Lecture 2: Time-Domain Analysis of Continuous-Time Systems

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Differential Equations

General Concepts and Definitions

In system dynamics, we use an **ordinary differential equation (ODE)** to explain a time-dependent behavior of the system. Let t be an **independent variable** and y be a **dependent variable**. And equation that involves y, t is called an ODE.

Examples:

$$3\ddot{y} + 7\dot{y} + 2t^2 y = 5 + \sin t$$

or

$$3\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 2t^2 y = 5 + \sin t$$

where x is the dependent variable.

- The input is $5 + \sin t$ and the response is $y(t)$.
- If the right-hand size is zero, the equation is said to be **homogeneous**;
- otherwise, it is **nonhomogeneous**.

- **Order** of ODE is the order of the highest derivative appearing in the differential equation.
- If y and its various derivatives $\dot{y}$, $\ddot{y}$,. . ., $y^{(n)}$ appear linearly in the equation, it is a **linear** differential equation; otherwise, it is **nonlinear**.

For example,

$$\frac{d^2y}{dt^2} + a^2y = \sin t, \qquad \text{Second-order, linear}$$

$$\left(\frac{dy}{dt}\right)^2 + 4y = \cos t, \qquad \text{First-order, nonlinear}$$

$$t^3\frac{d^3y}{dt^3} + 5x\frac{dy}{dx} + 6y = e^x, \qquad \text{Third-order, linear}$$

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + 2y = x. \qquad \text{Second-order, nonlinear}$$

The 2nd and the 4th are nonlinear because the terms $(dy/dt)^2$ and ydy/dt respectively.

Consider **Linear Time-Invariant Continuous-Time (LTIC)** Systems, for which the input $f(t)$ and the output $y(t)$ are related by linear differential equations of the form

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) = b_m\frac{d^m f}{dt^m} + b_{m-1}\frac{d^{m-1}f}{dt^{m-1}} + \cdots + b_1\frac{df}{dt} + b_0 f(t),$$

where all the coefficients a_i and c_i are constants.

- Theoretically the powers m and n can be take on any value.
- Practical noise considerations, require $m \leq n$.
- For the rest of this course we assume implicitly that $m \leq n$.

D-operator

$$
\begin{aligned}
Dy &\equiv \frac{dy}{dt}, \qquad Dy \text{ is taking first-order derivative of } y \text{ w.r.t. } t. \\
D^2y = D(Dy) &= \frac{d^2y}{dt^2} \\
\vdots \quad &= \quad \vdots \\
D^n y &= \frac{d^n y}{dt^n}, \ n \text{ is a positive interger.}
\end{aligned}
$$

Hence the D-operator is a differential operator; applying the D-operator on function $f(t)$ means differentiating $f(t)$ with respect to t, i.e.,

$$Df(t) = \frac{df(t)}{dt}.$$

The D-Operator

The following properties of the D-operator can be easily verified:

1. $D[y_1(t) + y_2(t)] = \frac{d}{dt}(y_1 + y_2) = \frac{dy_1}{dt} + \frac{dy_2}{dt} = Dy_1 + Dy_2;$
2. $D[cy(t)] = \frac{d}{dt}(cy) = c\frac{dy}{dt} = cDy, \qquad c =$ constant.
3. $D[c_1 y_1(t) + c_2 y_2(t)] = c_1 Dy_1 + c_2 Dy_2, \qquad c_1, c_2$ = constants.

Using the D-operator to the LTIC system, we can express the equation as

$$\left(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0\right) y(t) = \left(c_m D^m + c_{m-1}D^{m-1} + \cdots + c_1 D + c_0\right) f(t)$$

or

$$Q(D)y(t) = P(D)f(t)$$

Rewrite the following differential equations using the D-operator:

1. $6x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 3y = x^3 e^{2x}$

   Solution:

$$(6x^2 D^2 + 2xD - 3)y = x^3 e^{2x}, \qquad D \equiv \frac{d}{dx}$$

2. $5\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - \frac{dx}{dt} + 7x = 3\sin 8t$

   Solution:

$$(5D^3 + 2D^2 - D + 7)x = 3\sin 8t, \qquad D \equiv \frac{d}{dt}.$$

Total Response

Total Response

The response of the linear system (discussed above) can be expressed as the sum of two components: the zero-input component and the zero-state component (decomposition property).
Therefore

Total response = zero-input response + zero-state response

- the zero-input component is the system response when the input $f(t) = 0$ so that it is the result of internal system conditions (such as energy storages, initial conditions) alone.
- the zero-state component is the system response to the external input $f(t)$ when the system is in zero state, meaning the absence of all internal energy storages; that is all initial conditions are zero.

We can verify that the LTIC system has the decomposition property. If $y_0(t)$ is the zero-input response of the system, then, by definition

$$Q(D)y_0(t) = 0.$$

Total Response

If $y_i(t)$ is the zero-state response, then $y_i(t)$ is the solution of

$$Q(D)y_i(t) = P(D)f(t)$$

subject to zero initial conditions (zero-state). The addition of these two equations yields

$$Q(D)[y_0(t) + y_i(t)] = P(D)f(t).$$

Clearly, $y_0(t) + y(t)$ is the general solution of the linear system.

System Response to Internal Condition

The zero-input response $y_0(t)$ is the solution of the LTIC system when the input $f(t) = 0$ so that

$$\begin{aligned} Q(D)y_0(t) &= 0 \\ (D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)y_0(t) &= 0 \end{aligned} \tag{1}$$

- the last equation shows that a linear combination of $y_0(t)$ and its n successive derivatives is zero, not at some values of t but for all t.
- the result is possible if and only if $y_0(t)$ and all its n successive derivatives are of the same form. Other wise their sum can never add to zero for all values of t.

An exponential function $e^{\lambda t}$ is an only function has the property. Let us assume that

$$y_0(t) = ce^{\lambda t}$$

is a solution to Eq. (1).

System Response to Internal Condition

Zero-Input Response

Then

$$
\begin{aligned}
Dy_0(t) &= \frac{dy_0}{dt} = c\lambda e^{\lambda t} \\
D^2 y_0(t) &= \frac{d^2 y_0}{dt^2} = c\lambda^2 e^{\lambda t} \\
&\vdots \\
D^n y_0(t) &= \frac{d^n y_0}{dt^n} = c\lambda^n e^{\lambda t}
\end{aligned}
$$

Substituting these results in Eq. (1), we obtain

$$
c\left(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0\right) e^{\lambda t} = 0
$$

Zero-Input Response

For a nontrivial solution of this equation,

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \tag{2}$$

- this result means that $ce^{\lambda t}$ is indeed a solution of Eq. (1), provided that λ satisfies Eq. (2).
- this polynomial is identical to the polynomial $Q(D)$ in Eq. (1), with λ replacing D. Therefore $Q(\lambda) = 0$.
- $Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = 0$ **distinct roots**.
- λ has n solutions: $\lambda_1, \lambda_2, \ldots, \lambda_n$. Eq. (1) has n possible solutions: $c_1e^{\lambda_1 t}$,$c_2e^{\lambda_2 t}$,$\ldots$,$c_ne^{\lambda_n t}$, with $c_1, c_2, \ldots, c_n$ as arbitrary constants.

We can show that a general solution is given by the sum of these n solutions, so that

$$y_0(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + \cdots + c_ne^{\lambda_n t},$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants determined by n constraints (the auxiliary conditions) on the solution.

Zero-Input Response

- $Q(\lambda)$ is characteristic of the system, has nothing to do with the input.
- $Q(\lambda)$ is called the **characteristic polynomial** of the system.
- $Q(\lambda) = 0$ is called the **characteristic equation** of the system.

Distinct roots case.

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of the characteristic equation; they are called the **characteristic roots** of the system.
- we also called them **characteristic values, eigenvalues,** and **natural frequencies**.
- The exponentials $e^{\lambda_i t}(i = 1, 2, \ldots, n)$ in the zero-input response are the **characteristic modes** (also known as **modes** or **natural modes**) of the system.
- There is a characteristic mode for each characteristic root of the system, and the zero-input response is a linear combination of the characteristic modes of the system.
- The entire behavior of a system is dictated primarily by its characteristic modes.

Zero-Input Response: Repeated Roots

The solution of Eq. (1) assumes that the n characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. If there are **repeated roots**, the form of the solution is modified slightly. For example

$$(D-\lambda)^2 y_0(t) = (D^2 - 2\lambda D + \lambda^2) y_0(t) = 0,$$

by using distinct method, has

$$y_0(t) = c_1 e^{\lambda t} + c_2 e^{\lambda t} = (c_1 + c_2) e^{\lambda t} = c e^{\lambda t},$$

then there is an only one arbitrary constant. However, for a 2^{nd}-order differential equation, the solution must contain 2 arbitrary constants. To solve the problem, one can seek a second linearly independent solution. Try a solution of the form $y_0(t) = v(t) e^{\lambda t}$. Since

$$\begin{aligned} Dy_0 &= e^{\lambda t} Dv + \lambda v e^{\lambda t} = e^{\lambda t}(Dv + \lambda v), \\ D^2 y_0 &= e^{\lambda t} D^2 v + \lambda e^{\lambda t} Dv + \lambda^2 e^{\lambda t} v + \lambda e^{\lambda t} Dv \\ &= e^{\lambda t}(D^2 v + 2\lambda Dv + \lambda^2 v). \end{aligned}$$

Zero-Input Response: Repeated Roots

Substituting in the original equation yields

$$\begin{aligned} D^2 y_0 - 2\lambda D y_0 + \lambda^2 y_0 &= 0 \\ e^{\lambda t}\left(D^2 v + 2\lambda D v + \lambda^2 v\right) - 2\lambda e^{\lambda t}(Dv + \lambda v) + \lambda^2 v e^{\lambda t} &= 0 \\ e^{\lambda t} D^2 v &= 0 \end{aligned}$$

Hence $v(t)$ satisfies the differential equation $D^2 v = 0$. Integrating twice leads to

$$v(t) = c_1 + c_2 t.$$

The solution is then

$$y_0(t) = (c_1 + c_2 t)e^{\lambda t},$$

in which there two arbitrary constants.

- the root λ repeats twice. The characteristic modes in this case are $e^{\lambda t}$ and $te^{\lambda t}$.

Zero-Input Response: Repeated Roots

- for $(D-\lambda)^r y_0(t) = 0$ the characteristic modes are $e^{\lambda t}$, $te^{\lambda t}$, $t^2 e^{\lambda t}$, $\ldots$,$t^{r-t}e^{\lambda t}$, and that the solutions is

$$y_0(t) = (c_1 + c_2 t + \cdots + c_r t^{r-1})e^{\lambda t}.$$

Consequently, for a system with the characteristic polynomial

$$Q(\lambda) = \underbrace{(\lambda-\lambda_1)^r}_{r\text{repeated roots}} \overbrace{(\lambda-\lambda_{r+1})\cdots(\lambda-\lambda_n)}^{n-r\text{ distinct roots}}$$

the characteristic modes are $e^{\lambda_1 t}$,$te^{\lambda_1 t}$,$\ldots$,$t^{r-1}e^{\lambda_1 t}$,$\ldots$,$e^{\lambda_n t}$ and the solution is

$$y_0(t) = (c_1 + c_2 t + \cdots + c_r t^{r-1})e^{\lambda_1 t} + c_{r+1}e^{\lambda_{r+1} t} + \cdots + c_n e^{\lambda_n t}$$

Zero-Input Response: Complex roots

The procedure for handling complex roots is the same as that for real roots.

- for a real system, complex roots must occur in pairs of conjugates if the coefficients of the characteristic polynomial $Q(\lambda)$ are to be real.
- if $\alpha + j\beta$ is a characteristic root, $\alpha - j\beta$ must also be a characteristic root.
- the zero-input response corresponding to this pair of complex conjugate roots is

$$y_0(t) = c_1 e^{(\alpha+j\beta)t} + c_2 e^{(\alpha-j\beta)t}.$$

For a real system, the response $y_0(t)$ must also be real. This is possible only if c_1 and c_2 are conjugates. Let

$$c_1 = \frac{c}{2}e^{j\theta} \qquad \text{and} \qquad c_2 = \frac{c}{2}e^{-j\theta}$$

Zero-Input Response: Complex roots

This yields

$$\begin{aligned} y_0(t) &= \frac{c}{2}e^{j\theta}e^{(\alpha+j\beta)t} + \frac{c}{2}e^{-j\theta}e^{(\alpha-j\beta)t} \\ &= \frac{c}{2}e^{\alpha t}\left[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}\right] \\ &= ce^{\alpha t}\cos(\beta t+\theta) \end{aligned}$$

This form is more convenient because it avoids dealing with complex numbers.

Zero-Input Response: Example Distinct Roots

Find $y_0(t)$, the zero-input component of the response of an LTI system described by the following differential equation:

$$(D^2 + 3D + 2)y(t) = Df(t)$$

when the initial conditions are $y_0(0) = 0$, $\dot{y}_0(0) = -5$. Note that $y_0(t)$, being the zero-input component ($f(t) = 0$), is the solution of $(D^2 + 3D + 2)y_0(t) = 0$.

Solution: The characteristic polynomial of the system is
$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$ The characteristic roots of the system are $\lambda_1 = -1$ and $\lambda_2 = -2$, and the characteristic modes of the system are e^{-t} and e^{-2t}.
Consequently, the zero-input component of the loop current is

$$y_0(t) = c_1e^{-t} + c_2e^{-2t}$$

To determine the arbitrary constants c_1 and c_2, we differentiate above equation to obtain

$$\dot{y}_0(t) = -c_1e^{-t} - 2c_2e^{-2t}$$

Zero-Input Response: Example Distinct Roots

Setting $t = 0$ in both equations, and substituting the initial conditions $y_0(0) = 0$ and $\dot{y}(0) = -5$ we obtain

$$\begin{aligned} 0 &= c_1 + c_2 \\ -5 &= -c_1 - 2c_2. \end{aligned}$$

Solving these two simultaneous equations in two unknowns for c_1 and c_2 yields

$$c_1 = -5, \qquad c_2 = 5$$

Therefore

$$y_0(t) = -5e^{-t} + 5e^{-2t}$$

This is the zero-input component of $y(t)$ for $t \geq 0$.

Zero-Input Response: Example Distinct Roots

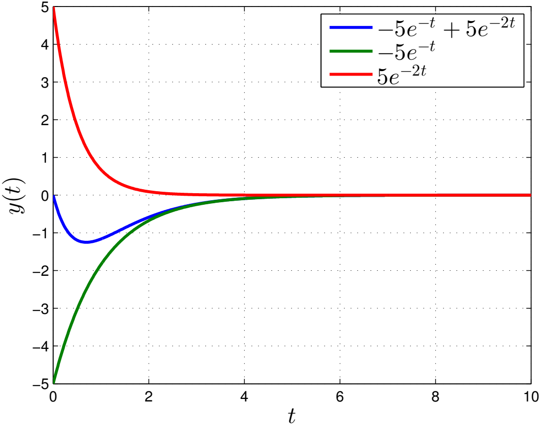


Figure 1: the plot of $y_0(t)$

For a system specified by

$$(D^2 + 6D + 9)y(t) = (3D + 5)f(t)$$

let us determine $y_0(t)$, the zero-input component of the response if the initial conditions are $y_0(0) = 3$ and $\dot{y}_0(0) = -7$.

Solution:

The characteristic polynomial is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$, and its characteristic roots are $\lambda_1 = -3, \lambda_2 = -3$ (repeated roots). Consequently, the characteristic modes of the system are e^{-3t} and te^{-3t}. The zero-input response, being a linear combination of the characteristic modes, is given by

$$y_0(t) = (c_1 + c_2 t)e^{-3t}.$$

The arbitrary constants c_1 and c_2 from the initial conditions $y_0(0) = 3$ and $\dot{y}(0) = -7$. From,

$$\dot{y}_0(t) = -3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t}$$

Zero-Input Response: Example repeated roots

Substituting the initial conditions, we obtain

$$\begin{aligned} 3 &= c_1 \\ -7 &= -3c_1 + c_2 \text{ and } c_2 = 2. \end{aligned}$$

Therefore

$$y_0(t) = (3 + 2t)e^{-3t}.$$

This is the zero-input component of $y(t)$ for $t \geq 0$.

Zero-Input Response: Example repeated roots

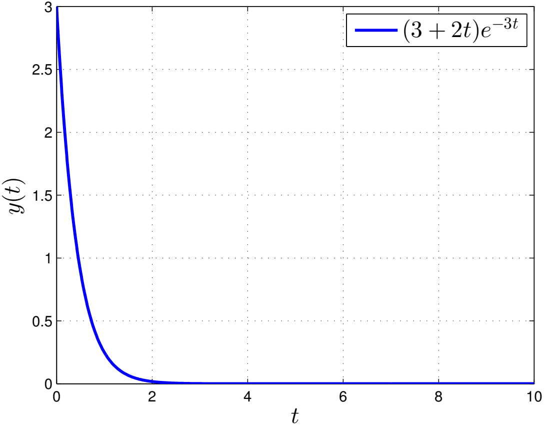


Figure 2: the plot of $y_0(t)$

Zero-Input Response: Example complex roots

Determine the zero-input response of an LTI system described by the equation:

$$(D^2 + 4D + 40)y(t) = (D + 2)f(t)$$

with initial conditions $y_0(0) = 2$ and $\dot{y}_0(0) = 16.78$.

Solution:
The characteristic polynomial is $\lambda^2 + 4\lambda + 40 = (\lambda + 2 - j6)(\lambda + 2 + j6)$. The characteristic roots are $-2 \pm j6$. The solution can be written either in the complex form or in the real form. The complex form is

Real form method:
Since $\alpha = -2$ and $\beta = 6$, the real form solution is

$$y_0(t) = ce^{-2t}\cos(6t + \theta)$$

where c and θ are arbitrary constants to be determined from the initial conditions $y_0(0) = 2$ and $\dot{y}_0(0) = 16.78$.

Zero-Input Response: Example complex roots

Differentiation of above equation yields

$$\dot{y}_0(t) = -2ce^{-2t}\cos(6t+\theta) - 6ce^{-2t}\sin(6t+\theta).$$

Setting $t = 0$ and then substituting initial conditions, we obtain

$$\begin{aligned} 2 &= c\cos\theta \\ 16.78 &= -2c\cos\theta - 6c\sin\theta. \end{aligned}$$

Solution of these two simultaneous equations in two unknowns $c\cos\theta$ and $c\sin\theta$ yields

$$\begin{aligned} c\cos\theta &= 2 \\ c\sin\theta &= -3.463. \end{aligned}$$

Squaring and then adding the two sides of the above equations yields

$$c^2 = (2)^2 + (-3.464)^2 = 16 \implies c = 4.$$

Zero-Input Response: Example complex roots

Next, dividing $c\sin\theta$ by $c\cos\theta$ yields

$$\tan\theta = \frac{-3.463}{2}$$

and

$$\theta = \tan^{-1}\left(\frac{-3.483}{2}\right) = -\frac{\pi}{3}$$

Therefore

$$y_0(t) = 4e^{-2t}\cos(6t - \frac{\pi}{3}).$$

Complex form method:

From

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{-(2-j6)t} + c_2 e^{-(2+j6)t}$$
$$= e^{-2t}\left(c_1 e^{j6t} + c_2 e^{-j6t}\right).$$

Using Euler's identities $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$, we obtain

$$y_0(t) = e^{-2t}\left(c_1(\cos 6t + j\sin 6t) + c_2(\cos 6t - j\sin 6t)\right)$$
$$= e^{-2t}\left((c_1 + c_2)\cos 6t + j(c_1 - c_2)\sin 6t\right) = e^{-2t}\left(K_1\cos 6t + K_2\sin 6t\right)$$

Since $y_0(t)$ is real, the coefficients of K_1 and K_2 must be real. This can be done by:

$$c_1 + c_2 = K_1 = 2a, \quad j(c_1 - c_2) = K_2 = -2b \Longrightarrow c_1 - c_2 = j2b,$$

a, b real constants or

$$c_1 = a + jb, \quad c_2 = a - jb$$

Zero-Input Response: Example Complex Roots

$$\dot{y}_0(t) = -2e^{-2t}\left(K_1\cos 6t + K_2\sin 6t\right) + e^{-2t}\left(-6K_1\sin 6t + 6K_2\cos 6t\right)$$

and

$$\dot{y}_0(0) = -2K_1 + 6K_2 = 16.78, \qquad y_0(0) = c_1 + c_2 = 2 \implies K_1 = 2, K_2 = 3.463.$$

Then,

$$\begin{aligned} y(t) &= e^{-2t}(2\cos 6t + 3.463\sin 6t) \\ &= 4e^{-2t}(0.5\cos 6t + 0.866\sin 6t), \qquad \cos\theta \le 1,\ \sin\theta \le 1 \\ &= 4e^{-2t}(\cos\frac{\pi}{3}\cos 6t + \sin\frac{\pi}{3}\sin 6t) \\ &= 4e^{-2t}\cos(6t - \frac{\pi}{3}) \end{aligned}$$

Zero-Input Response: Example Complex Roots

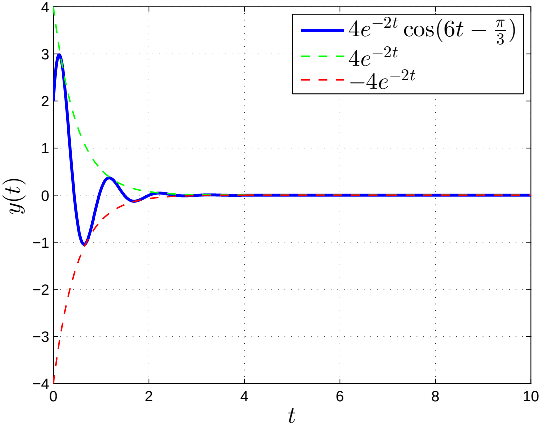
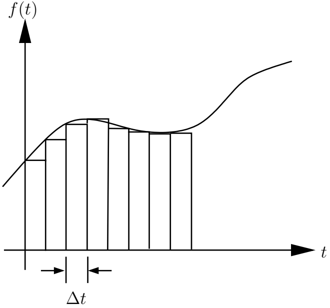


Figure 3: the plot of $y_0(t)$

System Response to External Input

The Unit Impulse Response $h(t)$

The impulse function $\delta(t)$ is also used in determining the response of a linear system to an arbitrary input $f(t)$.



We can approximate $f(t)$ with a sum of rectangular pulses of width Δt and of varying heights. The approximation improves as $\Delta t \to 0$, when the rectangular pulses become impulses. (Note : by using sampling property)

The Unit Impulse Response $h(t)$

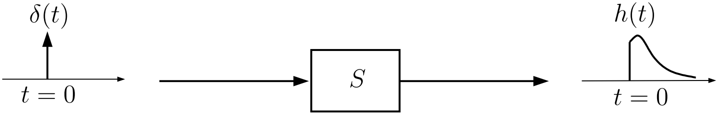
We can determine the system response to an arbitrary input $f(t)$, if we know the system response to an impulse input. The unit impulse response of an LTIC system described by the nth-order differential equation

$$Q(D)y(t) = P(D)f(t),$$

where $Q(D)$ and $P(D)$ are the polynomials. Generality, let $m = n$, we have

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)y(t) = \\ (b_n D^n + b_{n-1}D^{n-1} + \cdots + b_1 D + b_0)f(t)$$

The Unit Impulse Response $h(t)$



- an impulse input $\delta(t)$ appears momentarily at $t = 0$, and then it is gone forever.
- it generates energy storages; that is, it creates nonzero initial conditions instantaneously within the system at $t = 0^+$.
- the impulse response $h(t)$, therefore, must consist of the system's characteristic modes for $t \geq 0^+$ As a result

$$h(t) = \text{ characteristic mode terms} \qquad t \geq 0^+$$

The Unit Impulse Response $h(t)$: Characteristic modes

What happens at $t = 0$? At a single moment $t = 0$, there can at most be an impulse, so the form of the complete response $h(t)$ is given by

$$h(t) = A_0\delta(t) + \text{ characteristic mode terms} \qquad t \geq 0$$

Consider an LTIC system S specified by $Q(D)y(t) = P(D)f(t)$ or

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)y(t) = \\ (b_n D^n + b_{n-1}D^{n-1} + \cdots + b_1 D + b_0)f(t).$$

When the input $f(t) = \delta(t)$ the response $y(t) = h(t)$. Therefore, we obtain

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)h(t) = \\ (b_n D^n + b_{n-1}D^{n-1} + \cdots + b_1 D + b_0)\delta(t).$$

Substituting $h(t)$ with $A_0\delta(t)+$ characteristic modes, we have

$$A_0 D^n \delta(t) + \cdots = b_n D^n \delta(t) + \cdots .$$

The Unit Impulse Response $h(t)$: Characteristic modes cont.

Therefore, $A_0 = b_n$ and $h(t) = b_n\delta(t) +$ characteristic modes.

To find the characteristic mode terms, let us consider a system S_0 whose input $f(t)$ and the corresponding output $x(t)$ are related by

$$Q(D)x(t) = f(t).$$

Systems S and S_0 have the same characteristic polynomial. Moreover, S_0 has $P(D) = 1$, that is $b_n = 0$. Then the impulse response of S_0 consists of characteristic mode terms only without an impulse at

Let $y_n(t)$ is the response of S_0 to input $\delta(t)$. Therefore

$$\begin{aligned} Q(D)y_n(t) &= \delta(t) \\ (D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)y_n(t) &= \delta(t) \\ y_n^{(n)}(t) + a_{n-1}y_n^{(n-1)}(t) + \cdots + a_1 y_n^{(1)}(t) + a_0 y_n(t) &= \delta(t). \end{aligned}$$

The right-hand side contains a single impulse term $\delta(t)$. This is possible only if $y_n^{(n-1)}(t)$ has a unit jump discontinuity at $t = 0$, so that $y_n^{(n)}(t) = \delta(t)$.

The Unit Impulse Response $h(t)$: Characteristic modes cont.

The right-hand side contains a single impulse term $\delta(t)$. This is possible only if $y_n^{(n-1)}(t)$ has a unit jump discontinuity at $t = 0$, so that $y_n^{(n)}(t) = \delta(t)$. The lower-order terms cannot have any jump discontinuity because this would mean the presence of the derivatives of $\delta(t)$. Therefore, the n initial conditions on $y_n(t)$ are

$$y_n^{(n)}(0) = \delta(t),\ y_n^{(n-1)}(0) = 1$$
$$y_n(0) = y_n^{(1)}(0) = \cdots = y_n^{(n-2)}(0) = 0$$

In conclusion $y_n(t)$ is the zero-input response of the system S subject to initial conditions above.

Since

$$Q(D)x(t) = f(t) \ \Rightarrow \ P(D)Q(D)x(t) = P(D)f(t)$$
$$y(t) = P(D)x(t), \ \Rightarrow \ h(t) = P(D)[y_n(t)\mathbb{1}(t)],$$

where $y_n(t)$ is an characteristic mode of S_0 and we use $y_n(t)\mathbb{1}(t)$ because the impulse response is causal.

The Unit Impulse Response $h(t)$: Characteristic modes cont.

At the end,

$$h(t) = b_n\delta(t) + P(D)[y_n(t)\mathbb{1}(t)].$$

In gerneral, $m \leq n$, we can asserts that at $t = 0$, $h(t) = b_n\delta(t)$. Therefore,

$$\begin{aligned} h(t) &= b_n\delta(t) + P(D)y_n(t), \qquad t \geq 0 \\ &= b_n\delta(t) + [P(D)y_n(t)]\mathbb{1}(t), \end{aligned}$$

where c_n is the coefficient of the nth-order term in $P(D)$, and $y_n(t)$ is a linear combination of the characteristic modes of the system subject to the following initial conditions:

$$y_n^{(n-1)}(0) = 1, \text{ and } y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) = \cdots = y_n^{(n-2)}(0) = \cdots = 0$$

The Unit Impulse Response $h(t)$: Characteristic modes cont.

As an example, we can express this condition for various values of n (the system order) as follow:

$$n = 1 : y_n(0) = 1$$
$$n = 2 : y_n(0) = 0 \text{ and } \dot{y}_n(0) = 1$$
$$n = 3 : y_n(0) = \dot{y}_n(0) = 0 \text{ and } \ddot{y}_n(0) = 1$$
$$n = 4 : y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) = 0 \text{ and } \dddot{y}_n(0) = 1$$

and so on.

If the order of $P(D)$ is less than the order of $Q(D)$, $b_n = 0$, and the impulse term $b_n\delta(t)$ in $h(t)$ is zero.

The Unit Impulse Response $h(t)$: Example

Determine the unit impulse response $h(t)$ for a system specified by the equation

$$(D^2 + 3D + 2)y(t) = Df(t).$$

The system is a second-order system (n=2) having the characteristic polynomial

$$(\lambda^2 + 3\lambda + 2) = (\lambda + 1)(\lambda + 2) \text{ and } \lambda = -1, -2.$$

Therefore $y_n(t) = c_1 e^{-t} + c_2 e^{-2t}$ and $\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$.

To find the impulse response, we know that the initial conditions are

$$\dot{y}_n(0) = 1 \quad \text{and} \quad y_n(0) = 0.$$

Setting $t = 0$ and substituting the initial conditions, we obtain

$$0 = c_1 + c_2, \qquad 1 = -c_1 - 2c_2,$$

and $c_1 = 1$, $c_2 = -1$. Therefore $y_n(t) = e^{-t} - e^{-2t}$.

The Unit Impulse Response $h(t)$: Example cont.

From $P(D) = D$, so that

$$P(D)y_n(t) = Dy_n(t) = \dot{y}_n(t) = -e^{-t} + 2e^{-2t}.$$

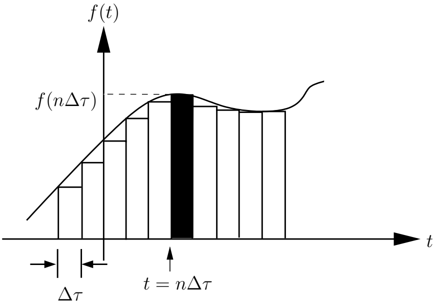
Also in this case, $c_n = c_2 = 0$ [the second-order term is absent in $P(D)$]. Therefore

$$h(t) = c_n\delta(t) + [P(D)y_n(t)]\mathbb{1}(t) = (-e^{-t} + 2e^{-2t})\mathbb{1}(t).$$

Zero-state Response

The zero-state response is the system response $y(t)$ to an input $f(t)$ when the system is in zero state; that is, when all initial conditions are zero.

- we use the superposition principle to derive a linear system's response to some arbitrary inputs $f(t)$.
- $f(t)$ is express in terms of impulses. $f(t)$ is a sum of rectangular pulses, each of width $\Delta\tau$.



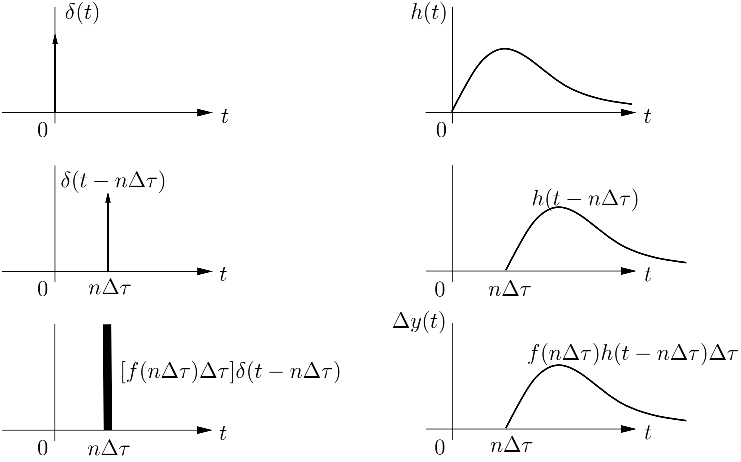
Zero-state Response

- As $\Delta\tau \to 0$, each pulse approaches an impulse having a strength equal to the area under the pulse. For example, the shaded rectangular pulse located at $t = n\Delta\tau$ will approach an impulse at the same location with strength $f(n\Delta\tau)\Delta\tau$ (area under pulse).
- This impulse can therefore be represented by $[f(n\Delta\tau)\Delta\tau]\delta(t - n\Delta\tau)$.
- the response to above input can be described by

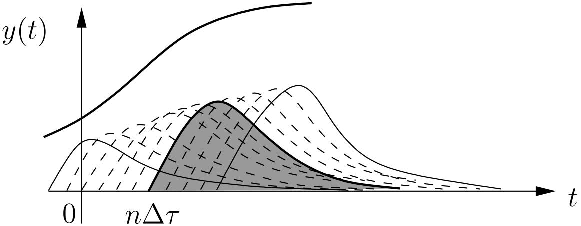
$$\begin{aligned}
\delta(t) &\Longrightarrow h(t) \\
\delta(t - n\Delta\tau) &\Longrightarrow h(t - n\Delta\tau) \\
\underbrace{[f(n\Delta\tau)\Delta\tau]\delta(t - n\Delta\tau)}_{\text{input}} &\Longrightarrow \underbrace{[f(n\Delta\tau)\Delta\tau]h(t - n\Delta\tau)}_{\text{output}}
\end{aligned}$$

Zero-state Response

Finding the system response to an arbitrary input $f(t)$



Zero-state Response



The total response $y(t)$ is obtained by summing all such components.

$$\lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} f(n\Delta\tau)\delta(t - n\Delta\tau)\Delta\tau \Longrightarrow \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} f(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau$$

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau \Longrightarrow y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

The Convolution Integral

The **convolution integral** of two functions $f_1(t)$ and $f_2(t)$ is denoted symbolically by $f_1(t) * f_2(t)$ and is defined as

$$f_1(t) * f_2(t) \triangleq \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

Some important properties of the convolution integral are given below:

1. **The Commutative Property:** Convolution operation operation is commutative; that is

$$\begin{aligned} f_1(t) * f_2(t) &= f_2(t) * f_1(t) \\ f_1(t) * f_2(t) &= \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau. \end{aligned}$$

If we let $x = t - \tau$ so that $\tau = t - x$ and $d\tau = -dx$, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau &= -\int_{\infty}^{-\infty} f_2(x) f_1(t-x) dx \\ &= \int_{-\infty}^{\infty} f_2(x) f_1(t-x) dx = f_2(t) * f_1(t) \end{aligned}$$

2. The Distributive Property:

$$\begin{aligned} f_1(t) * [f_2(t) + f_3(t)] &= \int_{-\infty}^{\infty} f_1(\tau)[f_2(t-\tau) + f_3(t-\tau)]d\tau \\ &= \int_{-\infty}^{\infty} \left[f_1(\tau)f_2(t-\tau) + f_1(\tau)f_3(t-\tau)\right] d\tau \\ &= f_1(t) * f_2(t) + f_1(t) * f_3(t) \end{aligned}$$

3. The Associative Property:

$$\begin{aligned} f_1(t) * [f_2(t) * f_3(t)] &= \int_{-\infty}^{\infty} f_1(\tau_1)[f_2 * f_3(t-\tau_1)]d\tau_1 \\ &= \int_{-\infty}^{\infty} f_1(\tau_1) \left[\int_{-\infty}^{\infty} f_2(\tau_2)f_3(t-\tau_1-\tau_2)d\tau_2\right] d\tau_1 \end{aligned}$$

Let $\lambda = \tau_1 + \tau_2$ and $d\lambda = d\tau_2$ (we consider τ_1 as a constant when we integrate a function with respect to τ_2). Then

$$\begin{aligned} f_1(t) * [f_2(t) * f_3(t)] &= \int_{-\infty}^{\infty} f_1(\tau_1) \left[\int_{-\infty}^{\infty} f_2(\lambda - \tau_1) f_3(t - \lambda) d\lambda \right] d\tau_1 \\ &= \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f_1(\tau_1) f_2(\lambda - \tau_1) d\tau_1 \right]}_{f_1 * f_2(\lambda)} f_3(t - \lambda) d\lambda \\ &= [f_1(t) * f_2(t)] * f_3(t) \end{aligned}$$

4 Convolution with an Impulse:

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau) d\tau.$$

It is obvious to see that $f(t) * \delta(t) = f(t)$ ($\delta(t - \tau)$ is an impulse located at $\tau = t$, the integral in the above equation is the value of $f(\tau)$ at $\tau = t$). Then

$$f(t - T) = \int_{-\infty}^{\infty} f(\tau)\delta(t - T - \tau) d\tau = f(t) * \delta(t - T).$$

5 **The Shift Property:**

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau = c(t).$$

Then

$$\begin{aligned}
f_1(t) * f_2(t-T) &= f_1(t) * f_2(t) * \delta(t-T) = c(t) * \delta(t-T) \\
&= c(t-T) \\
f_1(t-T) * f_2(t) &= f_1(t) * \delta(t-T) * f_2(t) = f_1(t) * f_2(t) * \delta(t-T) \\
&= c(t-T) \\
f_1(t-T_1) * f_2(t-T_2) &= f_1(t) * \delta(t-T_1) * f_2(t) * \delta(t-T_2) \\
&= f_1(t) * f_2(t) * \delta(t-T_1) * \delta(t-T_2) \\
&= c(t-T_1-T_2)
\end{aligned}$$

6. **The Width Property:** If the durations (width) of $f_1(t)$ and $f_2(t)$ are T_1 and T_2 respectively, then the duration of $f_1(t) * f_2(t)$ is $T_1 + T_2$.

The proof of this property follows readily from the graphical considerations discussed later.

Zero-state Response and Causality

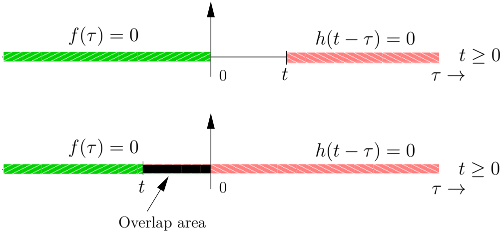
The (zero-state) response $y(t)$ of an LTIC system is

$$y(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau.$$

In practice, most systems are causal, so that their response cannot begin before the input starts. Furthermore, most inputs are also causal, which means they start at $t = 0$.

By definition, the response of a causal system cannot begin before its input begins. Consequently, the causal system's response to a unit impulse $\delta(t)$ (which is located at $t = 0$) cannot begin before $t = 0$. Therefore, a *causal system's unit impulse response* $h(t)$ is a *causal signal*.

Zero-state Response and Causality



- $f(t)$ is causal, $f(\tau) = 0$ for $\tau < 0$. If $h(t)$ is causal, $h(t-\tau) = 0$ for $t - \tau < 0$
- Therefore, the product $f(T)h(t-T) = 0$ everywhere except over the nonshaded interval $0 < \tau < t$. If t is negative, $f(\tau)h(t-\tau) = 0$ for all τ. Then,

$$y(t) = f(t) * h(t) = \begin{cases} \int_0^t f(\tau)h(t-\tau)d\tau & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

Zero-state Response and Causality: Examples

For an LTIC system with the unit impulse response $h(t) = e^{-2t}u(t)$, determine the response $y(t)$ for the input

$$f(t) = e^{-t}\mathbb{1}(t).$$

Here both $f(t)$ and $h(t)$ are causal. Hence, the system response is given by

$$\begin{aligned}
y(t) &= \int_0^t f(\tau)h(t-\tau)d\tau, \qquad t \geq 0 \\
&= \int_0^t e^{-\tau}e^{-2(t-\tau)}d\tau, \qquad t \geq 0 \\
&= e^{-2t}\int_0^t e^{\tau}d\tau = e^{-2t}\ e^{\tau}\Big|_0^t, \qquad t \geq 0 \\
&= e^{-2t}(e^t - 1) = e^{-t} - e^{-2t}, \qquad t \geq 0
\end{aligned}$$

Also, $y(t) = 0$ when $t < 0$. This result yields

$$y(t) = (e^{-t} - e^{-2t})\mathbb{1}(t).$$

Zero-state Response and Causality: Examples

Find the loop current $y(t)$ of the RLC circuit for the input $f(t) = 10e^{-3t}\mathbb{1}(t)$, when all the initial conditions are zero. If the loop equation of the circuit is

$$(D^2 + 3D + 2)y(t) = Df(t).$$

The impulse response $h(t)$ for this system, from the previous RLC example, is

$$h(t) = \left(2e^{-2t} - e^{-t}\right)\mathbb{1}(t).$$

The response $y(t)$ to the input $f(t)$ is

$$\begin{aligned} y(t) &= f(t) * h(t) = 10e^{-3t}\mathbb{1}(t) * \left[2e^{-2t} - e^{-t}\right]\mathbb{1}(t) \\ &= 10e^{-3t}\mathbb{1}(t) * 2e^{-2t}\mathbb{1}(t) - 10e^{-3t}\mathbb{1}(t) * e^{-t}\mathbb{1}(t) \\ &= 20\left[e^{-3t}\mathbb{1}(t) * e^{-2t}\mathbb{1}(t)\right] - 10\left[e^{-3t}\mathbb{1}(t) * e^{-t}\mathbb{1}(t)\right] \end{aligned}$$

Zero-state Response and Causality: Examples

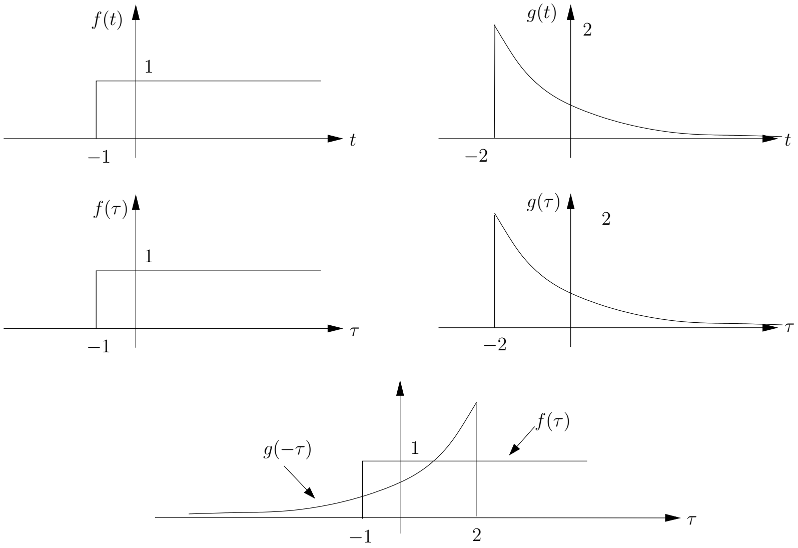
Using a pair 4 in the convolution table,

No	$f_1(t)$	$f_2(t)$	$f_1(t) * f_2(t) = f_2(t) * f_1(t)$
4	$e^{\lambda_1 t}\mathbb{1}(t)$	$e^{\lambda_2 t}\mathbb{1}(t)$	$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\mathbb{1}(t) \quad \lambda_1 \neq \lambda_2$

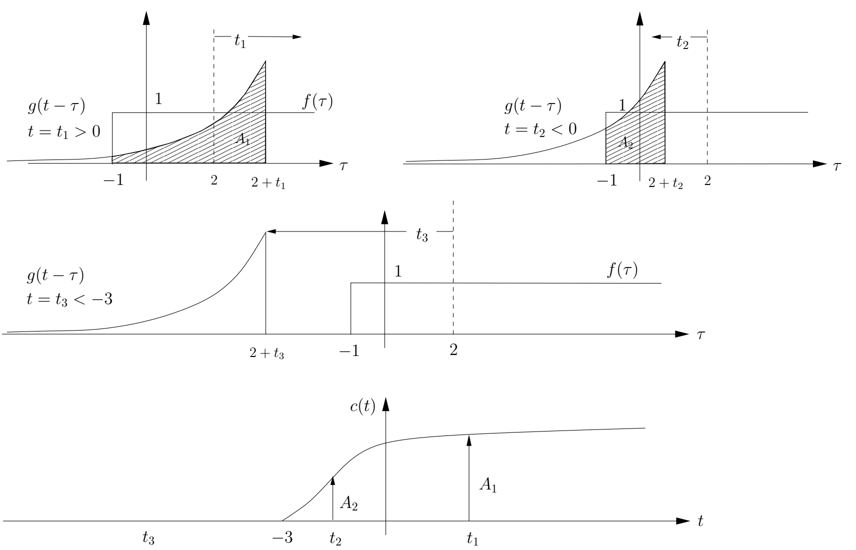
, yields

$$\begin{aligned} y(t) &= \frac{20}{-3-(-2)}\left[e^{-3t} - e^{-2t}\right]\mathbb{1}(t) - \frac{10}{-3-(-1)}\left[e^{-3t} - e^{-t}\right]\mathbb{1}(t) \\ &= -20\left(e^{-3t} - e^{-2t}\right)\mathbb{1}(t) + 5\left(e^{-3t} - e^{-t}\right)\mathbb{1}(t) \\ &= \left(-5e^{-t} + 20e^{-2t} - 15e^{-3t}\right)\mathbb{1}(t) \end{aligned}$$

Graphical Understanding of Convolution



Graphical Understanding of Convolution cont.



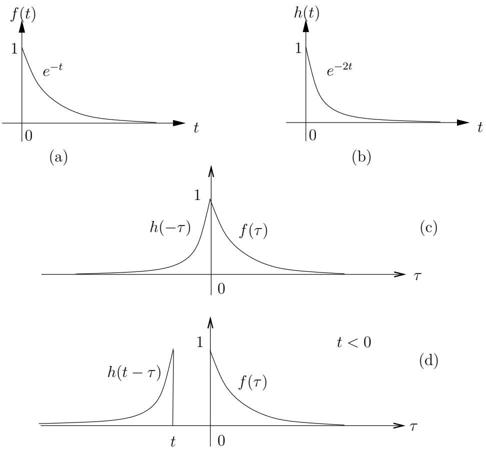
Graphical Understanding of Convolution cont.

Summary of the Graphical Procedure:

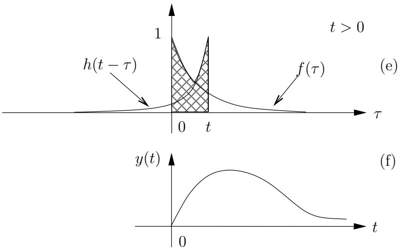
1. Keep the function $f(\tau)$ fixed.
2. Visualize the function $g(\tau)$ as a rigid wire frame, and rotate (or invert) this frame about the vertical axis ($\tau = 0$) to obtain $g(-\tau)$.
3. Shift the inverted frame along the τ axis by t_0 seconds. The shifted frame now represents $g(t_0 - \tau)$.
4. The area under the product of $f(\tau)$ and $g(t_0 - \tau)$ (the shifted frame) is $c(t_0)$, the value of the convolution at $t = t_0$.
5. Repeat this procedure, shifting the frame by different values (positive and negative) to obtain $c(t)$ for all values of t.

Graphical Understanding of Convolution cont.

Determine graphically $y(t) = f(t) * h(t)$ for $f(t) = e^{-t}\mathbb{1}(t)$ and $h(t) = e^{-2t}\mathbb{1}(t)$.



Graphical Understanding of Convolution cont.



The function $h(t-T)$ is now obtained by shifting $h(-T)$ by t. If t is positive, the shift is to the right (delay); if t is negative, the shift is to the left (advance). When $t < 0$, $h(-T)$ does not overlap $f(\tau)$, and the product $f(\tau)h(t-\tau) = 0$, so that

$$y(t) = 0, \qquad t < 0$$

Graphical Understanding of Convolution cont.

Figure (e) shows the situation for $t \geq 0$. Here $f(T)$ and $h(t-T)$ do overlap, but the product is nonzero only over the interval $0 \leq T \leq t$ (shaded interval). Therefore

$$y(t) = \int_0^t f(\tau)h(t-\tau)d\tau, \qquad t \geq 0.$$

Therefore $f(\tau) = e^{-\tau}$ and $h(t-\tau) = e^{-2(t-\tau)}$.

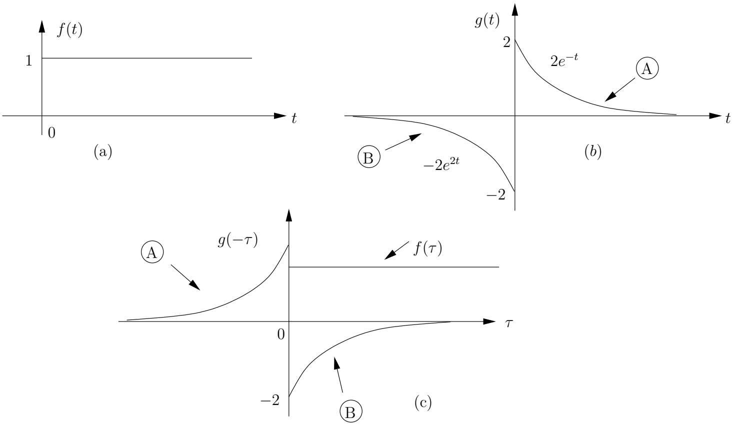
$$\begin{aligned} y(t) &= \int_0^t e^{-\tau} e^{-2(t-\tau)} d\tau \\ &= e^{-2t} \int_0^t e^{\tau} d\tau = e^{-2t} \; e^{\tau}|_0^t = e^{-2t}(e^t - 1) \\ &= e^{-t} - e^{-2t}, \qquad t \geq 0. \end{aligned}$$

Moreover, $y(t) = 0$ for $t < 0$, so that

$$y(t) = (e^{-t} - e^{-2t})\mathbb{1}(t).$$

Graphical Understanding of Convolution: Examples

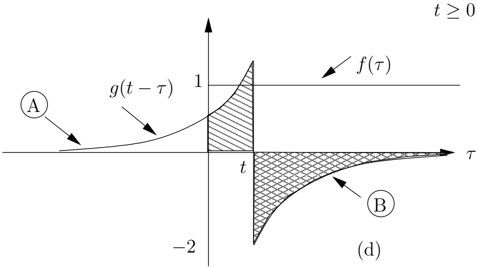
Find $f(t) * g(t)$ for the functions $f(t)$ and $g(t)$ shown in Figures below. Here $f(t)$ has a simpler mathematic description than that of $g(t)$, so it is preferable to invert $f(t)$. Hence, we shall determine $c(t) = g(t) * f(t)$.



Graphical Understanding of Convolution: Examples

Compute $c(t)$ for $t \geq 0$:

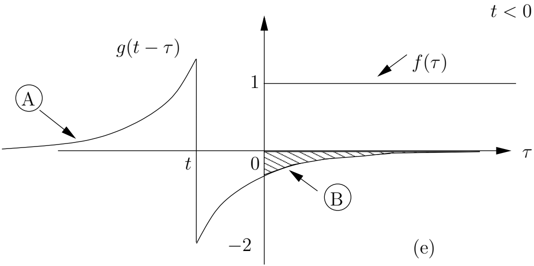
$$
\begin{aligned}
c(t) &= \int_0^\infty f(\tau)g(t-\tau)d\tau \\
&= \int_0^t 2e^{-(t-\tau)}d\tau + \int_t^\infty -2e^{2(t-\tau)}d\tau \\
&= 2(1-e^{-t}) - 1 \\
&= 1 - 2e^{-t}, \qquad t \geq 0.
\end{aligned}
$$



Graphical Understanding of Convolution: Examples

Compute $c(t)$ for $t < 0$:

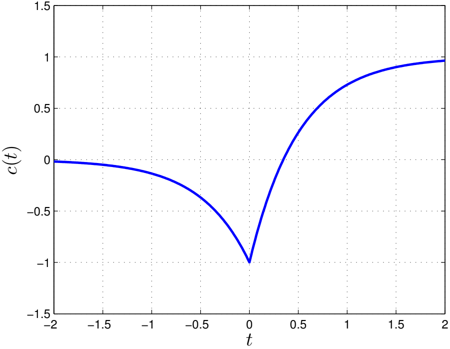
$$
\begin{aligned}
c(t) &= \int_0^\infty f(\tau)g(t-\tau)d\tau = \int_0^\infty g(t-\tau)d\tau \\
&= \int_0^\infty -2e^{2(t-\tau)}d\tau \\
&= -e^{2t}, \qquad t < 0
\end{aligned}
$$



Graphical Understanding of Convolution: Examples

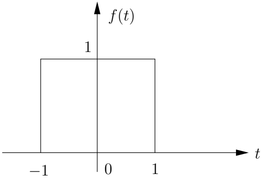
Therefore

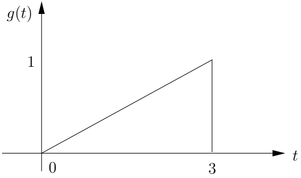
$$c(t) = \begin{cases} 1 - 2e^{-2t} & , t \geq 0 \\ -e^{2t} & , t < 0 \end{cases}$$

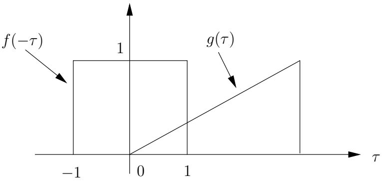


Graphical Understanding of Convolution: Examples

Find $f(t) * g(t)$ for the functions $f(t)$ and $g(t)$. $f(t)$ has a simpler mathematical description than that of $g(t)$. Hence we shall determine $g(t) * f(t)$.



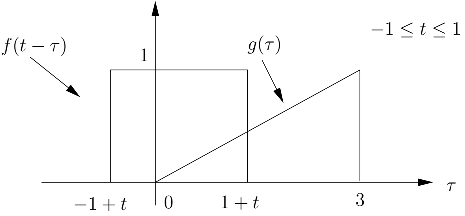




Graphical Understanding of Convolution: Examples

For $-1 \leq t \leq 1$:

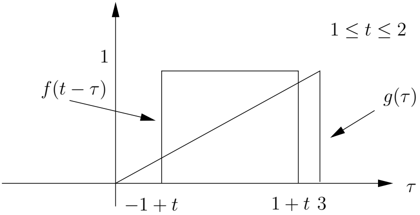
$$
\begin{aligned}
c(t) &= \int_0^{1+t} g(\tau) f(t-\tau) d\tau \\
&= \int_0^{1+t} \frac{1}{3}\tau d\tau \\
&= \frac{1}{6}(t+1)^2, \qquad -1 \leq t \leq 1
\end{aligned}
$$



Graphical Understanding of Convolution: Examples

For $1 \leq t \leq 2$:

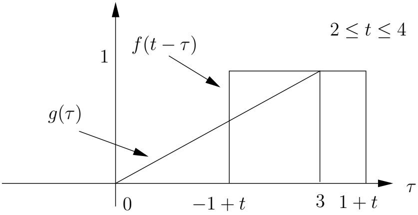
$$c(t) = \int_{-1+t}^{1+t} \frac{1}{3}\tau d\tau$$
$$= \frac{2}{3}t, \qquad 1 \leq t \leq 2$$



Graphical Understanding of Convolution: Examples

For $2 \leq t \leq 4$:

$$c(t) = \int_{-1+t}^{3} \frac{1}{3}\tau d\tau$$
$$= -\frac{1}{6}(t^2 - 2t - 8)$$



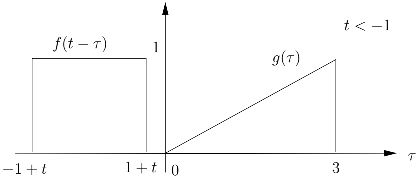
Graphical Understanding of Convolution: Examples

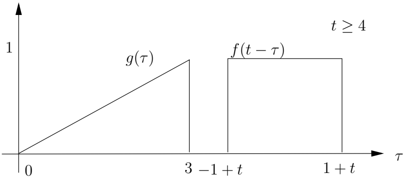
For $t \geq 4$:

$$c(t) = 0, \qquad t \geq 4.$$

For $t < -1$:

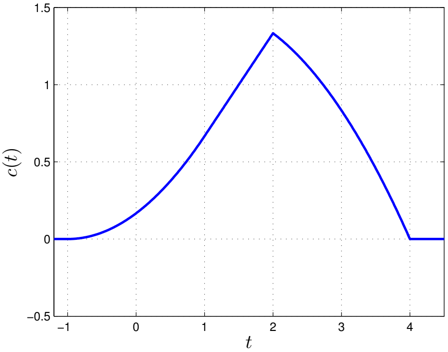
$$c(t) = 0, \qquad t < -1.$$





Graphical Understanding of Convolution: Examples

$$c(t) = \begin{cases} 0 & , t < -1 \\ \frac{1}{6}(t+1)^2 & , -1 \le t \le 1 \\ \frac{2}{3}t & , 1 \le t \le 2 \\ -\frac{1}{6}(t^2 - 2t - 8) & , 2 \le t \le 4 \\ 0 & , t \ge 4. \end{cases}$$



Total Response

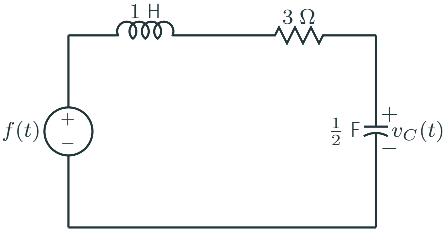
Total Response

The total response of a linear system can be expressed as the sum of its zero-input and zero-state components:

$$\text{Total Response} = \underbrace{\sum_{j=1}^{n} c_j e^{\lambda_j t}}_{\text{zero-input component}} + \underbrace{f(t) * h(t)}_{\text{zero-state component}}$$

For repeated roots, the zero-input component should be appropriately modified.

Total Response



For the series RLC circuit with the input $f(t) = 10e^{-3t}\mathbb{1}(t)$ and the initial conditions $y(0^-) = 0$, $v_C(0^-) = 5$, from the previous RLC examples, we obtain

$$\text{Total current } = \underbrace{(-5e^{-t} + 5e^{-2t})}_{\text{zero-input current}} + \underbrace{(-5e^{-t} + 20e^{-2t} - 15e^{-3t})}_{\text{zero-state current}}, \quad t \geq 0$$

Total Response

From the RLC circuit above, the characteristic modes were found to be e^{-t} and e^{-2t}. The zero-input response is composed exclusively of characteristic modes. However, the zero-state response contains also characteristic mode terms.

- If we lump all the characteristic mode terms in the total response together, giving us a component known as the **natural response** $y_n(t)$.
- The remainder, consisting entirely of noncharacteristic mode terms, is known as the **forced response** $y_\phi(t)$.

$$\text{Total current} = \underbrace{(-10e^{-t} + 25e^{-2t})}_{\text{natural response } y_n(t)} + \underbrace{(-15e^{-3t})}_{\text{forced response } y_\phi(t)}, \quad t \geq 0$$

Natural and Forced response cont.

The total system response is $y(t) = y_n(t) + y_\phi(t)$.

- $y_n(t)$ is the system's **natural response** (also known as the **homogeneous solution** or **complementary solution**).
- $y_\phi(t)$ is the system's **forced response** (also known as the **particular solution**).

Since $y(t)$ must satisfy the system equation,

$$Q(D)[y_n(t) + y_\phi(t)] = P(D)f(t)$$

or

$$Q(D)y_n(t) + Q(D)y_\phi(t) = P(D)f(t)$$

Natural and Forced response cont.

However $y_n(t)$ is composed entirely of characteristic modes. Therefore

$$Q(D)y_n(t) = 0$$

so that

$$Q(D)y_\phi(t) = P(D)f(t)$$

- The natural response, being a linear combination of the system's characteristic modes, has the same form as that of the zero-input response; only its arbitrary constants are different.

Forced response: The Method of Undetermined Coefficients

- The forced response of an LTIC system, when the input $f(t)$ is such that it yields only a finite number of independent derivatives.
- $e^{\zeta t}$ has only one independent derivative; the repeated differentiation of $e^{\zeta t}$ yields the same form as this input; that is, $e^{\zeta t}$.
- the repeated differentiation of t^r yields only r independent derivatives. For example, the input $at^2 + bt + c$, the suitable form for $y_\phi(t)$ in this case is, therefore

$$y_\phi(t) = \beta_2 t^2 + \beta_1 t + \beta_0.$$

The undetermined coefficients β_0, β_1, and β_2 are determined by substituting this expression for $y_\phi(t)$

$$Q(D) y_\phi(t) = P(D) f(t).$$

Forced response: The Method of Undetermined Coefficients cont.

	Input $f(t)$	Forced Response
1.	$e^{\zeta t}$ $\zeta \neq \lambda_i (i = 1, 2, \cdots, n)$	$\beta e^{\zeta t}$
2.	$e^{\zeta t}$ $\zeta = \lambda_i$	$\beta t e^{\zeta t}$
3.	k	β
4.	$\cos(\omega t + \theta)$	$\beta \cos(\omega t + \phi)$
5.	$(t^r + \alpha_{r-1} t^{r-1} + \cdots + \alpha_1 t + \alpha_0) e^{\zeta t}$	$(\beta_r t^r + \beta_{r-1} t^{r-1} + \cdots + \beta_1 t$ $+ \beta_0) e^{\zeta t}$

- $y_\phi(t)$ cannot have any characteristic mode terms.
- if the characteristic mode terms appearing in forced response, the correct form of the forced response must be modified to $t^i y_\phi(t)$.

Classical method: Examples

Solve the differential equation

$$(D^2 + 3D + 2)y(t) = Df(t)$$

if the input

$$f(t) = t^2 + 5t + 3$$

and the initial conditions are $y(0^+) = 2$ and $\dot{y}(0^+) = 3$.

Solution:

The characteristic polynomial of the system is

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2).$$

The natural response is then a linear combination of these modes, so that

$$y_n(t) = K_1 e^{-t} + K_2 e^{-2t}, \qquad t \geq 0.$$

The arbitrary constants K_1 and K_2 must be determined from the system's initial conditions.

Classical method: Examples

The forced response to the input $t^2 + 5t + 3$, is (from the previous table)

$$y_\phi(t) = \beta_2 t^2 + \beta_1 t + \beta_0.$$

$y_\phi(t)$ satisfies the system equation; that is

$$\begin{aligned}
(D^2 + 3D + 2)y_\phi(t) &= Df(t) \\
Dy_\phi(t) &= \frac{d}{dt}(\beta_2 t^2 + \beta_1 t + \beta_0) = 2\beta_2 t + \beta_1 \\
D^2 y_\phi(t) &= \frac{d^2}{dt^2}(\beta_2 t^2 + \beta_1 t + \beta_0) = 2\beta_2 \\
Df(t) &= \frac{d}{dt}\left[t^2 + 5t + 3\right] = 2t + 5.
\end{aligned}$$

Substituting these results yields

$$\begin{aligned}
2\beta_2 + 3(2\beta_2 t + \beta_1) + 2(\beta_2 t^2 + \beta_1 t + \beta_0) &= 2t + 5 \\
2\beta_2 t^2 + (2\beta_1 + 6\beta_2)t + (2\beta_0 + 3\beta_1 + 2\beta_2) &= 2t + 5
\end{aligned}$$

Classical method: Examples

Equating coefficients of similar powers of both sides of this expression yields

$$\begin{aligned} 2\beta_2 &= 0 \\ 2\beta_1 + 6\beta_2 &= 2 \\ 2\beta_0 + 3\beta_1 + 2\beta_2 &= 5. \end{aligned}$$

Solving these three equations for their unknowns, we obtain $\beta_0 = 1, \beta_1 = 1$, and $\beta_2 = 0$. Therefore

$$y_\phi(t) = t + 1, \qquad t > 0.$$

The total system response $y(t)$ is the sum of the natural of forced solutions. Therefore

$$\begin{aligned} y(t) &= y_n(t) + y_\phi(t) = K_1 e^{-t} + K_2 e^{-2t} + t + 1, \qquad t > 0 \\ \dot{y}(t) &= -K_1 e^{-t} - 2K_2 e^{-2t} + 1. \end{aligned}$$

Classical method: Examples

Setting $t = 0$ and substituting $y(0) = 2$ and $\dot{y}(0) = 3$ in these equations, we have

$$\begin{aligned} 2 &= K_1 + K_2 + 1 \\ 3 &= -K_1 - 2K_2 + 1. \end{aligned}$$

The solution of these two simultaneous equations is $K_1 = 4$ and $K_2 = -3$. Therefore

$$y(t) = 4e^{-t} - 3e^{-2t} + t + 1, \qquad t \geq 0.$$

Classical method: Examples

Solve the differential equation

$$(D^2 + 3D + 2)y(t) = Df(t)$$

if the initial conditions are $y(0^+) = 2$ and $\dot{y}(0^+) = 3$ and the input is
(a) $10e^{-3t}$ (b) 5 (c) e^{-2t} (d) $10\cos(3t + 30°)$
From the previous example, the natural response for this case is

$$y_n(t) = K_1 e^{-t} + K_2 e^{-2t}$$

(a) For input $f(t) = 10e^{-3t}, \zeta = -3$, and

$$\begin{aligned}
y_\phi(t) &= \beta e^{-3t} \\
(D^2 + 3D + 2)y_\phi(t) &= Df(t) \\
9\beta e^{-3t} - 9\beta e^{-3t} + 2\beta e^{-3t} &= -30e^{-3t} \\
2\beta &= -30, \qquad \beta = -15 \\
y_\phi(t) &= -15e^{-3t}
\end{aligned}$$

Classical method: Examples

$$y(t) = K_1 e^{-t} + K_2 e^{-2t} - 15e^{-3t}, \qquad t > 0$$
$$\dot{y}(t) = -K_1 e^{-t} - 2K_2 e^{-2t} + 45e^{-3t}, \qquad t > 0$$

The initial conditions are $y(0^+) = 2$ and $\dot{y}(0^+) = 3$. Setting $t = 0$ in the above equations and then substituting the initial conditions yields

$$K_1 + K_2 - 15 = 2 \qquad \text{and} \qquad -K_1 - 2K_2 + 45 = 3$$

Solution of these equations yields $K_1 = -8$ and $K_2 = 25$. Therefore

$$y(t) = -8e^{-t} + 25e^{-2t} - 15e^{-3t}, \qquad t > 0$$

Classical method: Examples

For input $f(t) = 5 = 5e^{0t}$, $\zeta = 0$, and $y_\phi(t) = \beta$.

$$(D^2 + 3D + 2)y_\phi(t) = Df(t)$$
$$0 + 0 + 2\beta = 0, \qquad \beta = 0$$

and

$$y(t) = K_1 e^{-t} + K_2 e^{-2t}, \qquad t > 0$$
$$\dot{y}(t) = -K_1 e^{-t} - 2K_2 e^{-2t}, \qquad t > 0$$

Setting $t = 0$ in the above equations and then substituting the initial conditions yields

$$K_1 + K_2 = 2 \qquad \text{and} \qquad -K_1 - 2K_2 = 3$$

Solution of this equations yields $K_1 = 7$ and $K_2 = -5$. Therefore

$$y(t) = 7e^{-t} - 5e^{-2t}, \qquad t > 0$$

Classical method: Examples

(c) Here $\zeta = -2$, which is also a characteristic root of the system. Hence $y_\phi(t) = \beta t e^{-2t}$ and

$$\begin{aligned}(D^2 + 3D + 2)y_\phi(t) &= Df(t)\\ D\left[\beta t e^{-2t}\right] &= \beta(1-2t)e^{-2t}\\ D^2\left[\beta t e^{-2t}\right] &= 4\beta(t-1)e^{-2t}\\ De^{-2t} &= -2e^{-2t}.\end{aligned}$$

Consequently

$$\begin{aligned}\beta(4t - 4 + 3 - 6t + 2t)e^{-2t} &= -2e^{-2t}\\ -\beta e^{-2t} &= -2e^{-2t}\end{aligned}$$

Therefore, $\beta = 2$ so that $y_\phi(t) = 2te^{-2t}$. The complete solution is $K_1 e^{-t} + K_2 e^{-2t} + 2te^{-2t}$.

Classical method: Examples

Then,

$$y(t) = K_1 e^{-t} + K_2 e^{-2t} + 2te^{-2t}, \qquad t > 0$$
$$\dot{y}(t) = -K_1 e^{-t} - 2K_2 e^{-2t} + 2e^{-2t} - 4te^{-2t}, \qquad t > 0$$

Setting $t = 0$ in the above equations and then substituting the initial conditions yields

$$K_1 + K_2 = 2 \qquad \text{and} \qquad -K_1 - 2K_2 = 1$$

Solution of this equations yields $K_1 = 5$ and $K_2 = -3$. Therefore

$$y(t) = 5e^{-t} - 3e^{-2t} + 2te^{-2t}, \qquad t > 0$$

Classical method: Examples

(d) For the input $f(t) = 10\cos(3t + 30^\circ)$, the forced response is $y_\phi(t) = \beta\cos(3t + \phi)$ and

$$\begin{aligned}
(D^2 + 3D + 2)y_\phi(t) &= Df(t) \\
D(\beta\cos(3t + \phi)) &= -3\beta\sin(3t + \phi) \\
D^2(\beta\cos(3t + \phi)) &= -9\beta\cos(3t + \phi) \\
D(10\cos(3t + 30^\circ)) &= -30\sin(3t + 30^\circ).
\end{aligned}$$

Consequently

$$\begin{aligned}
-9\beta\cos(3t + \phi) - 9\beta\sin(3t + \phi) + 2\beta\cos(3t + \phi) &= -30\sin(3t + 30^\circ) \\
\beta(-7\cos(3t + \phi) - 9\sin(3t + \phi)) &= -30\sin(3t + 30^\circ) \\
-\beta(C\sin(\theta_1)\cos(3t + \phi) + C\cos(\theta_1)\sin(3t + \phi)) &= -30\sin(3t + 30^\circ) \\
C = \sqrt{7^2 + 9^2} = 11.4018, \qquad \theta_1 = \tan^{-1}\left(\frac{7}{9}\right) &= 37.9^\circ \\
\beta = 30/11.4018 = 2.63, \qquad \phi + 37.9^\circ = 30^\circ \text{ and } \phi &= -7.9^\circ \\
y_\phi(t) &= 2.63\cos(3t - 7.9^\circ)
\end{aligned}$$

Classical method: Examples

Then

$$y(t) = K_1 e^{-t} + K_2 e^{-2t} + 2.63\cos(3t - 7.9^\circ)$$
$$\dot{y}(t) = -K_1 e^{-t} - 2K_2 e^{-2t} - 7.89\sin(3t - 7.9^\circ)$$

Setting $t = 0$ in the above equations and then substituting the initial conditions yields

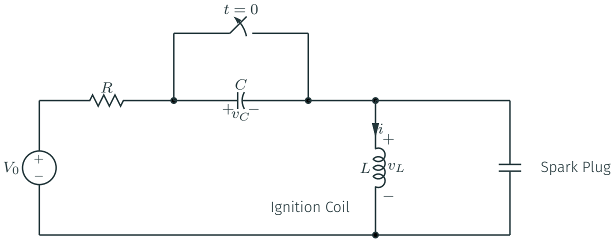
$$K_1 + K_2 = -0.6 \qquad \text{and} \qquad -K_1 - 2K_2 = 1.9$$

Solution of this equations yields $K_1 = 0.7$ and $K_2 = -1.3$. Therefore

$$y(t) = 0.7e^{-t} - 1.3e^{-2t} + 2.63\cos(3t - 7.9^\circ), \qquad t > 0.$$

Applications: Automobile Ignition Circuit

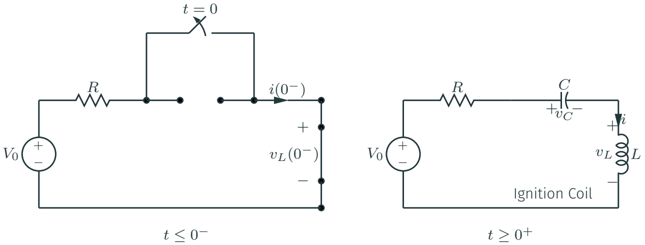
An automobile ignition system is modeled by the circuit shown in the following figure. The voltage source V_0 represents the battery and alternator. The resistor R models the resistance of the wiring, and the ignition coil is modeled by the inductor L. The capacitor C, known as the condenser, is in parallel with the switch, which is known as the electronic ignition. The switch has been closed for a long time prior to $t < 0^-$. Determine the inductor voltage v_L for $t > 0$.



For $V_0 = 12$ V, $R = 4\ \Omega$, $C = 1\ \mu$F, $L = 8$ mH, determine the maximal inductor voltage and the time when it is reached.

Applications: Automobile Ignition Circuit cont.

For $t < 0$, the switch is closed, the capacitor behaves as an open circuit and the inductor behaves as a short circuit as shown. Hence $i(0^-) = V_0/R, v_C(0^-) = 0$.



At $t = 0$, the switch is opened. Since the current in an inductor and the voltage across a capacitor cannot change abruptly, one has
$i(0^+) = i(0^-) = V_0/R = 3$ A, $v_C(0^+) = v_C(0^-) = 0$. The derivative $i'(0^+)$ is obtained from $v_L(0^+)$, which is determined by applying Kirchhoff's Voltage Law to the mesh at $t = 0^+$:

$$-V_0 + Ri(0^+) + v_C(0^+) + v_L(0^+) = 0 \implies v_L(0^+) = V_0 - Ri(0^+) = 0,$$

Applications: Automobile Ignition Circuit cont.

$$v_L(0^+) = L\frac{di(0^+)}{dt} \implies i'(0^+) = \frac{v_L(0^+)}{L} = 0.$$

For $t > 0$, applying Kirchhoff's Voltage Law to the mesh leads to

$$\begin{aligned}
-V_0 + Ri + \frac{1}{C}\int_{-\infty}^{t} idt + L\frac{di}{dt} &= 0 \\
L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} &= 0 \\
\frac{d^2 i}{dt^2} + 0.5 \times 10^3 \frac{di}{dt} + 1 \times 10^6 i &= 0 \\
(D^2 + 0.5 \times 10^3 D + 1 \times 10^6) i &= 0 \\
\lambda^2 + 0.5 \times 10^3 \lambda + 1 \times 10^6 &= 0
\end{aligned}$$

$$\lambda = -250 \pm 1.118 \times 10^4 j$$

$$
\begin{aligned}
i(t) &= ce^{-250t}\cos(1.118\times 10^4 t+\theta), \qquad i(0) = c\cos(\theta) = 3 \\
i'(t) &= -250ce^{-250t}\cos(1.118\times 10^4 t+\theta) \\
&\qquad - 1.118\times 10^4 ce^{-250t}\sin(1.118\times 10^4 t+\theta)
\end{aligned}
$$

Substituting $t = 0$, we obtain

$$
i'(0) = -250c\cos(\theta) - 1.118\times 10^4 c\sin(\theta) = 0
$$

and

$$
\begin{aligned}
-1.118\times 10^4 c\sin(\theta) &= 250c\cos(\theta), \\
\tan(\theta) &= \frac{250}{-1.118\times 10^4} = -0.0224, \\
\theta &= -0.0224 \text{ rad}, \qquad c = 3,
\end{aligned}
$$

Applications: Automobile Ignition Circuit cont.

Therefore, $i(t) = 3e^{-250t}\cos(1.118\times 10^4 t - 0.0224)$ and,

$$
\begin{aligned}
v(t) &= L\frac{di}{dt} \\
&= -6e^{-250t}\cos(1.118\times 10^4 t - 0.0224) - 268.32e^{-250t}\sin(1.118\times 10^4 t - 0.0224) \\
&= -268.39e^{-250t}\sin(1.118\times 10^4 t - 0.0224 + 0.0224) \\
&= -268.39e^{-250t}\sin(1.118\times 10^4 t)
\end{aligned}
$$

$v(t)$ is maximum when $1.118\times 10^4 t = \frac{\pi}{2}$, then

$$
t = \frac{1.5708}{1.118\times 10^4} = 1.405\times 10^{-4}\ \text{sec} = 140.5\ \mu s, \quad v_{\text{max}}(t) = -259\ \text{V}.
$$

Reference

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