Lecture 4: Transfer Functions and Electrical System

Dr.-Ing. Sudchai Boonto, Assistant Professor

Department of Control System and Instrument Engineering, KMUTT

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Transfer function

Transfer Function

A **Transfer Function** is the ratio of the output of a system to the input of a system, in the Laplace domain considering its initial conditions and equilibrium point to be zero.

A monic nth-order, linear, time-invariant differential equation,

$$\frac{d^n}{dt^n}y(t)+a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t)+\cdots+a_1\frac{d}{dt}y(t)+a_0y(t)=$$
$$c_m\frac{d^m}{dt^m}u(t)+c_{m-1}\frac{d^{m-1}}{dt^{m-1}}u(t)+\cdots+c_1\frac{d}{dt}u(t)+c_0u(t), \qquad n\geq m$$

Taking the Laplace transform to the both sides and set all initial conditions to be zero, the system becomes

$$\left(s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0\right)Y(s)=\left(c_ms^m+c_{m-1}s^{m-1}+\cdots+c_1s+c_0\right)U(s)$$
$$\frac{Y(s)}{U(s)}=\frac{c_ms^m+c_{m-1}s^{m-1}+\cdots+c_1s+c_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=G(s)$$

Transfer function

The concept of the transfer function is limited to linear, time-invariant (LTI) differential-equation systems.

- The transfer function of a system is a mathematical model of the system, in that it is an operational method of expressing the differential equation that relates the output variable to the input variable.
- The transfer function is a property of a system itself, unrelated to the magnitude and nature of the input or driving function.
- The transfer function includes the units necessary to relate the input to the output; however, it does not provide any information concerning the physical structure of the system. (The transfer function of many physically different systems can be identical.)
- If the transfunction of a system is known, the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system.

Transfer function: Example

Find the transfer function of the system represented by

$$\frac{d}{dt}y(t) + 2y(t) = u(t)$$

Taking the Laplace transform of both sides, and set all initial conditions to be zero, we have

$$sY(s) + 2Y(s) = U(s)$$

The tranfer function, $G(s)$ is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+2}$$

MATLAB Code I

```
num = [1];
den = [1 2];

sys = tf(num,den)
```

MATLAB Code II

```
% more convenience
s = tf('s');

sys = 1/(s+2);
```

System Response from the Transfer Function

Find the response, $y(t)$ to an $\mathbb{1}(t)$ input of a system $G(s) = 1/(s+2)$.
The Laplace transform of $\mathbb{1}(t)$ is $1/s$. Then

$$Y(s) = G(s)U(s) = \frac{1}{(s+2)}\frac{1}{s}$$

Using the partial fraction expansion, we get

$$Y(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Finally, taking the inverse Laplace transform of each term yields

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

We can use a MATLAB command `step(G)` to get the response of the system to the unit-step input, which gives the same result as directly time domain calculating with Matlab.

System Response from the Transfer Function

MATLAB Code I

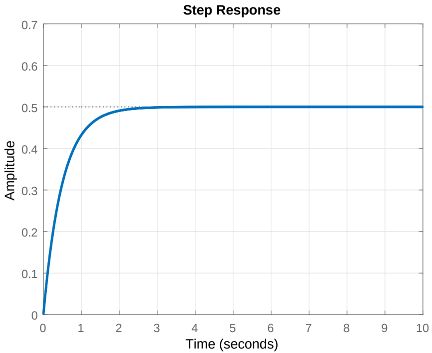
```
num = [1];
den = [1 2];

G = tf(num,den);
step(G,10);
```

MATLAB Code II

```
% more convenience
s = tf('s')
G = (1/(s+2))

step(G, 10)
```



Transfer function: Electrical Network

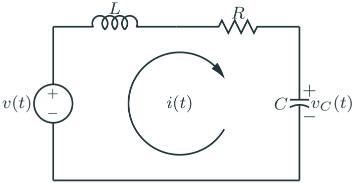
In feedback control system design, we use electrical networks to build analog controllers, analog filters, etc. These are RLC circuits and operational amplifier circuits.

Component	Voltage-current	Current-voltage	Impedance $Z(s) = V(s)/I(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{d}{dt}v(t)$	$\frac{1}{Cs}$
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	R
 Inductor	$v(t) = L\frac{d}{dt}i(t)$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	Ls

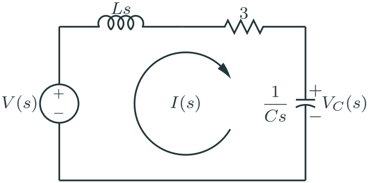
Note: All initial conditions are zero.

Transfer function: RLC series circuit

Find $v_c(t)$.



(a)



(b)

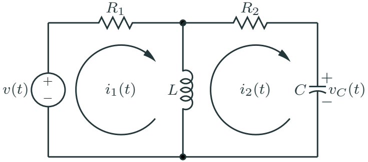
The Laplace transform of the mesh voltage with all zero conditions, is

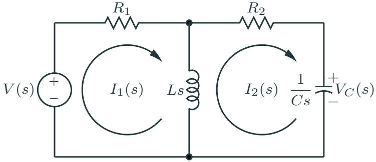
$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s), \qquad V_C(s) = \frac{1}{Cs} I(s)$$

$$\left(Ls + R + \frac{1}{Cs}\right) Cs V_C(s) = V(s)$$

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Transfer function: RLC circuit





$$\begin{bmatrix} R_1 + Ls & -Ls \\ -Ls & R_2 + Ls + \frac{1}{Cs} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V(s) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \frac{1}{(R_1 + Ls)(R_2 + Ls + \frac{1}{Cs}) - L^2 s^2} \begin{bmatrix} R_2 + Ls + \frac{1}{Cs} & Ls \\ Ls & R_1 + Ls \end{bmatrix} \begin{bmatrix} V(s) \\ 0 \end{bmatrix}$$

$$I_2(s) = \frac{LsV(s)}{(R_1 + Ls)(R_2 + Ls + \frac{1}{Cs}) - L^2 s^2}$$

$$\frac{I_2(s)}{V(s)} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1 R_2 C + L)s + R_1}$$

Transfer function: RLC circuit

MATLAB Code

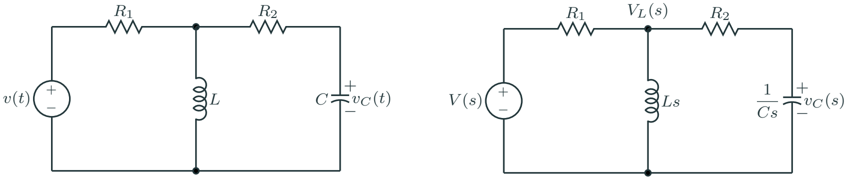
```
syms R1 R2 V L C s

A = [R1 + L*s, -L*s; -L*s, R2+L*s+1/(C*s)];
b = [V; 0];
I = A\b;
I2 = I(2, 1);

% Find the transfer function I2/V
sys = I2/V
```

$$\text{sys} = \frac{CLs^2}{R_1 + Ls + CLR_1 s^2 + CLR_2 s^2 + CR_1 R_2 s}$$

Transfer function



$$\frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0$$

$$\frac{V_C(s) - V_L(s)}{R_2} + CsV_C(s) = 0$$

$$V_L(s) = (R_2Cs + 1)V_C(s)$$

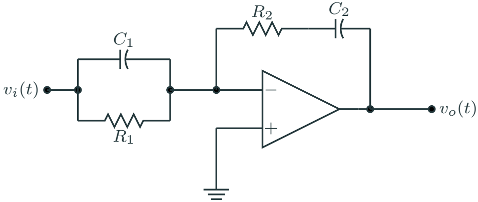
Substituting $V_L(s)$ to the first equation, we have

$$R_2Ls(R_2Cs + 1)V_C(s) + R_1R_2(R_2Cs + 1)V_C(s) + R_1R_2LCs^2V_C(s) = R_2LsV(s)$$

$$\frac{V_C(s)}{V(s)} = \frac{Ls}{(R_1 + R_2)LCs^2 + (L + R_1R_2C)s + R_1}$$

Transfer function: Inverting Amplifier

Instead of consider R, L and C separately, each composition RL, RC, LC, and RLC could be considered in terms of impedance as follow.



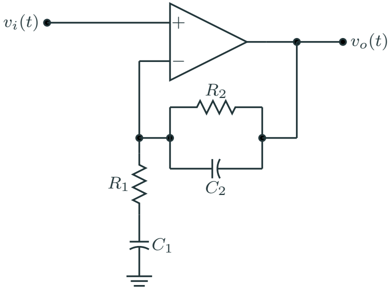
$$Z_1(s) = C_1||R_1 = \frac{R_1}{R_1C_1s+1}$$

$$Z_2(s) = R_2 + \frac{1}{Cs}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_2 + \frac{1}{Cs}}{\frac{R_1}{R_1C_1s+1}} = -\frac{(R_1C_1s+1)(R_2C_2s+1)}{R_1C_2s}$$

This circuit is called a PID controller.

Transfer function: Noninverting Amplifier

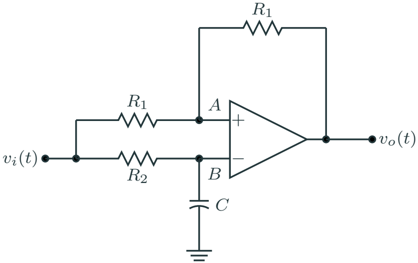


$$Z_1(s) = R_1 + \frac{1}{C_1 s}$$

$$Z_2(s) = R_2 || C_2 = \frac{R_2}{R_2 C_2 s + 1}$$

$$\frac{V_o(s)}{V_i(s)} = 1 + \frac{Z_2(s)}{Z_1(s)} = 1 + \frac{\frac{R_2}{R_2 C_2 s + 1}}{\frac{R_1 C_1 s + 1}{C_1 s}} = 1 + \frac{R_2 C_1 s}{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}$$

$$= \frac{R_1 R_2 C_1 C_2 s^2 + (R_1 C_1 + R_2 C_1 + R_2 C_2) s + 1}{(R_1 R_2 C_1 C_2 s^2 + (R_1 C_1 + R_2 C_2) s + 1}$$



The voltage at point A is

$$\frac{v_A - v_i}{R_1} + \frac{v_A - v_o}{R_1} = 0$$
$$v_A = \frac{1}{2}\left[v_i + v_o\right]$$
$$V_A(s) = \frac{1}{2}\left[V_i(s) + V_o(s)\right]$$

The voltage at point B is, using s-domain and voltage divier,

$$V_B(s) = \frac{1/Cs}{R_2 + 1/Cs}V_i(s) = \frac{1}{R_2Cs+1}V_i(s) = V_A(s)$$
$$\frac{1}{R_2Cs+1}V_i(s) = \frac{1}{2}\left[V_i(s) + V_o(s)\right]$$
$$V_o(s) + V_i(s) = \frac{2}{R_2Cs+1}V_i(s)$$
$$V_o(s) = \left[\frac{2}{R_2Cs+1} - 1\right]V_i(s) = -\frac{R_2Cs-1}{R_2Cs+1}V_i(s) = -\frac{s - \frac{1}{R_2C}}{s + \frac{1}{R_2C}}V_i(s)$$

Reference

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3. Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini, "*Feedback Control of Dyanmic Systems*",4^{th} edition, Prentice Hall, 2002
4. Katsuhiko Ogata, "*System Dynamics*," 4^{th} edition, Prentice Hall, 2004