ \frac{mv^2}{2} - \frac{mv_0^2}{2} + mg(x - x_0) &= 0 \end{aligned}$$

Speed of a Falling Object

An object with a mass of $m = 2$ kg drops from a height of 30 m above the ground. Determine its speed after it drops 20 m to a platform that is 10 m above the ground.

Solution: The distance from the ground is $x_0 = 30$ m and $x = 10$ m at the platform. Since $v_0 = 0$, we have

$$\frac{m}{2}(v^2 - 0) + mg(10 - 30) = 0 \qquad \Rightarrow \qquad v^2 = 40g \qquad (g = 32.2 \text{ m/sec}^2)$$

We obtain $v = \sqrt{644} = 25.4$ m/sec. This is the speed of the object when it reaches the platform.
Note that if we had chosen to measure x from the platform instead of the ground, then $v_0 = 0$ m/s, $x_0 = 20$ m, and $x = 0$ mat the platform. We have

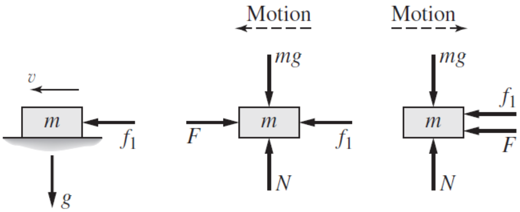
$$\frac{m}{2}(v^2 - 0) + mg(0 - 20) = 0 \qquad \Rightarrow \qquad v^2 = 40g$$

Non-conservative force

The dry friction force is non-conservative because the work done by the force depends on the path taken. The dry friction force F is directly proportional to the force N normal to the frictional surface. Thus $F = \mu N$. The proportionality constant is μ, the **coefficient of friction**.

- The dry friction force that exists before motion begins is called **static friction** (sometimes shortened to **stiction**). The static friction coefficient have the value μ_s.
- The friction after motion begin is the **dynamic friction**. The dynamic friction coefficient μ_d describes the friction after motion begins. The dynamic friction is also called **sliding** friction, **kinetic** friction, or **Coulomb** friction.
- In general, $\mu_s > \mu_d$, which explains why it is more difficult to start an object sliding than keep it moving.
- Normally μ is refer to μ_d.
- Coulomb friction cannot be derived from a potential energy function, the conservation of mechanical energy principle does not apply.
- The friction force dissipates the energy as heat, and thus mechanical energy, which consists of kinetic plus potential energy, is not conserved.
- The total energy is conserved.

Motion with Friction



The free body diagrams above have two cases: $v > 0$ and $v < 0$. The normal force N is the weight mg. Thus the friction force F is μN, or $F = \mu mg$. If $v > 0$, the equation of motion is

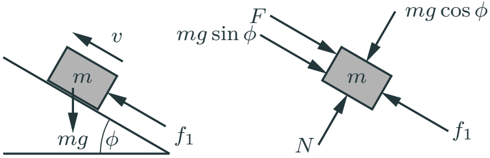
$$m\dot{v} = f_1 - \mu mg, \qquad v > 0$$

Dry friction always opposes the motion. For $v < 0$,

$$m\dot{v} = f_1 + \mu mg, \qquad v < 0$$

Motion with Friction on an Inclined Plane

For the mass $m = 2$ kg, $\phi = 30°$, $v(0) = 3$ m/s, and $\mu = 0.5$. Determine whether the mass comes to rest if (a) $f_1 = 50$ N and (b) $f_1 = 5$ N.



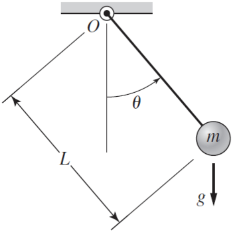
Solution: Because the velocity is initially positive [$v(0) = 3$], we use equation

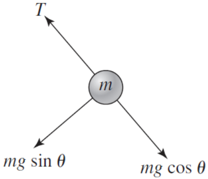
$$\begin{aligned} m\dot{v} &= f_1 - mg\sin\phi - \mu mg\cos\phi \qquad v > 0 \\ 2\dot{v} &= f_1 - 18.3 \end{aligned}$$

(a) $f_1 = 50$ N and thus $\dot{v} = (50 - 18.3)/2 = 15.85$ and the acceleration is positive. Thus, because $v(0) > 0$, the speed is always positive for $t \geq 0$ and the mass never comes to rest.

(b) $f_1 = 5$ N , $\dot{v} = (5 - 18.3)/2 = -6.65$, and thus the mass is decelerating. Because $v(t) = -6.65t + 3$, the speed becomes zero at $t = 3/6.65 = 0.45$s.

Rotation About a Fixed Axis: Pendulum





From $\Sigma M = J\alpha$, we have

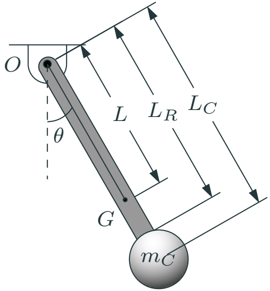
$$
\begin{aligned}
mL^2\ddot{\theta} &= -mgL\sin\theta \\
L\ddot{\theta} &= -g\sin\theta \\
\ddot{\theta} &= -\frac{g}{L}\sin\theta
\end{aligned}
$$

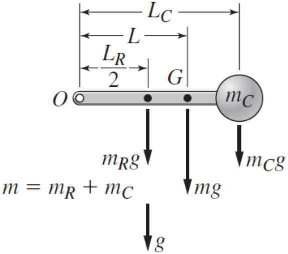
If $\theta << 1$, we have

$$\ddot{\theta} = -\frac{g}{L}\theta$$

Rotation About a Fixed Axis: Example

The pendulum shown in Figure below consists of a concentrated mass m_C (the bob) a distance L_C from point O, attached to a rod of length L_R and inertia J_{RG} about its mass center. (a) Obtain its equation of motion. (b) Discuss the case where the rod's mass m_R is small compared to the concentrate mass. (c) Determine the equation of motion for small angles θ.





Rotation About a Fixed Axis: Example

- The inertia of the concentrated mass m_C about point O is

$$J_m = m_C L_C^2$$

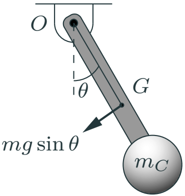
- From the parallel axis theorem, the rod's inertia about point O is

$$J_{RO} = J_{RG} + m_R \left(\frac{L_R}{2}\right)^2$$

Thus the entire pendulum's inertia about point O is

$$J_O = J_{RO} + m_C L_C^2 = J_{RG} + m_R \left(\frac{L_R}{2}\right)^2 + m_C L_C^2$$

Rotation About a Fixed Axis: Example



The moment equation is

$$\Sigma M = J\dot{\omega}$$

Then

$$J_O\ddot{\theta} = -mgL\sin\theta$$

The distance L between point O and the mass center G of the entire pendulum is not given, but can be calculated as

$$mgL = (m_C)gL_C + (m_R)g\frac{L_R}{2} \quad \text{where } m = m_C + m_R$$

Solve for L to obtain

$$L = \frac{m_C L_C + m_R(L_R/2)}{m_C + m_R}$$

Rotation About a Fixed Axis: Example

(b) IF we neglect the rod's mass m_R compared to the concentrated mass m_C, then we can take $m_R = J_{RG} = 0$, $m = m_C$, $L = L_C$ and $J_O = mL^2$. In this case, the equation of motion reduced to

$$mL^2\ddot{\theta} = -mgL\sin\theta$$

$$L\ddot{\theta} + g\sin\theta = 0$$

(c) For small angles, $\sin\theta \approx \theta$ if θ is in radians. Substituting this approximation into above equation gives

$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

Energy and Rotational Motion

The work done by a moment M causing a rotation through an angle θ is

$$W = \int_0^\theta M d\theta$$

From, the dynamic equation

$$M = J\dot{\omega} \Rightarrow M\omega dt = J\frac{d\omega}{dt}\omega dt$$
$$M\frac{d\theta}{dt}dt = J\frac{d\omega}{dt}\omega dt = Md\theta = J\omega d\omega$$

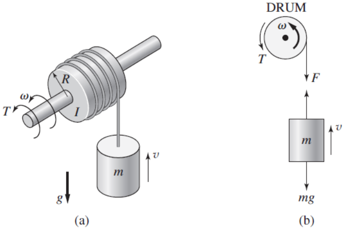
Integrating both sides gives

$$\int_0^\theta J\omega d\omega = \frac{1}{2}J\omega^2 = \int_0^\theta M d\theta$$

It shows that the work done by the moment M produces the kinetic energy of rotation:

$$\text{KE} = \frac{1}{2}J\omega^2$$

Rotation About a Fixed Axis: Example



A motor supplies a torque T to turn a drum of radius R and inertial I about its axis of rotation. The rotating drum lifts a mass m by means of a cable that wraps around the drum. The drum's speed is ω. Discounting the mass of the cable, use the values $m = 40$ kg, $R = 0.2$ m, and $J = 0.8$ kg· m^2. Find the acceleration $\dot{v}$ if the torque $T = 300$ N· m.

Use $J\dot{\omega} = M$ we have

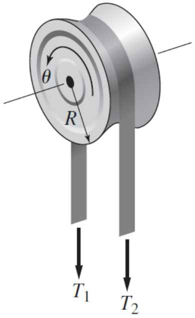
$$0.8\dot{\omega} = 300 - 0.2F$$
$$40\dot{v} = F - 40(9.81)$$

Solve above equation for F and

$$0.8\dot{\omega} = 300 - 8\dot{v} - 8(9.81)$$

Note that $v = R\omega = 0.2\omega$ to obtain $\dot{v} = 0.2\dot{\omega}$. Then we have $\dot{v} = 18.46$ m/s^2.

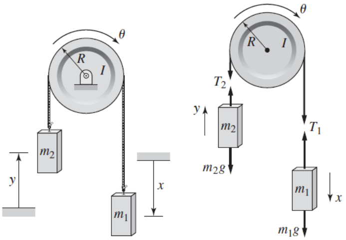
Pulley Dynamics



A pulley of inertial J whose center is fixed to a support. Then we have

$$J\ddot{\theta} = R(T_1 - T_2)$$

Pulley Dynamics : Energy-based analysis



- If pulley inertial is negligible then it is obvious that m_1 will lift m_2 if $m_1 > m_2$. How does a nonnegligible pulley inertial J change the result?
- If the pulley cable is inextensible, the $x = y$ and $\dot{x} = \dot{y}$. If the cable does not slip, then $\dot{\theta} = \dot{x}/R$.
- If the system starts at rest at $x = y = 0$, then the kinetic energy is initially zero. We take the potential energy to be zero at $x = y = 0$.

Thus

$$\text{KE} + \text{PE} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}J\dot{\theta}^2 + m_2 g y - m_1 g x = 0$$

Substituting $y = x$, $\dot{y} = \dot{x}$, and $\dot{\theta} = \dot{x}/R$ into the equation, then

$$\frac{1}{2}\left(m_1 + m_2 + \frac{J}{R^2}\right)\dot{x}^2 + (m_2 - m_1)gx = 0, \qquad \dot{x} = \sqrt{\frac{2(m_1 - m_2)gx}{m_1 + m_2 + J/R^2}}$$

The pulley inertia does decrease the speed with which m_1 lifts m_2.

Pulley Dynamics : Newton's law

It is inconvenient to use an energy-based analysis to compute $x(t)$ or the tensions in the cable. For the free body diagram of the previous slide, Newton's law for the mass m_1 and m_2 give

$$m_1\ddot{x} = m_1 g - T_1, \qquad m_2\ddot{y} = T_2 - m_2 g$$

By using $J\ddot{\theta} = R(T_1 - T_2)$ and $x = y$ then

$$\begin{aligned} T_1 &= m_1 g - m_1\ddot{x} = m_1(g - \ddot{x}) \\ T_2 &= m_2\ddot{y} + m_2 g = m_2(\ddot{y} + g) = m_2(\ddot{x} + g) \end{aligned}$$

Then

$$J\ddot{\theta} = (m_1 - m_2)gR - (m_1 + m_2)R\ddot{x}$$

Since $x = R\theta$, then we have

$$\left(m_1 + m_2 + \frac{J}{R^2}\right)\ddot{x} = (m_1 - m_2)g$$

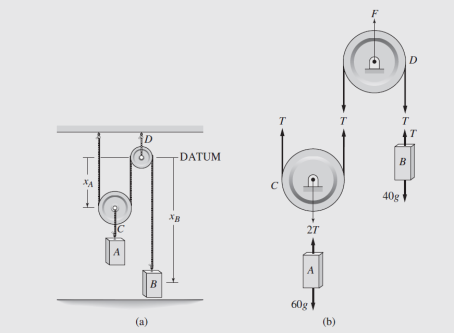
Pulley Dynamics : Newton's law

We can solve it for $\ddot{x}$ and substitute the result to the previous equations to find T_1 and T_2.
To find $\dot{x}$ and $x(t)$, we can do by using direct integration. Let

$$\begin{aligned} A = \ddot{x} &= \frac{(m_1 - m_2)gR^2}{(m_1 + m_2)R^2 + J} \\ \dot{x}(t) &= At + \dot{x}(0) \\ x(t) &= \frac{A}{2}t^2 + \dot{x}(0)t + x(0) \end{aligned}$$

Note to get the solution of $\dot{x}$, we have

$$\left(m_1 + m_2 + \frac{J}{R^2}\right)\ddot{x} = (m_1 - m_2)g \;\Rightarrow\; \left(m_1 + m_2 + \frac{J}{R^2}\right)\frac{d\dot{x}}{dx}\frac{dx}{dt} = (m_1 - m_2)g$$
$$\left(m_1 + m_2 + \frac{J}{R^2}\right)d\dot{x}\dot{x} = (m_1 - m_2)gdx \;\Rightarrow\; \frac{1}{2}\left(m_1 + m_2 + \frac{J}{R^2}\right)\dot{x}^2 = (m_1 - m_2)gx$$



The two masses shown in Fig are released from rest. The mass of block A is 60 kg; the mass of block B is 40 kg. Disregards the masses of the pulleys and rope. Block A is heavier than block B, but will block B rise or fall? Find out by determining the acceleration of block B by (a) using free body diagrams and (b) using conservation of energy.

a) From the free body diagram, we have

$$2T - 60g = 60\ddot{x}_A(*), \qquad 40g - T = 40\ddot{x}_B(**)$$

If B goes down a distance x_B, the A will goes up a distance $x_B/2$. Then $\ddot{x}_B = 2\ddot{x}_A$.

From $(*)$, $T = 30g + 15\ddot{x}_B$ and substitute into $(**)$ then $40g - 30g - 15\ddot{x}_B = 40\ddot{x}_B$.

$$55\ddot{x}_B = 10g$$
$$\ddot{x}_B = \frac{2}{11}g = 1.784 \text{ m/s}^2 (B \text{ is going downward.})$$

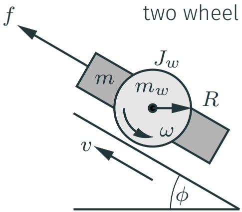
b) Choosing gravitational potential energy to be zero at the datum line, the energy in the system is (we need to find $\ddot{x}_B$.)

$$\frac{1}{2}m_A\dot{x}_A^2 + \frac{1}{2}m_B\dot{x}_B^2 + m_A g x_A - m_B g x_B = k$$
$$m_A\dot{x}_A\ddot{x}_A + m_B\dot{x}_B\ddot{x}_B + m_A g\dot{x}_A - m_B g\dot{x}_B = 0$$

Since $\dot{x}_A = \dot{x}_B/2$, and $\ddot{x}_A = \ddot{x}_B/2$. Hence

$$m_A\frac{\dot{x}_B}{2}\frac{\ddot{x}_B}{2} + m_B\dot{x}_B\ddot{x}_B + m_A g\frac{\dot{x}_B}{2} - m_B g\dot{x}_B = 0$$
$$\left(m_A\frac{\ddot{x}_B}{4} + m_B\ddot{x}_B\right) + \left(m_A\frac{g}{2} - m_B g\right) = 0$$
$$\left(\frac{60}{4} + 40\right)\ddot{x}_B = \left(\frac{60}{2} - 40\right)g, \qquad \ddot{x}_B = 1.784 \text{ m/s}^2$$

Pulley Dynamics :



A tractor pulls a cart up a slope, starting from rest and accelerating to 20 m/s in 15 s. The force in the cable is f, and the body of the cart has a mass m. The cart has two identical wheels, each with radius R, mass m_w, and inertia J_w about the wheel center. The two wheels are coupled with an axle whose mass is negligible. Assume that the wheels do not slip or bounce. Derive an expression for the force f using kinetic energy equivalence.

Solution: Form the assumption of no slip and no bounce means that the wheel rotation is directly coupled to the cart translation. We the wheel rotation θ can be calculated from the cart translation x , because $x = R\theta$ if the wheels do not slip or bounce.

Equivalent Mass and Inertia

The kinetic energy of the system is

$$\mathsf{KE} = \frac{1}{2}mv^2 + \frac{1}{2}\left(2m_w v^2\right) + \frac{1}{2}\left(2J_w\omega^2\right)$$

Because $v = R\omega$, we obtain

$$\begin{aligned}\mathsf{KE} &= \frac{1}{2}\left(m + 2m_w + 2\frac{J_w}{R^2}\right)v^2 \\ &= \frac{1}{2}m_e v^2,\end{aligned}$$

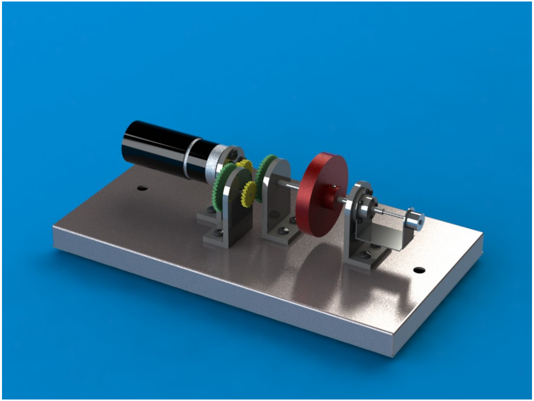
where $m_e = m + 2m_w + 2\frac{J_w}{R^2}$ and from the free body diagram we have

$$m_e\dot{v} = f - (m + 2m_w)g\sin\theta$$

The acceleration is $\dot{v} = 20/15 = 4/3$ m/s^2. We have

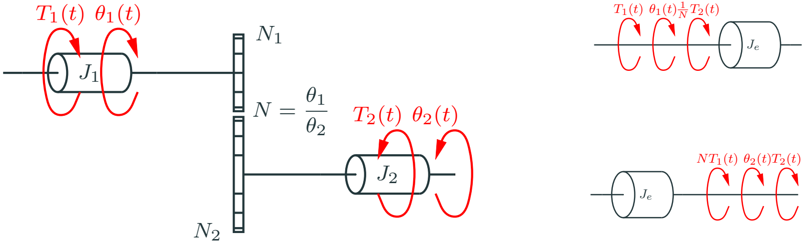
$$f = \frac{4}{3}m_e + (m + 2m_w)g\sin\theta$$

Mechanical Drives



Picture from grabcad.com

Gears transform an input motion, force, or torque into another motion, force, or torque at the output. For example, a gear pair can be used to reduce speed and increase torque.



A pair of **Spur-gear** shown in Fig. The input shaft is connected to a motor that produces a torque T_1 at a speed $\dot{\theta}_1 = \omega_1$, and drive the output shaft. One use of such a system is to increase the effective motor torque. The gear ratio N is defined as the ratio of the input rotation θ_1 to the output rotation θ_2. Thus

$$N = \frac{N_2}{N_1} = \frac{\theta_1}{\theta_2} = \frac{r_2}{r_1} = \frac{\omega_1}{\omega_2} = \frac{T_2}{T_1}$$

Consider the spur gears shown in the previous page. Derive the expression for the equivalent inertia J_e felt on the input shaft.

Solution: Let J_1 and J_2 be the total moments of inertia on the shafts. The kinetic energy of the system is then

$$\begin{aligned}\mathsf{KE} &= \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 \\ &= \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\left[\frac{N_1}{N_2}\right]^2\omega_1^2 \\ &= \frac{1}{2}\left(J_1 + \left[\frac{N_1}{N_2}\right]^2 J_2\right)\omega_1^2 = \frac{1}{2}J_e\omega_1^2\end{aligned}$$

Therefore the equivalent inertia felt on the input shaft is

$$J_e = J_1 + \left[\frac{N_1}{N_2}\right]^2 J_2$$

This mean that the dynamics of the system can be described by the model $J_e\dot{\omega}_1 = T_e$.

For the geared system from the previous example, the inertias is kg.m^2 are $J_1 = 0.1$, for the motor shaft and $J_2 = 0.4$ for the load shaft. The motor speed ω_1 is five times faster than the load speed ω_2, so this device is called a *speed reducer*. Obtain the equation of motion (a) in terms of ω_1 and (b) in terms of ω_2, assuming that the motor torque T_1 and load torque T_2 are given.

Solution: We have $N = N_2/N_1 = \omega_1/\omega_2 = 5$ (not equal 4 due to the load shaft effect).

(a) Referencing both inertias to shaft 1 gives the equivalent inertia (from the previous example).

$$J_e = J_1 + \frac{1}{25} J_2 = 0.116$$

The ω_1 speed is effected by both T_1 and T_2 but the moment felt on shaft 1 due to T_2 is only T_2/N. Then the motion equation is

$$J_e \dot{\omega}_1 = 0.116 \dot{\omega}_1 = T_1 + \frac{T_2}{5}$$

Mechanical Drives: Speed Reducer

(b) Using the kinetic energy of the system as

$$\begin{aligned}\text{KE} &= \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 = \frac{1}{2}J_1(5\omega_2)^2 + \frac{1}{2}J_2\omega_2^2 \\ &= \frac{1}{2}(2.9)\omega_2^2\end{aligned}$$

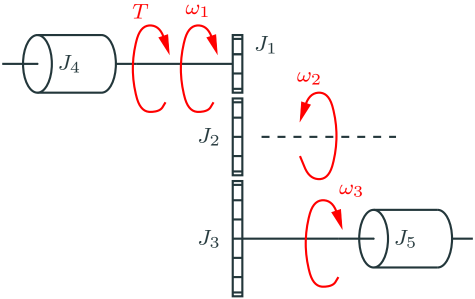
Thus the equivalent inertia referenced to shaft 2 is $J_e = 2.9$, and the equation of motion is

$$2.9\dot{\omega}_2 = T_e = NT_1 + T_2 = 5T_1 + T_2$$

Note that with a speed reducer

- the load speed is slower than the motor speed,
- the effect of the motor torque on the load shaft is increased by a factor equal to the gear ratio N.

Mechanical Drives: A Three-Gear System



Assume that the shaft inertias are small. The remaining inertias in kg·m^2 are $J_1 = 0.005$, $J_2 = 0.01$, $J_3 = 0.02$, $J_4 = 0.04$, and $J_5 = 0.2$. The speed ratios are

$$\frac{\omega_1}{\omega_2} = \frac{3}{2}, \qquad \frac{\omega_2}{\omega_3} = 2$$

Obtain the equation of motion in terms of ω_3. The torque T is given.

Mechanical Drives: A Three-Gear System

Solution: Note that

$$\omega_1 = \left(\frac{3}{2}\right)\omega_2 = \left(\frac{3}{2}\right)2\omega_3 = 3\omega_3$$

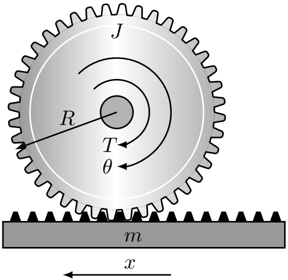
The ratio of the speed $\omega_1/\omega_2 < J_2/J_1$, then the system is a speed reducer. The kinetic energy is

$$\begin{aligned}
\text{KE} &= \frac{1}{2}J_4\omega_1^2 + \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2 + \frac{1}{2}J_5\omega_3^2 \\
&= \frac{1}{2}\left(J_4 + J_1\right)\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}\left(J_3 + J_5\right)\omega_3^2 \\
&= \frac{1}{2}\left(J_4 + J_1\right)9\omega_3^2 + \frac{1}{2}4\omega_3^2 + \frac{1}{2}\left(J_3 + J_5\right)\omega_3^2 \\
&= \frac{1}{2}(0.665)\omega_3^2
\end{aligned}$$

Since $\omega_1 = 3\omega_3$, the torque is increased by a factor of 3. Thus the equation of motion is

$$0.665\dot{\omega}_3 = 3T$$

Mechanical Drives: Rack-and-Pinion



A rack-and-pinion is used to convert rotation into translation. The input shaft rotates through the angle θ as a result of the torque T produced by a motor. The pinion rotates and causes the rack to translate. Derive the expression for the equivalent inertia J_e felt on the input shaft. The mass of the rack is m, the inertia of the pinion is J, and its mean radius is R.

Solution: The kinetic energy of the system is (neglecting the inertia of hte shaft)

$$\text{KE} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2, \qquad x = R\theta$$

We have the expression for KE as

$$\text{KE} = \frac{1}{2}m\left(R\dot{\theta}\right)^2 + \frac{1}{2}J\dot{\theta}^2 = \frac{1}{2}\left(mR^2 + J\right)\dot{\theta}^2$$

Mechanical Drives: Rack-and-Pinion

Thus the equivalent inertia felt on the shaft is

$$J_e = mR^2 + J$$

and the model of the system's dynamics is

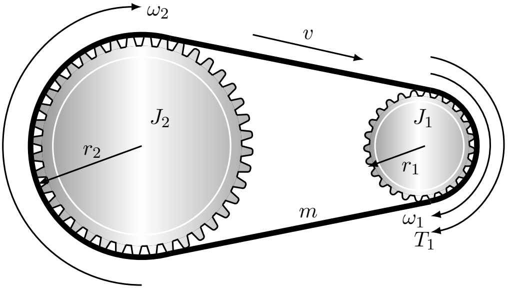
$$J_e \ddot{\theta} = T,$$

which can be expressed in terms of x as

$$J_e \ddot{x} = RT$$

Note: We have no load torque.

Mechanical Drives: Belt Drive



Belt drives and chain drives shown in Figure. The input shaft is connected to a device that produces a torque T_1 at a speed ω_1, and drives the output shaft. The mean sprocket radii are r_1 and r_2 and their inertias are J_1 and J_2. The belt mass is m. Derive the expression for the equivalent inertia J_e felt on the input shaft. **Note:** We have no load torque.

Mechanical Drives: Belt Drive

Solution:

The kinetic energy of the system is

$$\text{KE} = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}mv^2$$

If the belt does not stretch, the translational speed of the belt is $v = r_1\omega_1 = r_2\omega_2$. Thus we can express KE as

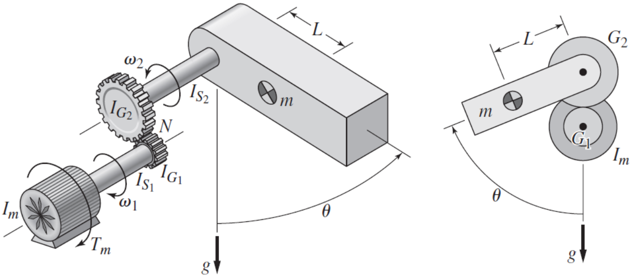
$$\text{KE} = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\left(\frac{r_1\omega_1}{r_2}\right)^2 + \frac{1}{2}m\left(r_1\omega_1\right)^2 = \frac{1}{2}\left[J_1 + J_2\left(\frac{r_1}{r_2}\right)^2 + mr_1^2\right]\omega_1^2$$

Therefore, the equivalent inertia felt on the input shaft is

$$J_e = J_1 + J_2\left(\frac{r_1}{r_2}\right)^2 + mr_1^2$$

This means that the dynamics of the system can be described by the model

$$J_e\dot{\omega}_1 = T_1.$$



A single link of a robot arm is shown in Figure. The arm mass is m and its center of mass is located a distance L from the joint, which is driven by a motor torque T_m through a pair of spur gears. The values of m and L depend on the payload being carried in the hand and thus can be different for each application. The gear ratio is $N = 2$ (the motor shaft has the greater speed). The motor and gear rotation axes are fixed by bearings. To control the motion of the arm we need to have its equation of motion. Obtain this equation in terms of the angle θ. The given values for the motor, shaft, and gear inertias are (Here we use I instead of J.)

$$I_m = 0.05 \text{ kg} \cdot \text{m}^2 \qquad I_{G_1} = 0.025 \text{ kg} \cdot \text{m}^2 \qquad I_{S_1} = 0.01 \text{kg} \cdot \text{m}^2$$
$$I_{G_2} = 0.1 \text{ kg} \cdot \text{m}^2 \qquad I_{S_2} = 0.02 \text{ kg} \cdot \text{m}^2$$

Solution: Here we need to model the system as a single inertia rotating about the motor shaft with a speed ω_1. To find the equivalent inertia about this shaft we first obtain the expression for the kinetic energy of the total system and express it in terms of the shaft speed ω_1. Note that the mass m is translating with a speed $L\omega_2$ $(L\theta = x)$

$$\text{KE} = \frac{1}{2}\left(I_m + I_{S_1} + I_{G_1}\right)\omega_1^2 + \frac{1}{2}\left(I_{S_2} + I_{G_2}\right)\omega_2^2 + \frac{1}{2}m\left(L\omega_2\right)^2 .$$

Since $\omega_2 = \omega_1/N = \omega_1/2$, thus

$$\text{KE} = \frac{1}{2}\left[I_m + I_{S_1} + I_{G_1} + \frac{1}{4}\left(I_{S_2} + I_{G_2} + mL^2\right)\right]\omega_1^2$$

Therefore, the equivalent inertia referenced to the motor shaft is

$$I_e = I_m + I_{S_1} + I_{G_1} + \frac{1}{4}\left(I_{S_2} + I_{G_2} + mL^2\right) = 0.115 + 0.25mL^2$$

Mechanical Drives: Robot-Arm-Link

The equation of motion for this equivalent inertia can be obtained by noting that the gravity moment $mgL\sin\theta$, which acts on shaft 2, is also felt on the motor shaft, but reduced by a factor of N due to the gear pair. (Load torque makes speed of system reduce.) Thus

$$I_e\dot{\omega}_1 = T_m - \frac{1}{N}mgL\sin\theta$$

But $\omega_1 = N\omega_2 = N\dot{\theta}$, thus

$$\begin{aligned} I_e N\ddot{\theta} &= T_m - \frac{1}{N}mgL\sin\theta \\ \left(0.23 + 0.5mL^2\right)\ddot{\theta} &= T_m - 4.9mL\sin\theta \end{aligned}$$

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