Lecture 3: Laplace Transform and Its Applications

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Outline

- Motivation
- The Laplace Transform
- The Laplace Transform Properties
- Application to Zero-input and Zero-State Response
- Analysis of Electrical Networks

Motivation

The **Laplace Transform** convert *integral* and *differential* equations into *algebraic* equations.

It can applies to

- general signal, not just sinusoids
- handles transient conditions

It can be used to analyze

- Linear Constant Coefficient Ordinary Differential Equation (LCCODE) or LTI system
- complicated RLC circuits with sources
- complicated systems with integrators, differentiators, gains

The Unilateral Laplace transform

We will be interested in signals defined for $t > 0$.

Laplace Transform. Let $f(t), t > 0$, be a given signal (function). The *Unilateral Laplace transform* of a signal (function) $f(t)$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt,$$

for those $s \in \mathbb{C}$ for which the integral exists.

- F is a complex-values function of complex numbers
- s is called the *(complex) frequency variable*, with units $\sec^{-1}$; t is called the *time variable* (in sec); st is unitless.
- For convenience, we will call the unilateral laplace transform as the laplace transform.

The Laplace transform: Example

Exponential function: $f(t) = e^t$

$$F(s) = \int_0^\infty e^t e^{-st} dt = \int_0^\infty e^{(1-s)t} dt = \frac{1}{1-s} e^{(1-s)t} \Big|_0^\infty = \frac{1}{s-1}$$

provide we can say $e^{(1-s)t} \to 0$ as $t \to \infty$, which is true for $\mathrm{Re}\, s > 1$:

$$|e^{(1-s)t}| = \underbrace{|e^{-j(\mathrm{Im}\, s)t}|}_{=1} |e^{(1-\mathrm{Re}\, s)t}| = e^{(1-\mathrm{Re}\, s)t}$$

- the *integral* defining $F(s)$ exists for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$. This condition is called **region of convergence (ROC)** of $F(s)$.
- however the resulting *formula* for $F(s)$ makes sense for all $s \in \mathbb{C}$ excepts $s = 1$.

The Laplace transform: Example

Constant or **unit step function**: $f(t) = \mathbb{1}(t)$ (for $t \geq 0$)

$$F(s) = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

provided we can say $e^{-st} \to 0$ as $t \to \infty$, which is true for $\operatorname{Re} s > 0$ since

$$|e^{-st}| = |e^{-j(\operatorname{Im} s)t}||e^{-(\operatorname{Re} s)t}| = e^{-(\operatorname{Re} s)t}$$

- the *integral* defining $F(s)$ makes sense for all s with $\operatorname{Re} s > 0$.
- however the resulting *formula* for $F(s)$ makes sense for all s except $s = 0$.

The Laplace transform: Example

Sinusoid : first express $f(t) = \cos\omega t$ as

$$f(t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}$$

now we can find F as

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}\left(\frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}\right) dt \\ &= \frac{1}{2}\int_0^\infty e^{(-s+j\omega)t} dt + \frac{1}{2}\int_0^\infty e^{(-s-j\omega)t} dt \\ &= \frac{1}{2}\frac{1}{s-j\omega} + \frac{1}{2}\frac{1}{s+j\omega} \\ &= \frac{s}{s^2+\omega^2} \end{aligned}$$

(valid for $\mathrm{Re}\, s > 0$; final formula for $s \neq \pm j\omega$)

The Laplace transform: Example

Powers of t: $f(t) = t^n, \ (n \geq 1)$

$$F(s) = \int_0^\infty t^n e^{-st} dt = t^n \left(\frac{-e^{-st}}{s} \right) \Bigg|_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt$$
$$= \frac{n}{s} \mathcal{L}(t^{n-1})$$

provided $t^n e^{-st} \to 0$ if $t \to \infty$, which is true for $\mathrm{Re}\, s > 0$. Applying the formular recursively, we obtain

$$F(s) = \frac{n!}{s^{n+1}}$$

valid for $\mathrm{Re}\, s > 0$; final formula exists for all $s \neq 0$.

The Laplace transform: Impulses

If $f(t)$ contains impulses at $t = 0$ we choose to *include* them in the integral defining $F(s)$:

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

example: impulse function, $f(t) = \delta(t)$

$$F(s) = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = \left. e^{-st}\right|_{t=0} = 1 \text{ sampling property}$$

Similarly for $f(t) = \delta^{(k)}(t)$ we have

$$F(s) = \int_{0^-}^{\infty} \delta^{(k)}(t)e^{-st}dt = (-1)^k \left. \frac{d^k}{dt^k}e^{-st}\right|_{t=0} = \left. s^k e^{-st}\right|_{t=0} = s^k$$

The Laplace transform: Multiplication by t

Let $f(t)$ be a signal and define

$$g(t) = tf(t) \quad \text{then we have} \quad G(s) = -\frac{d}{ds}F(s)$$

To verify formula, just differentiate both sides of

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

with respect to s to get

$$\begin{aligned} \frac{d}{ds}F(s) &= \int_0^\infty (-t)e^{-st} f(t) dt = \int_0^\infty (-t) f(t) e^{-st} dt \\ &= -\int_0^\infty t f(t) e^{-st} dt = -G(s) \end{aligned}$$

The Laplace transform: Multiplication by t

Examples:

- $f(t) = e^{-t},\ g(t) = te^{-t}$

$$\mathcal{L}\left\{te^{-t}\right\} = -\frac{d}{ds}\frac{1}{s+1} = \frac{1}{(s+1)^2}$$

- $f(t) = te^{-t},\ g(t) = t^2e^{-t}$

$$\mathcal{L}\left\{t^2e^{-t}\right\} = -\frac{d}{ds}\frac{1}{(s+1)^2} = \frac{2}{(s+1)^3}$$

- in general

$$\mathcal{L}\left\{t^k e^{-\lambda t}\right\} = \frac{k!}{(s+\lambda)^{k+1}}$$

Inverse Laplace transform

In principle we can recover $f(t)$ from $F(s)$ via

Inverse Laplace Transform.

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

where σ is large enough that $F(s)$ is defined for $\mathrm{Re}\, s \geq \sigma$.

In practical, no one uses this formula!.

Inverse Laplace Transform

Finding the inverse Laplace transform by using the standard formula

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

is difficult and tedious.

- We can find the inverse transforms from the transform table.
- All we need is to express $F(s)$ as a sum of simpler functions of the forms listed in the Laplace transform table.
- Most of the transforms $F(s)$ of practical interest are rational functions: that is ratios of polynomials in s.
- Such functions can be expressed as a sum of simpler functions by using partial fraction expansion.

Inverse Laplace Transform

Example: Find the inverse Laplace transform of $\dfrac{7s-6}{s^2-s-6}$.

$$F(s) = \frac{7s-6}{(s+2)(s-3)} = \frac{k_1}{s+2} + \frac{k_2}{s-3}$$

Using a "cover up" method:

$$k_1 = \left.\frac{7s-6}{s-3}\right|_{s=-2} = \frac{-14-6}{-2-3} = 4$$

$$k_2 = \left.\frac{7s-6}{s+2}\right|_{s=3} = \frac{21-6}{3+2} = 3$$

Therefore

$$F(s) = \frac{7s-6}{(s+2)(s-3)} = \frac{4}{s+2} + \frac{3}{s-3}$$

Inverse Laplace Transform

Using the table of Laplace transforms, we obtain

$$f(t) = \mathcal{L}^{-1}\left\{\frac{4}{s+2} + \frac{3}{s-3}\right\}$$
$$= (4e^{-2t} + 3e^{3t}), \qquad t \geq 0.$$

Example: Find the inverse Laplace transform of $F(s) = \dfrac{2s^2+5}{s^2+3s+2}$.
$F(s)$ is an improper function with $m = n$. In such case we can express $F(s)$ as a sum of the coefficient c_n (the coefficient of the highest power in the numerator) plus partial fractions corresponding to the denumerator.

$$F(s) = \frac{2s^2+5}{(s+1)(s+2)} = 2 + \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

where

$$k_1 = \left.\frac{2s^2+5}{s+2}\right|_{s=-1} = \frac{2+5}{-1+2} = 7$$

and

$$k_2 = \left.\frac{2s^2+5}{s+1}\right|_{s=-2} = \frac{8+5}{-2+1} = -13$$

Therefore $F(s) = 2 + \frac{7}{s+1} - \frac{13}{s+2}$. From the table, we obtain

$$f(t) = 2\delta(t) + 7e^{-t} - 13e^{-2t}, \qquad t \geq 0.$$

Example: Find the inverse Laplace transform of $F(s) = \dfrac{6(s+34)}{s(s^2+10s+34)}$

$$\begin{aligned} F(s) &= \frac{6(s+34)}{s(s^2+10s+34)} = \frac{6(s+34)}{s(s+5-j3)(s+5+j3)} \\ &= \frac{k_1}{s} + \frac{k_2}{s+5-j3} + \frac{k_2^*}{s+5+j3} \end{aligned}$$

Note that the coefficients (k_2 and k_2^*) of the conjugate terms must also be conjugate. Now

$$\begin{aligned} k_1 &= \left.\frac{6(s+34)}{s^2+10s+34}\right|_{s=0} = \frac{6\times 34}{34} = 6 \\ k_2 &= \left.\frac{6(s+34)}{s(s+5+j3)}\right|_{s=-5+j3} = \frac{29+j3}{-3-j5} = -3+j4 \\ k_2^* &= -3-j4 \end{aligned}$$

To use the Laplace transform table, we need to express k_2 and k_2^* in polar form

$$-3+j4 = \sqrt{3^2+4^2}e^{j\tan^{-1}(4/-3)} = 5e^{j\tan^{-1}(4/-3)}$$

Inverse Laplace Transform

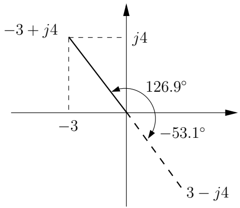
From the Figure below, we observe that

$$k_2 = -3 + j4 = 5e^{j126.9^\circ} \text{ and } k_2^* = 5e^{-j126.9^\circ}$$

Therefore
$$F(s) = \frac{6}{s} + \frac{5e^{j126.9^\circ}}{s+5-j3} + \frac{5e^{-j126.9^\circ}}{s+5+j3}$$

From the table pair 10b

$$f(t) = \left[6 + 10e^{-5t}\cos(3t + 126.9^\circ)\right] \mathbb{1}(t)$$



$$F(s) = \frac{6(s+34)}{s(s^2+10s+34)} = \frac{k_1}{s} + \frac{As+B}{s^2+10s+34}$$

We have already determined that $k_1 = 6$ by the (Heaviside) "cover-up" method. Therefore

$$\frac{6(s+34)}{s(s^2+10s+34)} = \frac{6}{s} + \frac{As+B}{s^2+10s+34}$$

Clearing the fractions by multiplying both sides by $s(s^2+10s+34)$ yields

$$\begin{aligned} 6(s+34) &= 6(s^2+10s+34) + s(As+B) \\ &= (6+A)s^2 + (60+B)s + 204 \end{aligned}$$

Now, equating the coefficients of s^2 and s on both sides yields

$$0 = (6+A) \implies A = -6$$

$$6 = 60+B \implies B = -54$$

and

$$F(s) = \frac{6}{s} + \frac{-6s - 54}{s^2 + 10s + 34}$$

Now from the table, the parameters for this inverse are
$A = -6, B = -54, a = 5, c = 34$, and $b = \sqrt{c - a^2} = 3$, and

$$r = \sqrt{\frac{A^2c + B^2 - 2ABa}{c - a^2}} = 10, \qquad \theta = \tan^{-1}\frac{Aa - B}{A\sqrt{c - a^2}} = 126.9^\circ$$
$$b = \sqrt{c - a^2}$$

Therefore

$$f(t) = \left[6 + 10e^{-5t}\cos(3t + 126.9^\circ)\right]\mathbb{1}(t)$$

which agrees with the previous result.

$$F(s) = \frac{6(s+34)}{s(s^2+10s+34)} = \frac{6}{s} + \frac{As+B}{s^2+10s+34}$$

This step can be accomplished by multiplying both sides of the above equation by s and then letting $s \to \infty$. This procedure yields

$$0 = 6 + A \implies A = -6.$$

Therefore

$$\frac{6(s+34)}{s(s^2+10s+34)} = \frac{6}{s} + \frac{-6s+B}{s^2+10s+34}$$

To find B, we let s take on any convenient value, say $s = 1$, in this equation to obtain

$$\frac{210}{45} = 6 + \frac{B-6}{45}$$

Inverse Laplace Transform

Multiplying both sides of this equation by 45 yields

$$210 = 270 + B - 6 \implies B = -54$$

a deduction which agrees with the results we found earlier.

Inverse Laplace Transform

Example: Find the inverse Laplace transform of $F(s) = \dfrac{8s+10}{(s+1)(s+2)^3}$

$$F(s) = \frac{8s+10}{(s+1)(s+2)^3} = \frac{k_1}{s+1} + \frac{a_0}{(s+2)^3} + \frac{a_1}{(s+2)^2} + \frac{a_2}{a+2}$$

where

$$k_1 = \left.\frac{8s+10}{(s+2)^3}\right|_{s=-1} = 2$$

$$a_0 = \left.\frac{8s+10}{(s+1)}\right|_{s=-2} = 6$$

$$a_1 = \left\{\frac{d}{ds}\left[\frac{8s+10}{(s+1)}\right]\right\}_{s=-2} = -2$$

$$a_2 = \frac{1}{2}\left\{\frac{d^2}{ds^2}\left[\frac{8s+10}{(s+1)}\right]\right\}_{s=-2} = -2$$

Note : the general formula is

$$a_n = \frac{1}{n!}\left\{\frac{d^n}{ds^n}\left[(s-\lambda)^r F(s)\right]\right\}_{s=\lambda}$$

Inverse Laplace Transform

Therefore

$$F(s) = \frac{2}{s+1} + \frac{6}{(s+2)^3} - \frac{2}{(s+2)^2} - \frac{2}{s+2}$$

and

$$f(t) = \left[2e^{-t} + (3t^2 - 2t - 2)e^{-2t}\right] \mathbb{1}(t)$$

Alternative Method: A Hybrid of Heaviside and Clearing Fractions: Using the values $k_1 = 2$ and $a_0 = 6$ obtained earlier by the Heaviside "cover-up" method, we have

$$\frac{8s+10}{(s+1)(s+2)^3} = \frac{2}{s+1} + \frac{6}{(s+2)^3} + \frac{a_1}{(s+2)^2} + \frac{a_2}{s+2}$$

Inverse Laplace Transform

We now clear fractions by multiplying both sides of the equation by $(s+1)(s+2)^3$. This procedure yields

$$\begin{aligned} 8s+10 &= 2(s+2)^3 + 6(s+1) + a_1(s+1)(s+2) + a_2(s+1)(s+2)^2 \\ &= (2+a_2)s^3 + (12+a_1+5a_2)s^2 + (30+3a_1+8a_2)s + (22+2a_1+4a_2) \end{aligned}$$

Equating coefficients of s^3 and s^2 on both sides, we obtain

$$\begin{aligned} 0 &= (2+a_2) \implies a_2 = -2 \\ 0 &= 12+a_1+5a_2 = 2+a_1 \implies a_1 = -2 \end{aligned}$$

Equating the coefficients of s^1 and s^0 serves as a check on our answers.

$$\begin{aligned} 8 &= 30+3a_1+8a_2 \\ 10 &= 22+2a_1+4a_2 \end{aligned}$$

Substitution of $a_1 = a_2 = -2$, obtained earlier, satisfies these equations.

Alternative Method: A Hybrid of Heaviside and Short-Cuts: Using the values $k_1 = 2$ and $a_0 = 6$, determined earlier by the Heaviside method, we have

$$\frac{8s+10}{(s+1)(s+2)^3} = \frac{2}{s+1} + \frac{6}{(s+2)^3} + \frac{a_1}{(s+2)^2} + \frac{a_2}{s+2}$$

There are two unknowns, a_1 and a_2. If we multiply both sides by s and then let $s \to \infty$, we eliminate a_1. This procedure yields

$$0 = 2 + a_2 \implies a_2 = -2$$

Therefore

$$\frac{8s+10}{(s+1)(s+2)^3} = \frac{2}{s+1} + \frac{6}{(s+2)^3} + \frac{a_1}{(s+2)^2} - \frac{2}{s+2}$$

There is now only one unknown, a_1. This value can be determined readily by equal to any convenient value, say $s = 0$. This step yields

$$\frac{10}{8} = 2 + \frac{3}{4} + \frac{a_1}{4} - 1 \implies a_1 = -2.$$

The Laplace transform properties

The Laplace transform is *linear*: if $f(t)$ and $g(t)$ are any signals, and a is any scalar, we have

$$\mathcal{L}\left\{af(t)\right\} = aF(s), \qquad \mathcal{L}\left\{(f(t)+g(t))\right\} = F(s) + G(s)$$

i.e., homogeneity and superposition hold.

Example:

$$\begin{aligned}\mathcal{L}\left\{3\delta(t) - 2e^t\right\} &= 3\mathcal{L}\left\{\delta(t)\right\} - 2\mathcal{L}\left\{e^t\right\} \\ &= 3 - \frac{2}{s-1} \\ &= \frac{3s-5}{s-1}\end{aligned}$$

The Laplace transform properties

The Laplace transform is *one-to-one*: if $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$ then $f(t) = g(t)$.

- $F(s)$ determines $f(t)$
- inverse Laplace transform $\mathcal{L}^{-1}\{f(t)\}$ is well defined.

Example:

$$\mathcal{L}^{-1}\left\{\frac{3s-5}{s-1}\right\} = 3\delta(t) - 2e^t$$

in other words, the only function $f(t)$ such that

$$F(s) = \frac{3s-5}{s-1}$$

is $f(t) = 3\delta(t) - 2e^t$.

The Laplace transform properties: Time delay

This property states that if

$$f(t) \Longleftrightarrow F(s)$$

then for $T \geq 0$

$$f(t-T) \Longleftrightarrow e^{-sT}F(s)$$

(If $g(t)$ is $f(t)$, delayed by T seconds), then we have $G(s) = e^{-sT}F(s)$.
Derivation:

$$\begin{aligned} G(s) = \int_0^\infty e^{-st}g(t)dt &= \int_0^\infty e^{-st}f(t-T)dt \\ &= \int_0^\infty e^{-s(\tau+T)}f(\tau)d\tau = e^{-sT}F(s) \end{aligned}$$

The Laplace transform properties: Time delay

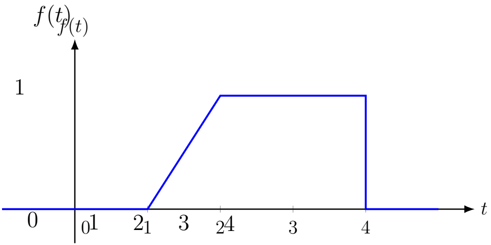
To avoid a pitfall, we should restate the property as follow:

$$f(t)\mathbb{1}(t) \Longleftrightarrow F(s)$$

then

$$f(t-T)\mathbb{1}(t-T) \Longleftrightarrow e^{-sT}F(s), \qquad T \geq 0.$$

The Laplace transform properties: Time delay example



Find the Laplace Transform of $f(t)$ depicted in Figure above.

The signal can be described as

$$
\begin{aligned}
f(t) &= (t-1)[\mathbb{1}(t-1) - \mathbb{1}(t-2)] + [\mathbb{1}(t-2) - \mathbb{1}(t-4)] \\
&= (t-1)\mathbb{1}(t-1) - (t-1)\mathbb{1}(t-2) + \mathbb{1}(t-2) - \mathbb{1}(t-4) \\
&= (t-1)\mathbb{1}(t-1) - (t-2)\mathbb{1}(t-2) - \mathbb{1}(t-4)
\end{aligned}
$$

The Laplace transform properties: Time delay example

Since $t \iff \frac{1}{s^2}$ yields

$$(t-1)\mathbb{1}(t-1) \iff \frac{1}{s^2}e^{-s} \text{ and } (t-2)\mathbb{1}(t-2) \iff \frac{1}{s^2}e^{-2s}$$

Also $\mathbb{1}(t) \iff \frac{1}{s}$ yields

$$\mathbb{1}(t-4) \iff \frac{1}{s}e^{-4s}$$

Therefore

$$F(s) = \frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-4s}$$

The Laplace transform properties: Time delay example

Find the inverse Laplace transform of

$$F(s) = \frac{s + 3 + 5e^{-2s}}{(s+1)(s+2)}$$

The $F(s)$ can be separated in two parts

$$F(s) = \underbrace{\frac{s+3}{(s+1)(s+2)}}_{F_1(s)} + \underbrace{\frac{5e^{-2s}}{(s+1)(s+2)}}_{F_2(s)e^{-2s}}$$

where

$$F_1(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$
$$F_2(s) = \frac{5}{(s+1)(s+2)} = \frac{5}{s+1} - \frac{5}{s+2}$$

Therefore

$$
\begin{aligned}
f_1(t) &= \left(2e^{-t} - e^{-2t}\right) \\
f_2(t) &= 5\left(e^{-t} - e^{-2t}\right)
\end{aligned}
$$

Since

$$
\begin{aligned}
F(s) &= F_1(s) + F_2(s)e^{-2s} \\
f(t) &= f_1(t) + f_2(t-2) \\
&= \left(2e^{-t} - e^{-2t}\right)\mathbb{1}(t) + 5\left[e^{-(t-2)} - e^{-2(t-2)}\right]\mathbb{1}(t-2)
\end{aligned}
$$

The Laplace transform properties: Time scaling

Define a signal $g(t)$ by $g(t) = f(at)$, where $a > 0$; then

$$G(s) = \frac{1}{a}F(\frac{s}{a}).$$

time are scaled by a, then frequencies are scaled by $1/a$.

$$G(s) = \int_0^\infty f(at)e^{-st}dt = \frac{1}{a}\int_0^\infty f(\tau)e^{-\frac{s}{a}\tau}d\tau = \frac{1}{a}F(\frac{s}{a}),$$

where $\tau = at$.
Example: $\mathcal{L}\left\{e^t\right\} = \dfrac{1}{s-1}$ so

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{a}\frac{1}{\frac{s}{a}-1} = \frac{1}{s-a}$$

The Laplace transform properties: Exponential scaling

Let $f(t)$ be a signal and a a scale, and define $g(t) = e^{at} f(t)$; then

$$G(s) = F(s-a)$$

Proof:

$$G(s) = \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-(s-a)t} f(t) dt = F(s-a)$$

Example: $\mathcal{L}\left\{\cos t\right\} = \frac{s}{s^2+1}$, and hence

$$\mathcal{L}\left\{e^{-t}\cos t\right\} = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$$

The Laplace transform properties: Exponential scaling

Example: Consider $F(s) = \dfrac{-6s - 54}{s^2 + 10s + 34}$. By using the exponential exponential scaling, we obtain

$$\frac{-6s - 54}{s^2 + 10s + 34} = \frac{-6(s+5) - 24}{(s+5)^2 + 9} = \frac{-6(s+5)}{(s+5)^2 + 3^2} + \frac{-8(3)}{(s+5)^2 + 3^2}$$

Then,

$$\begin{aligned} f(t) &= -6e^{-5t}\cos 3t - 8e^{-5t}\sin 3t \\ &= 10e^{-5t}\cos(3t + 127^\circ) \end{aligned}$$

You can do this inverse Laplace transform using only standard Laplace transform table.

The Laplace transform properties: Derivative

If signal $f(t)$ is continuous at $t = 0$, then

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0);$$

- time-domain differentiation becomes multiplication by frequency variable s (as with phasors)
- plus a term that includes initial condition (i.e., $-f(0)$)

higher-order derivatives: applying derivative formula twice yields

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s\mathcal{L}\left\{\frac{df(t)}{dt}\right\} - \frac{df(t)}{dt}$$
$$= s(sF(s) - f(0)) - \frac{df(0)}{dt} = s^2 F(s) - sf(0) - \frac{df(0)}{dt}$$

similar formulas hold for $\mathcal{L}\left\{f^{(k)}\right\}$.

The Laplace transform properties: Derivation of derivative formula

Start from the defining integral

$$G(s) = \int_0^\infty \frac{df(t)}{dt} e^{-st} dt$$

integration by parts yields

$$\begin{aligned} G(s) &= e^{-st} f(t) \Big|_0^\infty - \int_0^\infty f(t)(-se^{-st}) dt \\ &= f(t)e^{-s\infty} - f(0) + sF(s) \end{aligned}$$

we recover the formula

$$G(s) = sF(s) - f(0)$$

The Laplace transform properties: Derivative example

1. $f(t) = e^t$, so $f'(t) = e^t$ and

$$\mathcal{L}\left\{f(t)\right\} = \mathcal{L}\left\{f'(t)\right\} = \frac{1}{s-1}$$

   by using $\mathcal{L}\left\{f'(t)\right\} = s\dfrac{1}{s-1} - 1$, which is the same.

2. $\sin \omega t = -\frac{1}{\omega}\frac{d}{dt}\cos \omega t$, so

$$\mathcal{L}\left\{\sin \omega t\right\} = -\frac{1}{\omega}\left(s\frac{s}{s^2+\omega^2} - 1\right) = \frac{\omega}{s^2+\omega^2}$$

3. $f(t)$ is a unit ramp, so $f'(t)$ is a unit step

$$\mathcal{L}\left\{f'(t)\right\} = s\left(\frac{1}{s^2}\right) - 0 = \frac{1}{s}$$

The Laplace transform properties: Integral

Let $g(t)$ be the running integral of a signal $f(t)$, i.e.,

$$g(t) = \int_0^t f(\tau)d\tau$$

then $G(s) = \frac{1}{s}F(s)$, i.e., *time-domain integral* become division by frequency variable s.

Example: $f(t) = \delta(t)$ is a unit impulse function, so $F(s) = 1$; $g(t)$ is the unit step

$$G(s) = \frac{1}{s}.$$

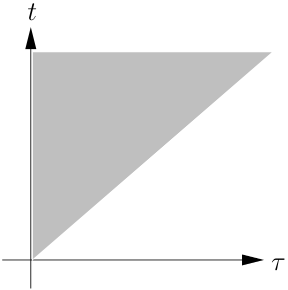
Example: $f(t)$ is a unit step function, so $F(s) = 1/s$; $g(t)$ is the unit ramp function ($g(t) = t$ for $t \geq 0$),

$$G(s) = \frac{1}{s^2}$$

The Laplace transform properties: Derivation of integral

$$G(s) = \int_{t=0}^{\infty} \left(\int_{\tau=0}^{t} f(\tau) d\tau \right) e^{-st} dt$$

here we integrate horizontally first over the triangle $0 \le \tau \le t$.



Let's switch the order, integrate vertically first:

$$\begin{aligned} G(s) &= \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} f(\tau) e^{-st} dt d\tau \\ &= \int_{\tau=0}^{\infty} f(\tau) \left(\int_{t=\tau}^{\infty} e^{-st} dt \right) d\tau \\ &= \int_{\tau=0}^{\infty} f(\tau) \frac{1}{s} e^{-s\tau} d\tau = \frac{F(s)}{s} \end{aligned}$$

The Laplace transform properties: Convolution

The convolution of signals $f(t)$ and $g(t)$, denoted $h(t) = f(t) * g(t)$, is the signal

$$h(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

In terms of Laplace transforms:

$$H(s) = F(s)G(s)$$

The Laplace transform turns convolution into multiplication.

The Laplace transform properties: Convolution cont.

Let's show that $\mathcal{L}\left\{f(t) * g(t)\right\} = F(s)G(s)$:

$$\begin{aligned} H(s) &= \int_{t=0}^{\infty} e^{-st} \left(\int_{\tau=0}^{t} f(\tau)g(t-\tau)d\tau \right) dt \\ &= \int_{t=0}^{\infty} \int_{\tau=0}^{t} e^{-st} f(\tau)g(t-\tau)d\tau dt \end{aligned}$$

where we integrate over the triangle $0 \leq \tau \leq t$. By changing the order of the integration and changing the limits of integration yield

$$H(s) = \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} e^{-st} f(\tau)g(t-\tau)dtd\tau$$

The Laplace transform properties: Convolution cont.

Change variable t to $\bar{t} = t - \tau; d\bar{t} = dt$; region of integration becomes $\tau \geq 0, \bar{t} \geq 0$

$$
\begin{aligned}
H(s) &= \int_{\tau=0}^{\infty} \int_{\bar{t}=0}^{\infty} e^{-s(\bar{t}+\tau)} f(\tau) g(\bar{t}) d\bar{t} d\tau \\
&= \left(\int_{\tau=0}^{\infty} e^{-s\tau} f(\tau) d\tau \right) \left(\int_{\bar{t}=0}^{\infty} e^{-s\bar{t}} g(\bar{t}) d\bar{t} \right) \\
&= F(s) G(s)
\end{aligned}
$$

The Laplace transform properties: Convolution cont.

Example: Using the time convolution property of the Laplace transform, determine $c(t) = e^{at}\mathbb{1}(t) * e^{bt}\mathbb{1}(t)$. From the convolution property, we have

$$C(s) = \frac{1}{s-a}\frac{1}{s-b} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

The inverse transform of the above equation yields

$$c(t) = \frac{1}{a-b}(e^{at} - e^{bt}), \qquad t \geq 0.$$

Applications: Solution of Differential Equations

Solve the second-order linear differential equation

$$(D^2 + 5D + 6)y(t) = (D + 1)f(t)$$

if the initial conditions are $y(0^-) = 2, \dot{y}(0^-) = 1$, and the input $f(t) = e^{-4t}\mathbb{1}(t)$.
The equation is

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{df}{dt} + f(t).$$

Let

$$y(t) \Longleftrightarrow Y(s).$$

Then

$$\frac{dy}{dt} \Longleftrightarrow sY(s) - y(0^-) = sY(s) - 2.$$

and

$$\frac{d^2y}{dt^2} \Longleftrightarrow s^2Y(s) - sy(0^-) - \dot{y}(0^-) = s^2Y(s) - 2s - 1.$$

Moreover, for $f(t) = e^{-4t}\mathbb{1}(t)$,

$$F(s) = \frac{1}{s+4}, \text{ and } \frac{df}{dt} \Longleftrightarrow sF(s) - f(0^-) = \frac{s}{s+4} - 0 = \frac{s}{s+4}.$$

Taking the Laplace transform, we obtain

$$\left[s^2Y(s) - 2s - 1\right] + 5\left[sY(s) - 2\right] + 6Y(s) = \frac{s}{s+4} + \frac{1}{s+4}$$

Collecting all the terms of $Y(s)$ and the remaining terms separately on the left-hand side, we obtain

$$(s^2 + 5s + 6)Y(s) - (2s + 11) = \frac{s+1}{s+4}$$

Therefore

$$(s^2 + 5s + 6)Y(s) = (2s + 11) + \frac{s+1}{s+4} = \frac{2s^2 + 20s + 45}{s+4}$$

and

$$\begin{aligned} Y(s) &= \frac{2s^2 + 20s + 45}{(s^2 + 5s + 6)(s+4)} \\ &= \frac{2s^2 + 20s + 45}{(s+2)(s+3)(s+4)} \end{aligned}$$

Expanding the right-hand side into partial fractions yields

$$Y(s) = \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4}$$

The inverse Laplace transform of the above equation yields

$$y(t) = \left(\frac{13}{2} e^{-2t} - 3e^{-3t} - \frac{3}{2} e^{-4t} \right) \mathbb{1}(t).$$

Zero-Input and Zero-State Components of Response

- The Laplace transform method gives the total response, which includes zero-input and zero-state components.
- The initial condition terms in the response give rise to the zero-input response.

For example in the previous example, $(s^2+5s+6)Y(s)-(2s+11)=\dfrac{s+1}{s+4}$
so that

$$(s^2+5s+6)Y(s) = \underbrace{(2s+11)}_{\text{initial condition terms}} + \underbrace{\frac{s+1}{s+4}}_{\text{input terms}}$$

Zero-Input and Zero-State Components of Response

Therefore

$$Y(s) = \underbrace{\frac{2s+11}{s^2+5s+6}}_{\text{zero-input component}} + \underbrace{\frac{s+1}{(s+4)(s^2+5s+6)}}_{\text{zero-state component}}$$

$$= \left[\frac{7}{s+2} - \frac{5}{s+3}\right] + \left[\frac{-1/2}{s+2} + \frac{2}{s+3} - \frac{3/2}{s+4}\right]$$

Taking the inverse transform of this equation yields

$$y(t) = \underbrace{(7e^{-2t} - 5e^{-3t})\mathbb{1}(t)}_{\text{zero-input response}} + \underbrace{(-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t})\mathbb{1}(t)}_{\text{zero-state response}}$$

Analysis of Electrical Networks

- It is possible to analyze electrical networks directly without having to write the integro-differential equation.
- This procedure is considerably simpler because it permits us to treat an electrical network as if it was a resistive network.
- To do such a procedure, we need to represent a network in “frequency domain” where all the voltages and currents are represented by their Laplace transforms.

Analysis of Electrical Networks

zero initial conditions case:

If $v(t)$ and $i(t)$ are the voltage across and the current through an inductor of L henries, then

$$v(t) = L\frac{di(t)}{dt} \iff V(s) = sLI(s), \qquad i(0) = 0.$$

Similarly, for a capacitor of C farads, the voltage-current relationship is

$$i(t) = C\frac{dv(t)}{dt} \iff V(s) = \frac{1}{Cs}I(s), \qquad v(0) = 0.$$

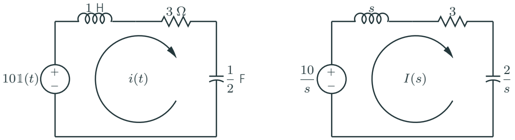
For a resistor of R ohms, the voltage-current relationship is

$$v(t) = Ri(t) \iff V(s) = RI(s).$$

- Thus, in the “frequency domain,” the voltage-current relationships of an inductor and a capacitor are algebraic;
- These elements behave like resistors of “resistance” Ls and $1/Cs$, respectively.
- The generalized “resistance” of an element is called its **impedance** and is given by the ratio $V(s)/I(s)$ for the element (under zero initial conditions).
- The impedances of a resistor of R ohms, and inductor of L henries, and a capacitance of C farads are R, Ls, and $1/Cs$, respectively.
- The Kirchhoff’s laws remain valid for voltages and currents in the frequency domain.

Analysis of Electrical Networks

Find the loop current $i(t)$ in the circuit, if all the initial conditions are zero.



In the first step, we represent the circuit in the frequency domain shown in the right hand side. The impedance in the loop is

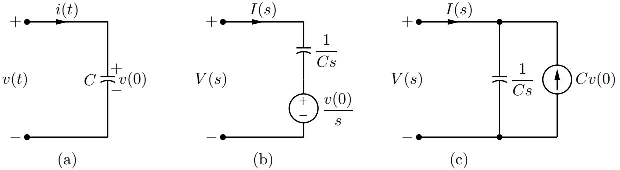
$$Z(s) = s + 3 + \frac{2}{s} = \frac{s^2 + 3s + 2}{s}$$

The input voltage is $V(s) = 10/s$. Therefore, the loop current $I(s)$ is

$$I(s) = \frac{V(s)}{Z(s)} = \frac{10/s}{(s^2 + 3s + 2)/s} = \frac{10}{s^2 + 3s + 2} = \frac{10}{(s+1)(s+2)} = \frac{10}{s+1} - \frac{10}{s+2}$$

The inverse transform of the equation yields: $i(t) = 10(e^{-t} - e^{-2t})\mathbb{1}(t)$.

A capacitor C with an initial voltage $v(0)$ can be represented in the frequency domain by an uncharged capacitor of impedance $1/Cs$ in series with a voltage source of value $v(0)/s$ or as the same uncharged capacitor in parallel with a current source of value $Cv(0)$.

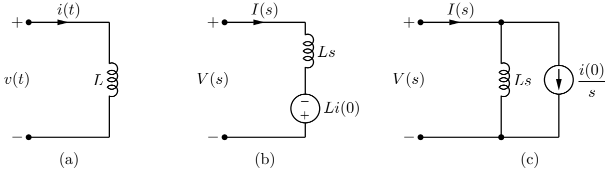


$$i(t) = C\frac{dv}{dt} \iff I(s) = C[sV(s) - v(0)]$$

Rearranging the equation, we obtain

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s} \text{ or } V(s) = \frac{1}{Cs}\left[I(s) + Cv(0)\right]$$

An inductor L with an initial voltage $i(0)$ can be represented in the frequency domain by an inductor of impedance Ls in series with a voltage source of value $Li(0)$ or by the same inductor in parallel with a current source of value $i(0)/s$.

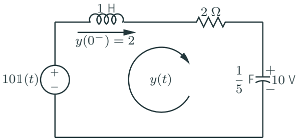


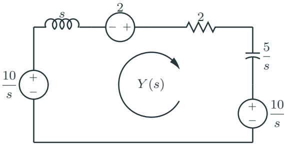
$$v(t) = L\frac{di}{dt} \Longleftrightarrow V(s) = L[sI(s) - i(0)]$$

Rearranging the equation, we obtain

$$V(s) = sLI(s) - Li(0) \text{ or } V(s) = Ls\left[I(s) - \frac{i(0)}{s}\right]$$

Find the loop current $i(t)$ in the circuit, if $y(0) = 2$ and $v_C(0) = 10$.





The right hand side figure shows the frequency-domain representation of the circuit. Applying mesh analysis we have

$$-\frac{10}{s} + sY(s) - 2 + 2Y(s) + \frac{5}{s}Y(s) + \frac{10}{s} = 0$$

$$Y(s) = \frac{2}{s + 2 + \frac{5}{s}}$$

$$= \frac{2s}{s^2 + 2s + 5}$$

$$Y(s) = \frac{2s}{s^2 + 2s + 5} = \frac{2(s+1)}{(s+1)^2 + 2^2} - \frac{2}{(s+1)^2 + 2^2)}$$

Therefore

$$y(t) = e^{-t}(2\cos 2t - \sin 2t) = e^{-t}(C\cos\theta\cos 2t - C\sin\theta\sin 2t),$$

since

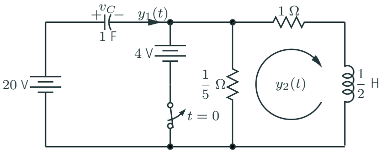
$$C = \sqrt{2^2+1} = \sqrt{5}, \qquad \theta = \tan^{-1}\frac{2}{4} = 26.6^\circ$$

then

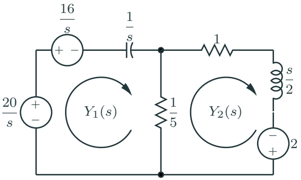
$$y(t) = \sqrt{5}e^{-t}\cos(2t + 26.6^\circ)\mathbb{1}(t).$$

Analysis of Electrical Networks

The switch in the circuit is in the closed position for a long time before $t = 0$, when it is opened instantaneously. Find the currents $y_1(t)$ and $y_2(t)$ for $t \geq 0$.



(a)



(b)

When the switch is closed and the steady-state conditions are reached, the capacitor voltage $v_C = 16$ volts, and the inductor current $y_2 = 4$ A. The right hand side circuit shows the transformed version of the circuit in the left hand side. Using mesh analysis, we obtain

$$\frac{Y_1(s)}{s} + \frac{1}{5}\left[Y_1(s) - Y_2(s)\right] = \frac{4}{s}$$
$$-\frac{1}{5}Y_1(s) + \frac{6}{5}Y_2(s) + \frac{s}{2}Y_2(s) = 2$$

Rewriting in matrix form, we have

$$\begin{bmatrix} \frac{1}{s} + \frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{6}{5} + \frac{s}{2} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{4}{s} \\ 2 \end{bmatrix}$$

Therefore,

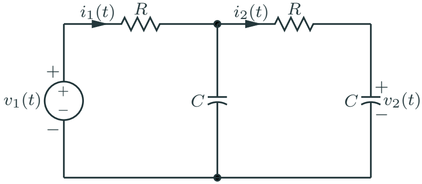
$$\begin{aligned} Y_1(s) &= \frac{24(s+2)}{s^2+7s+12} \\ &= \frac{24(s+2)}{(s+3)(s+4)} = \frac{-24}{s+3} + \frac{48}{s+4} \\ Y_2(s) &= \frac{4(s+7)}{s^2+7s+12} = \frac{16}{s+3} - \frac{12}{s+4}. \end{aligned}$$

Finally,

$$\begin{aligned} y_1(t) &= (-24e^{-3t} + 48e^{-4t})\mathbb{1}(t) \\ y_2(t) &= (16e^{-3t} - 12e^{-4t})\mathbb{1}(t) \end{aligned}$$

Transfer Functions of Linear Continuous-Times Systems

Transfer Function. The transfer function of a linear time-invariant continuous-time system (LTICT) is the ratio of the Laplace transforms of the output and the input under zero initial conditions.



The loop equation for zero initial conditions,

$$\left(R + \frac{1}{sC}\right) I_1(s) - \frac{1}{sC} I_2(s) = V_1(s)$$

$$-\frac{1}{sC} I_1(s) + \left(R + \frac{2}{sC}\right) I_2(s) = 0$$

Solving the equations, we obtain

$$I_2(s) = \frac{sCV_1}{s^2C^2R^2 + sCR + 1} = sCV_2(s)$$

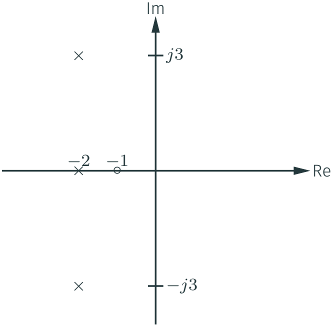
$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{s^2C^2R^2 + sCR + 1}$$

Transfer Functions of Linear Continuous-Times Systems

- Poles and zeros are the roots of the denominator and numerator polynomials,respectively, of a rational function.
- The poles of the transfer function are also its **natural frequencies**.
- The zeros of a transfer function can be considered as the frequencies at which there will be no output; in other words, inputs at these frequencies will be *blocked* by the system.

$$H(s) = \frac{20(s+1)}{(s+2)(s^2+4s+13)}$$

- We have a zero at -1 and three poles at $-2, -2 \pm j3$.
- We can reconstruct the transfer function from the pole-zero map, except the scale factor.



Laplace Transforms of Causal Repeating Functions

How can we take the Laplace transform of a causal function, which repeats every T seconds for $t > 0$?

- Let us denote this function as $x(t)$ and define $X_1(s)$ as the Laplace transform of the first cycle of the function. This implies that

$$X_1(s) = \int_{0^-}^{T^-} x(t)e^{-st}dt$$

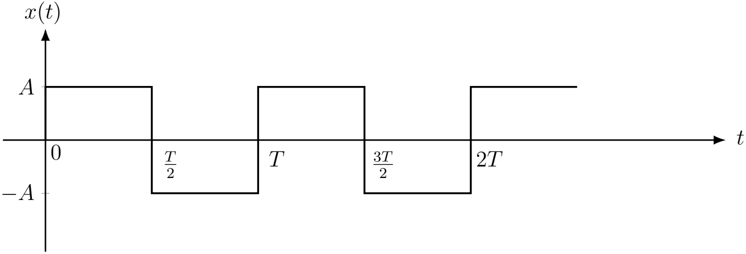
- Using the fact that all subsequent complete cycles of the function can be obtained by shifting the first cycle by $T,\ 2T,\ 3T, \ldots,$, we can write the following expression for the Laplace transform of the entire function $x(t)$:

$$\begin{aligned} X(s) &= X_1(s)\left(1 + e^{-sT} + e^{-sT} + e^{-3sT} + \cdots\right) \\ &= \frac{X_1(s)}{1 - e^{-sT}} \end{aligned}$$

- The last line follows from the properties of a geometric series:

$$\sum_{n=0}^{\infty} e^{-nsT} = \frac{1}{1 - e^{-sT}}$$

Laplace Transforms of Causal Repeating Functions



The first cycle of this waveform can be expressed as

$$x_1(t) = A\mathbb{1}(t) - 2A\mathbb{1}(t - \frac{T}{2}) + A\mathbb{1}(t - T)$$

Taking the Laplace transform we get

$$X_1(s) = \frac{A}{s}\left(1 - 2e^{-sT/2} + e^{-sT}\right) \Rightarrow X(s) = \frac{X_1(s)}{1 - e^{-sT}}$$

$$X(s) = \frac{A}{s}\frac{\left(1 - 2e^{-sT/2} + e^{-sT}\right)}{1 - e^{-sT}} = \frac{A}{s}\frac{(1 - e^{-sT/2})(1 - e^{-sT/2})}{(1 - e^{-sT/2})(1 + e^{-sT/2})} = \frac{A}{s}\frac{(1 - e^{-sT/2})}{(1 + e^{-sT/2})}$$

Laplace Transforms of Causal Repeating Functions

Another method:

$$x(t) = A\mathbb{1}(t) - 2A\mathbb{1}(t - \frac{T}{2}) + 2A\mathbb{1}(t-T) - 2A\mathbb{1}(t - \frac{3T}{2}) + 2A\mathbb{1}(t-2T) - \cdots$$

$$\begin{aligned} X(s) &= \frac{A}{s} - \frac{2A}{s}e^{-s(T/2)} + \frac{2A}{s}e^{-sT} - \frac{2A}{s}e^{-s(3T/2)} + \frac{2A}{s}e^{-s(2T)} + \cdots \\ &= \frac{A}{s} - \frac{2A}{s}e^{-s(T/2)}\left(1 - e^{-s(T/2)} + e^{-sT} - e^{-s(3T/2)} + e^{-s(2T)} - \cdots\right) \end{aligned}$$

Let

$$\begin{aligned} S &= 1 - e^{-s(T/2)} + e^{-sT} - e^{-s(3T/2)} + e^{-s(2T)} - \cdots \\ -e^{-s(T/2)}S &= -e^{-s(T/2)} + e^{-sT} - e^{-s(3T/2)} + e^{-s(2T)} - \cdots \\ (1 + e^{-s(T/2)})S &= 1 \;\Rightarrow\; S = \frac{1}{1 + e^{-s(T/2)}} \end{aligned}$$

Then

$$X(s) = \frac{A}{s}\left(1 - 2e^{-s(T/2)}\left(\frac{1}{1+e^{-s(T/2)}}\right)\right) = \frac{A}{s}\frac{(1-e^{-sT/2})}{(1+e^{-sT/2})}$$

Laplace Transforms of Causal Repeating Functions

The response of the system to the causal repeating input in Laplace domain is

$$Y(s) = H(s)X(s) = \frac{H(s)X(s)}{1 - e^{-sT}}$$

The roots of the system is not only the root of the denominator of the product $H(s)X(s)$ but also the roots of the equation

$$1 - e^{-sT} = 0.$$

It has an **infinite number of roots**, located at $s = j2\pi n/T$, where n is any positive or negative integer. Then we cannot find $y(t)$ using the inverse Laplace transforms. To overcome this problem:

- Express $Y(s)$ in the following from:

$$Y(s) = H(s)X_1(s)\left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots\right]$$

Laplace Transforms of Causal Repeating Functions

- If we just obtain the inverse Laplace transform of $H(s)X_1(s)$, then the remaining terms are obtained by shifting in time.
- If we define

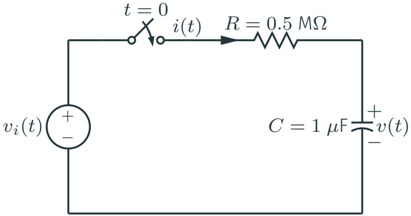
$$y_1(t) = \mathcal{L}^{-1}\left[H(s)X_1(s)\right]$$

then during the interval $(n-1)T < t < nT$, the output can be expressed as

$$y(t) = y_1(t)\mathbb{1}(t) + y_1(t-T)\mathbb{1}(t-T) + \cdots + y_1(t-nT+T)\mathbb{1}(t-nT+T)$$

Laplace Transforms of Causal Repeating Functions

The square wave is applied to the RC circuit. The amplitude of the waveform is 20 V and the period T is 2 sec. The switch is closed at $t = 0$ and the initial voltage across the capacitor is 10 V.



$$v(t) + RC\frac{dv}{dt} = v_i(t)$$

$$(1 + sRC)V(s) = V_i(s) + RCv(0)$$

with the given values, $RC = 0.5$, $v(0) = 10$, and

$$V_i(s) = \frac{20(1 - 2e^{-s} + e^{-2})}{s(1 - e^{-2s})}$$

Solving for $V(s)$, the Laplace transform of the voltage across the capacitor, we obtain

$$V(s) = \frac{10}{s+2} + \frac{40(1 - 2e^{-s} + e^{-2s})}{s(s+2)(1 - e^{-2s})}$$

Laplace Transforms of Causal Repeating Functions

By using a long division, we have

$$\frac{1-2e^{-s}+e^{-2s}}{1-e^{-2s}} = \big(1-2e^{-s}+e^{-2s}\big)(1+e^{-2s}+e^{-4s}+\cdots\big)$$
$$= 1-2e^{-s}+2e^{-2s}-2e^{-3s}+2e^{-4s}-\cdots$$

Then

$$V(s) = \frac{10}{s+2} + \frac{40}{s(s+2)}\left(1-2e^{-s}+2e^{-2s}-2e^{-3s}+2e^{-4s}-\cdots\right)$$
$$= V_1(s) + \widetilde{V}_2(s)\left(1-2e^{-s}+2e^{-2s}-2e^{-3s}+2e^{-4s}-\cdots\right) = V_1(s)+V_2(s)$$
$$v_1(t) = 10e^{-2t}$$
$$\widetilde{v}_2(t) = 20-20e^{-2t}$$
$$v_2(t) = 20-20e^{-2t} - 40(1-e^{-2(t-1)})\mathbb{1}(t-1)$$
$$+40(1-e^{-2(t-2)})\mathbb{1}(t-2) - 40(1-e^{-2(t-3)})\mathbb{1}(t-3)+\cdots$$

Laplace Transforms of Causal Repeating Functions

If we consider one nth period, that is $2(n-1) < t < 2n$, we have for example
$3 < t < 4$
($\mathbb{1}(t) = \mathbb{1}(t-1) = \mathbb{1}(t-2) = \mathbb{1}(t-3) = 1$,$\mathbb{1}(t-4) = 0$)

$$\begin{aligned} v_2(t) &= 20 - 20e^{-2t} - 40(1 - e^{-2(t-1)}) + 40(1 - e^{-2(t-2)}) - 40(1 - e^{-2(t-3)}) \\ &= -20 + 20e^{-2t} - 40e^{-2t}\left(1 - e^2 + e^4 - e^6\right) \end{aligned}$$

If $n < t < n+1$

$$v_2(t) = (-1)^n 20 + 20e^{-2t} - 40e^{-2t}\left(1 - e^2 + e^4 - e^6 + \cdots + (-e^2)^n\right)$$

Then

$$v(t) = 10e^{-2t} + (-1)^n 20 + 20e^{-2t} - 40e^{-2t}\left[\frac{1 - (-e^2)^{n+1}}{1 - e^2}\right]$$

Laplace Transforms of Causal Repeating Functions

- The total response of the system during the period $nT < t < (n+1)T$ can usually be expressed in a compact form by using some algebraic properties of geometric series.
- We could not easily determine the steady-state component. The problem is caused by the fact that we cannot simply assume t to be very large and drop all terms multiplied by negative exponentials.
- for example from the last example if we do that we will get a square wave as an output, which is not correct.
- To find the steady-state response, one way can do as follow:
  - Find the Laplace transform of zero-state response of the system to only the first cycle of the repetitive input.
  - Find the transient component from the residues at the poles of the system transfer function. These poles must lie strictly in the left half of the s-plane for the system to have a steady-state response.
  - The steady-state response is

$$y_{ss}(t) = \mathcal{L}^{-1}\left[H(s)X_1(s)\right] - \sum_{i=1}^{n} A_i e^{-p_i t},$$

where $X_1(s)$ is the Laplace transform of the first cycle of $x(t)$.

From the last example we have the Laplace transform of the first cycle of $v_i(t)$ is given by:

$$V_{i1}(s) = \frac{20(1 - 2e^{-s} + e^{-2})}{s}$$

The Laplace transform of the zero-state response is by giving $v(0) = 0$ and

$$H(s) = \frac{V(s)}{V_i(s)} = \frac{2}{s+2}$$

The zero-state response of the first cycle is

$$\begin{aligned} v_s(t) &= \mathcal{L}^{-1}\left[H(s)V_{i1}(s)\right] = \mathcal{L}^{-1}\left[\frac{40(1 - 2e^{-s} + e^{-2s}}{s(s+2)}\right] \\ &= 20(1 - e^{-2t})\mathbb{1}(t) - 40\left[1 - e^{-2(t-1)}\right]\mathbb{1}(t-1) + 20\left[1 - e^{-2(t-2)}\right]\mathbb{1}(t-2) \end{aligned}$$

Laplace Transforms of Causal Repeating Functions

To calculate the transient component of the complete (all periods) zero-state response, we consider the residue at the pole at $s = -2$ as follow:

$$H(s)V_i(s) = \frac{40(1-2e^{-s}+e^{-2s})}{s(s+2)(1-e^{-2s})} = \frac{A}{s} + \frac{B}{s+2}$$

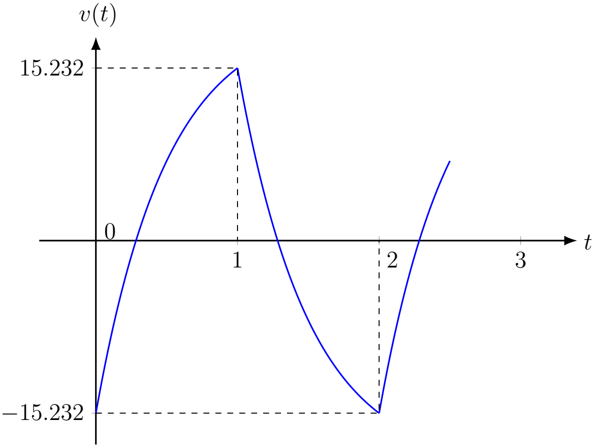
$$B = \left.\frac{40(1-2e^{-s}+e^{-2s})}{s(1-e^{-2s})}\right|_{s=-2} = 15.232$$

Then the steady-state output during the first cycle is given by

$$v_{ss}(t) = v_1(t) - Be^{-2t} = (20 - 35.232e^{-2t})\mathbb{1}(t) - 40[1-e^{-2(t-1)}]\mathbb{1}(t-1), \quad 0 < t < 2$$

The last term of $v_s(t)$ is equal zero during the first cycle $0 < t < 2$.

Laplace Transforms of Causal Repeating Functions



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