Introduction

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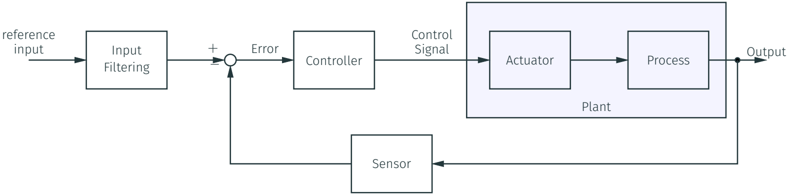
January 12, 2025

Big Picture

In control and automation field



System modeling is an important part of the control system. The topic is how to build mathematical models of plants and determine the plant's response before constructing real systems.

Introduction to System Dynamics

Definition:

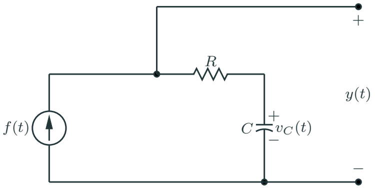
- A system transforms **input signals** into **output signals** (or **response**)
- A system is a function mapping input signals into output signals.

We concentrate on systems with one input and one output signal, i.e., **single-input, single-output (SISO)** systems.

Notation:

- $y = Su$ or $y = S(u)$ me ans the system S acts on input signal u to produce output signal y
- $y = Su$ does not, in general, mean multiplication.

RC circuit

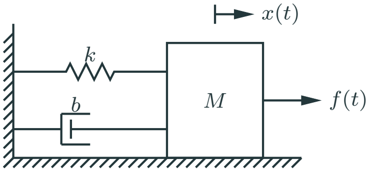


The output voltage $y(t)$ is given by

$$y(t) = Rf(t) + \frac{1}{C}\int_{-\infty}^{t} f(\tau)d\tau$$

$$\frac{d}{dt}y(t) = R\frac{d}{dt}f(t) + \frac{1}{C}f(t)$$

Spring Mass System



From the Newton's law $\Sigma F = ma$, we have

$$M\frac{d^2}{dt^2}x(t) + b\frac{d}{dt}x(t) + kx(t) = f(t)$$

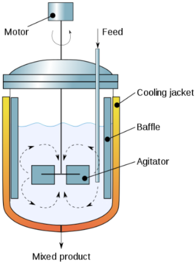
Drag force

The drag force on an object moving through a liquid or a gas is a function of the velocity:

$$D = \frac{1}{2}\rho A C_D v^2,$$

where ρ is the mass density of the fluid, A is the object's cross-sectional area normal to the relative flow, v is the object's velocity relative to the fluid, and C_D is the *drag coefficient*.

Mixer Transient Energy Balance



https://commons.wikimedia.org/wiki/File:Agitated_vessel.svg

Using energy balance technique to determine the dynamic equation.

$$mc_p\frac{dT}{dt} = \dot{m}_{in}c_p\left(T_{in} - T_{ref}\right) - \dot{m}_{out}c_p\left(T_{out} - T_{ref}\right) + UA(T_\infty - T) + W_s + rV\Delta H_r$$

Static Elements: The element is a **static** element if the present value of an element's output depends only on the present value of its input.

- Ex. the current flowing through a resistor depends only on the present value of the applied voltage, or $v(t) = Ri(t)$. The resistor is thus a static element.
- However, no physical element can respond instantaneously, the concept of a static element is an approximation. It is widely used because it is simple and can be represented in terms of algebraic equations rather than differential equations.

Dynamic Elements: The element is a **dynamic** element if the present value of an element's output depends on the past values of its input.

- Ex. the present position of a bike depends on what its velocity has been from the start.

Dynamic System is the one in which the current effects (outputs) are the result of present and previous causes (inputs).
Static system is the one in which the current effects (outputs) depend only on current causes (inputs).

- A static system contains all static elements.
- Any system that contains at least one dynamic element must be a dynamic system.

Modelling of Systems

The modelling process of engineering system dynamics:

- Identifying the *fundamental properties* of an actual system.
- The *minimum set of variables* necessary to fully define the *system configuration* is formed of the *degree of freedom (DOF)*.
- The key of this selection is a schematic or diagram, which pictorially identifies the parameters and the variables, such as the *free-body diagram* that corresponds to the dynamics of the system.
- It is necessary to utilize an appropriate *modelling procedure* that will result in the *mathematical model* of the system.
- A mathematical model describing the dynamic behavior of an engineering system consists of a differential equation combining parameters with known functions, unknown functions, and derivatives.
- The next step involves *solving the mathematical model* through adequate mathematical procedures that deliver the *solution*, that is, expressions (equations) of variables as functions of the system parameters and time (of frequency), and the reflect they *system response* or *behavior*.

Modelling of Systems

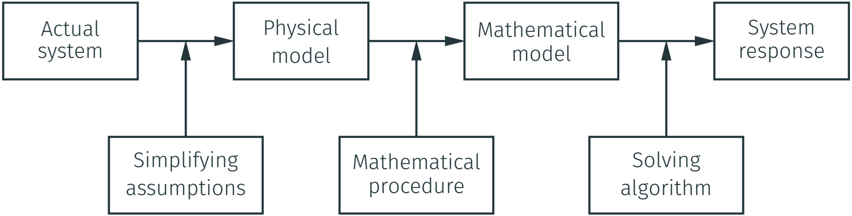
Modelling of Systems

Dynamic models are essential for understanding the system dynamics in open-loop (manual mode) or for closed-loop (automatic) control. These models are either derived from data (empirical) or from more fundamental relationships (first principles, physics-based) that rely on knowledge of the process. A combination of the two approaches is often used in practice where the form of the equations is developed from fundamental balance equations and unknown or uncertain parameters are adjusted to fit process data.

In engineering,

- there are four common balance equations from conservation principles mass, momentum, energy, and species
- An alternative to physics-based models is to use input-output data to develop empirical dynamic models such as first-order or second-order systems.

Modelling of Systems



Flow in a Process Connecting an Actual Dynamic System to Its Response

1. Simplifying the problem sufficiently and applying the appropriate fundamental principles is call **modelling**
2. The resulting mathematical description is called a **mathematical model**, or just a **model**.
3. When the modelling has been finished, we need to solve the mathematical model to obtain the required answer.

Modeling of systems

https://blogs.nasa.gov/spacex/2019/06/25/side-boosters-have-landed/

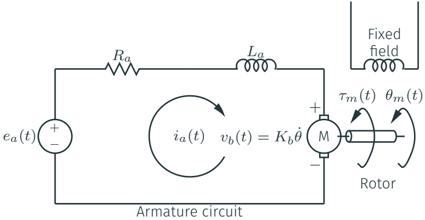
- We want to design a rocket that can self-landing.
- We don't have a rocket yet, so we cannot experiment to obtain the answer.
- There are a lot of variations in climate during the self-landing.
- We need the model of the rocket to predict its landing behavior.
- We can quickly simulate the model without investing much time and money to build the rocket.
- If the operations in the simulation fail, the broken rocket doesn't harm life or money.

Modeling of systems: go back to academic toy problem

- Rather than setting out to control this motor's speed by performing a series of trial-and-error test.
- We will begin with the dynamic system.

The mathematical model is

$$J\dot{\omega}(t) + B\omega(t) = T(t)$$



- ω is a angular velocity of the motor in rpm.
- J is a mass moment of inertia.
- $T(t)$ is a torque input.

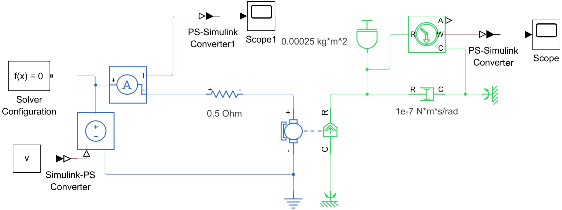
Modeling of systems: go back to academic toy problem

Relating the electrical current $i(t)$ to the motor's inductance L, resistance R, velocity $\omega(t)$ and the applied voltage $v(t)$, we two mathematical representation of the entire motor.

$$L\frac{di(t)}{dt} + Ri(t) + k_b\omega(t) = v(t)$$

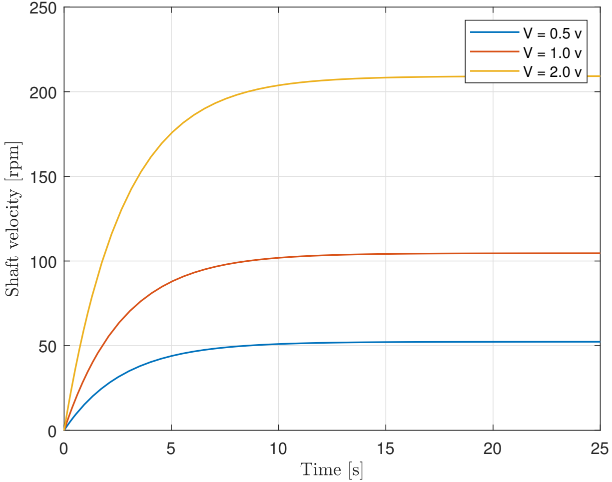
$$J\dot{\omega}(t) + B\omega(t) = k_i i(t)$$

Here k_b and k_i are the back electromotive force (EMF) constant and the motor's torque constant, respectively.
To test the motor's model we can use MATLAB\SIMSCAPE as shown below:



Modeling of systems: go back to academic toy problem

The simulation results:



Classical of the model

: Lumped-Parameters vs Distributed-Parameters Systems

- A **distributed model** has variables as functions of time and spatial coordinated(s), which we can define by partial differential equations (PDEs). We also call a distributed model an infinite-dimensional system because its response is expressed in terms of an infinite number of coordinates.
- A **lumped model** has variables as functions of time only, which can be described by ordinary differential equations (ODEs). We also call the lumped model a finite-dimensional system because its response is expressed by a finite number of coordinates.

Linear Models vs Nonlinear Models

- A **linear model** has variables that are governed by linear differential equations. A linear differential equation about a variable is one which only involves a linear combination of the variable and its derivatives.

$$m\ddot{x} + c\dot{x} + kx = f,$$

where m, c, and k are the mass, damping, and spring parameters of the system, and f is an external force.

Classical of the model

- A nonlinear model has variables that are governed by nonlinear differential equations. A nonlinear differential equation about a variable is one which involves the products and nonlinear functions of the variable and its derivatives. A spring-mass-damper system including dry friction is described by

$$m\ddot{x} + c\dot{x} + kx + \mu N \mathrm{sgn}(\dot{x}) = f,$$

  where μ is a kinetic friction coefficient, N is a normal force, and $\mathrm{sgn}(\dot{x})$ is the sign function. Because $\mathrm{sgn}(\dot{x})$ is a nonlinear function of $\dot{x}$.

Time-Invariant Models vs Time-Variant Models

- A time-invariant system (TI) is a system that is described by differential equations having only constant coefficients.
- A time-varying system (TV) is a system that is described by differential equations containing at least one coefficients that change with time.

$$m\ddot{x} + c\dot{x} + kx = f \quad (TI) \qquad m\ddot{x} + c\dot{x} + (k_0 + \varepsilon \sin \omega t)x = f \quad (TV)$$

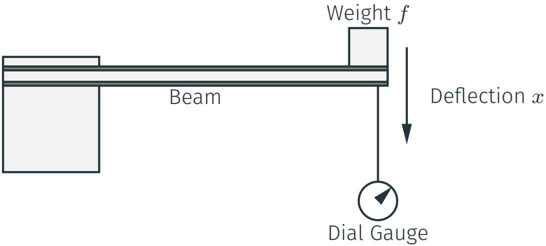
Modeling of systems

Applications of System Modelling:

- Control systems: robotics, mechatronics, precision engineering, etc.
- Mechanical systems: design active suspension systems, robot arms, etc.
- Electrical and Electromechanical systems: Grid-connected system, electronic driver, mechanical conveyer, etc.
- Fluid systems: backhoe truck, hydraulic servomotor.
- Thermal systems.

Developing Linear Models from Data

A Cantilever Bean Deflection Model

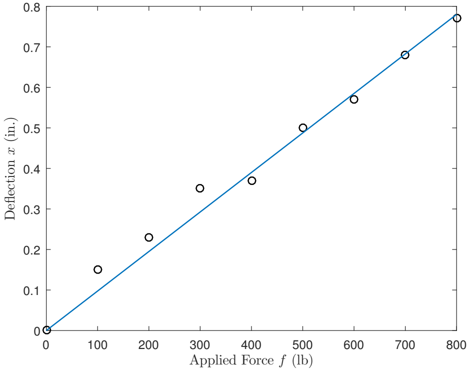


The deflection of a cantilever beam is the distance its end moves in response to a force applied at the end. The following table gives the measured deflection x that was produced in a particular beam by the given applied force f.

Force f (lb)	0	100	200	300	400	500	600	700	800
Deflection x (in)	0	0.15	0.23	0.35	0.37	0.5	0.57	0.68	0.77

What is the model of this cantilever beam deflection?

A Cantilever Bean Deflection Model



- the function is $x = af$, where

$$a = \frac{0.77 - 0}{800 - 0} = 9.625 \times 10^{-4} \text{ in./lb}$$

A Cantilever Bean Deflection Model

Matlab code

```
f = [0,100,200,300,400,500,600,700,800];
defl = [0 ,0.15 ,0.23 ,0.35 ,0.37 ,0.5 ,0.57 ,0.68 ,0.77];

a = (defl(end)-defl(1))/(f(end)-f(1));

defl_hat = a*f;
plot(f,defl,'ko',f,defl_hat)
xlabel('Applied Force $f$ (lb)');
ylabel('Deflection $x$ (in.)');
```

To learn how to code MATLAB please consult the MATLAB onramp at

`https://matlabacademy.mathworks.com/details/matlab-onramp/gettingstarted`

Interpolation and Extrapolation

Once we have a function relating input and output,

- **Interpolation:** is a process of using the model to make predictions for conditions that lie inside the data range of the original data.
  - We can use the beam model to estimate the deflection when the applied force is 550 lb.
  - We can be fairly confident of this prediction because we have data below and above 500 lb and we have seen that our model describes this data very well.
- **Extrapolation:** is a process of using the model to make predictions for conditions that lie outside the original data range.
  - Extrapolation might be used i the beam application to predict how much the force would be required to bend the beam 1.2 in.
  - We must be careful when using extrapolation because we usually have no reason to believe that the mathematical model is valid beyond the range of the original data.
  - For example, if we continue to bend the beam, eventually the force is no longer proportional to the deflection, and it becomes much greater than that predicted by the linear model.

Linear Functions

Superposition and linear functions

- $f : \mathbb{R}^n \to \mathbb{R}$ means f is a function mapping n-vectors to numbers
- f satisfies superposition property if for any numbers α, β and n-vectors x, y

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

- a function that satisfies superposition is called **linear**
- a function that is linear plus a constant is called **affine**
- the function is **affine** if and only if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \qquad \alpha, \beta \text{ with } \alpha + \beta = 1$$

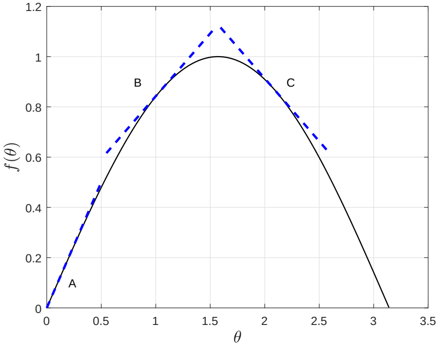
- sometimes (ignorant) people refer to affine functions as linear

Note: if $f(x) = ax + b$ then

$$\begin{aligned} f(\alpha x + (1-\alpha)y) &= a\left(\alpha x + (1-\alpha)y\right) + b \\ &= \alpha f(x) - \alpha b + (1-\alpha) f(y) - (1-\alpha)b + b \\ &= \alpha f(x) + (1-\alpha) f(y) \end{aligned}$$

Linearization

Not all element descriptions are in the form of data. Often we know the analytical form of the model, and if the model is nonlinear, we can obtain a linear model that is an accurate approximation over a limited range of the independent variable.



- The models of many mechanical systems involve the sine function $\sin\theta$, which is nonlinear.
- Obtain three linear approximations of $f(\theta) = \sin\theta$, one value near $\theta = 0$, one near $\theta = \pi/3$ rad, and one near $\theta = 2\pi/3$ rad.
- The slope of the sine function is its derivative, $d\sin\theta/d\theta = \cos\theta$, which is not constant.

Linearization

- **case A:** $\theta = 0$, the slope is $\cos 0 = 1$, and thus the straight line passing through the point with a slope of 1 is

$$f(\theta) = 1(\theta - 0) + \sin 0 = \theta$$

- **case B:** $\theta = \frac{\pi}{3}$, we have $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$. The straight line passing through the point with a slope of $\cos \frac{\pi}{3} = 0.5$ is

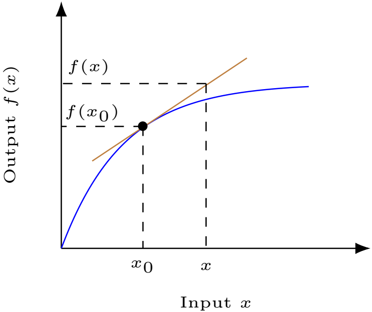
$$f(\theta) = 0.5 \left(\theta - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}$$

- **case C:** $\theta = \frac{2\pi}{3}$, we have $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$. The straight line passing through the point with a slope of $\cos \frac{2\pi}{3} = -0.5$ is

$$f(\theta) = -0.5 \left(\theta - \frac{2\pi}{3}\right) + \frac{\sqrt{3}}{2}$$

Linearization

The linearization can be developed with an analytical approach based on the Taylor series.



The first-order Taylor approximation of f , near point x_0:

$$\hat{f}(x) = f(x_0) + \left(\frac{\partial f}{\partial x}\right)_{x_0} (x - x_0)$$

note: It is $y = mx + c$!

If you use this technique to calculate the linear approximation of the previous example, you will get the same results.

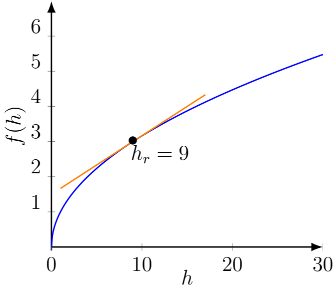
Linearization: Example

The models of many fluid systems involve the square-root function $\sqrt{h}$, which is nonlinear. Obtain a linear approximation of $f(h) = \sqrt{h}$ valid near $h = 9$.

$$f(h) = f(h_r) + \left.\frac{d\sqrt{h}}{dh}\right|_{h_r} (h - h_r),$$

where $h_r = 9$. This gives the linear approximation

$$f(h) = \sqrt{9} + \left.\frac{1}{2}h^{-\frac{1}{2}}\right|_{h_r} (h - 9) = 3 + \frac{1}{6}(h - 9)$$



Basic Curve Fitting

The **curve fitting** is used to describe the process of finding a curve, and the function generating the curve, to describe a given set of data. **Parameter estimation** is the process of obtaining values for the parameters, or coefficients, in the function that describes the data.

The simple three function types can often describe physical phenomena:

- The **affine** function $y(x) = mx + b$. Note that $y(0) = b$. Ex. the linear function describes the voltage-current relation for a resistor ($v = iR$) and the velocity versus time relation for an object with constant acceleration ($v = at + v_0$).
- The **power** function $y(x) = bx^m$. Note that $y(0) = 0$ if $m \geq 0$, and $y(0) = \infty$ if $m < 0$. Ex. the distance d traveled by a falling object versus time is described by a power function ($d = 0.5gt^2$)
- The **exponential** function $y(x) = b(10)^{mx}$ or its equivalent form $y = be^{mx}$, where e is the base of the natural logarithm ($\ln e = 1$). Note that $y(0) = b$ for both forms. Ex. The temperature change ΔT of a cooling object can be described by an exponential function ($\Delta T = \Delta T_0 e^{ct}$).

Basic Curve Fitting

- The affine(linear) function $y = mx + b$ gives a straight line when plotted on rectilinear axes. The slope is m and the y intercept is b.
- The power function $y = bx^m$ gives a straight line when plotted on log-log axes.

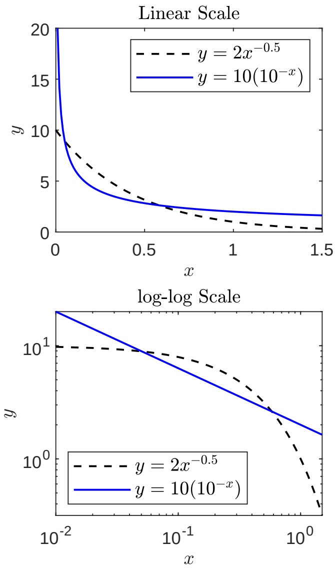
$$\ln(y) = \ln(bx^m) = \ln(b) + m\ln(x) \qquad \Rightarrow \qquad \hat{y} = m\hat{x} + \hat{b}$$

  where $(\hat{\cdot})$ operator means $\log(\cdot)$.
- The exponential function $y = be^{mx}$, give a straight line when plotted on semilog axes with a logarithmic y axis

$$\ln(y) = \ln(be^{mx}) = \ln(b) + mx \qquad \Rightarrow \qquad \hat{y} = mx + \hat{b}$$

  where $(\hat{\cdot})$ operator means $\log(\cdot)$.

Basic Curve Fitting



Basic Curve Fitting

The curve fitting procedure is

- Examine the data near the origin. The exponential functions $y = b(10)^{mx}$ and $y = be^{m}$ can never pass through the origin. The linear function $y = mx + b$ can pass through the origin only if $b = 0$. The power function $y = bx^{m}$ can pass through the origin but only if $m > 0$.
- Plot the data using rectilinear scales. If it forms a straight line, then it can be represented by the linear function, and you are finished. If you have data at $x = 0$, then
  1. If $y(0) = 0$, try the power function, or
  2. If $y(0) \neq 0$, try the exponential function.

  If data is not given for $x = 0$, proceed to the next step.
- If you suspect a power function, plot the data using log-log scales. Only a power function will form a straight line. If you suspect an exponential function, plot it using semilog scales. Only an exponential function will form a straight line.

Basic Curve Fitting

To obtain the coefficients of each function:

- For the linear function $y = mx + b$, the slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  If we know m, we can find b from $b = y_1 - mx_1$.

- For the power function $y = bx^m$, we select

$$y_1 = bx_1^m \text{ and } y_2 = bx_2^m \quad \Rightarrow \quad \frac{y_2}{y_1} = \left(\frac{x_2}{x_1}\right)^m$$

$$\ln(y_2) - \ln(y_1) = m(\ln(x_2) - \ln(x_1)) \quad \Rightarrow \quad m = \frac{\ln(y_2) - \ln(y_1)}{\ln(x_2) - \ln(x_1)}$$

  One m is known, b can determined by $y_1 = bx_1^m$ and $b = y_1 x_1^{-m}$.

Basic Curve Fitting

To obtain the coefficients of each function:

- Fro the exponential function $y = be^{mx}$, we can select

$$y_1 = be^{mx_1} \text{ and } y_2 = be^{mx_2} \quad \Rightarrow \quad \frac{y_2}{y_1} = \left(\frac{e^{mx_2}}{e^{mx_1}}\right)$$

$$\ln(y_2) - \ln(y_1) = \ln\left(e^{mx_2}\right) - \ln\left(e^{mx_1}\right) = m(x_2 - x_1)$$

$$m = \frac{\ln(y_2) - \ln(y_1)}{x_2 - x_1}$$

Once m is known, we can find b from $b = y_1 e^{-mx_1}$.

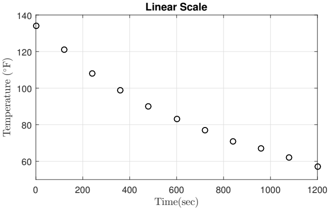
Basic Curve Fitting: example

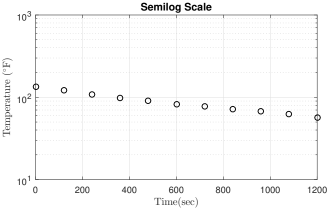
Water in a glass measuring cup was allowed to cool after being heated to 204°F. The ambient air temperature was 70°F. The measured water temperature at various times is given in the following table.

Time (sec)	0	120	240	360	480	600
Temperature (°F)	204	191	178	169	160	153

Time (sec)	720	840	960	1080	1200
Temperature (°F)	147	141	137	132	127

Obtain a functional description of the water temperature versus time. **The temperature data is subtracted by 70°** due to the ambient temperature offset or $\Delta T = T - 70°$F.





Basic Curve Fitting: example

The data can be described by an exponential function, we plot the data on a semilog plot, which is shown on the right hand side. The straight line has shown we can use the exponential function to describe the relative temperature. From $T = be^{mt}$ with $t_1 = 1200$ at $T_1 = 127 - 70 = 57$ and $t_2 = 120$ at $T_2 = 191 - 70 = 121$ then

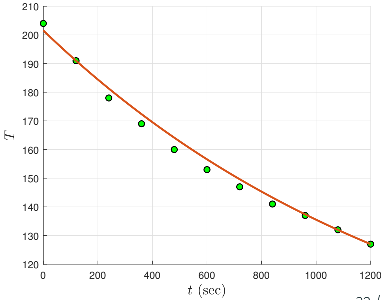
$$T_1 = be^{mt_1}, \qquad T_2 = be^{mt_2} \;\Rightarrow\; \frac{T_1}{T_2} = e^{m(t_1 - t_2)}$$

$$m = \frac{1}{1200 - 120} \ln \frac{57}{121} = -6.9698 \times 10^{-4} \qquad b = 121e^{6.9698\times10^{-4}(120)} = 131.5554$$

Thus the estimated function is

$$\Delta T = 131.5554e^{-6.9698\times10^{-4}t}$$

$$T_{est} = \Delta T + 70 = 131.5554e^{-6.9698\times10^{-4}t} + 70$$



Basic Curve Fitting: example

Matlab code

```
tt = 0:120:1200;
Temp = [204, 191, 178, 169, 160, 153, 147, 141, 137, 132, 127];
offset = 70;
TempT = Temp - offset;

ts = 0:0.1:1200;
m1 = (log(TempT(end)) - log(TempT(2)))/(tt(end)-tt(2));

b = TempT(2)*exp(-m1*tt(2));
Ta = b*exp(m1 * ts) + 70;

scatter(tt, Temp, 'go', 'filled', 'MarkerEdgeColor', 'black', 'linewidth',1 );
      grid;
ylabel('$T$', 'interpreter', 'latex', 'fontsize', 14);
xlabel('$t$ (sec)', 'interpreter', 'latex', 'fontsize', 14);
hold on

plot(ts, Ta, 'linewidth', 2)
hold off
```

Basic Curve Fitting: example

A hole 6 mm in diameter was made in a translucent milk container. A series of marks 1 cm apart was made above the hole. While adjusting the tap flow to keep the water height constant, the time for the outflow to fill a 250-ml cup was measured (1 ml = 10^{-6} m^3). This was repeated for several heights. The data are given in the following table.

Height h (cm)	11	10	9	8	7	6	5	4	3	2	1
Time t (s)	7	7.5	8	8.5	9	9.5	11	12	14	19	26

Obtain a function description of the value outflow rate f as a function of water height h above the hole.

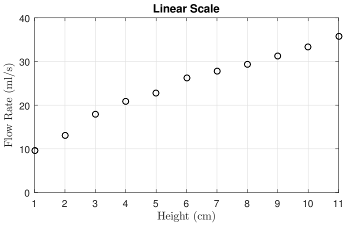
First, obtain the flow rate data in ml/s by dividing the 250 ml volume by the time to fill:

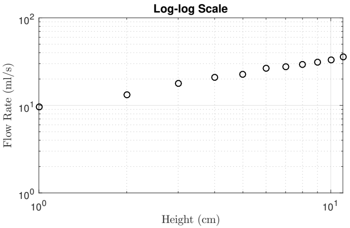
$$f = \frac{250}{t}$$

we have

f (ml/s)	33.7	33.3	31.3	29.4	27.8	26.3	22.7	20.8	17.9	13.2	9.6
h (cm)	11	10	9	8	7	6	5	4	3	2	1

Basic Curve Fitting: example





The log-log plot shown that the data lie close to a straight line, so we can use the power function to describe the flow rate as a function of height. Thus we can write

$$f = bh^m$$

We use the two points from the plot $(1, 9.6)$ and $(8, 29.4)$. Then

$$f_1 = bh_1^m, \quad f_2 = bh_2^m \qquad \Rightarrow \frac{f_2}{f_1} = \left(\frac{h_2}{h_1}\right)^m \qquad \Rightarrow \qquad \log(29.4/9.6) = m(\log(8/1))$$

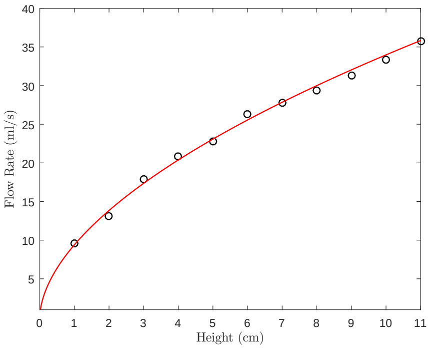
$m = \dfrac{\log(29.4/9.6)}{\log(8/1)} = 0.5382$ and $b = 9.6(1)^{-0.5382} = 9.6$

Basic Curve Fitting: example

Thus the estimated function is

$$f = 9.6h^{0.5382},$$

where f is the outflow rate in ml/s.



The quality of a curve fit

Sum of the Square of the residuals (errors)

$$J = \sum_{i=1}^{n} [f(x_i) - y_i]^2$$

This criterion compares the quality of the curve fit for two or more functions used to describe the same data.

Sum of the Squares of the deviation from the mean

$$S = \sum_{i=1}^{n} [y_i - \bar{y}]^2 ,$$

where $\bar{y}$ is the mean value of the data set y_i.

r-squared value

$$r^2 = 1 - \frac{J}{S}$$

To accept the model the r^2 must greater than 0.99.

The quality of a curve fit

Consider the water in glass example, we have a model as

$$T_{est} = 131.55e^{-6.97\times10^{-4}t} + 70,$$

Using MATLAB code below

Matlab code

```
% Quality of a curve fit
T_r = [204, 191, 178, 169, 160, 153, 147, 141, 137, 132, 127];
t = 0:120:1200;

T_est = 131.56*exp(-6.97e-4 * t) + 70;
T_mean = mean(T_r)

J = sum((T_est - T_r).^2)
S = sum((T_r - T_mean).^2)
r2 = 1 - J/S
```

We have $J = 70.56, S = 6.24 \times 10^3$, and $r^2 = 0.9887$. The model is not good enough!

Reference

1. William J. Palm III, "*System Dynamics*,4^{nd} edition, McGraw-Hill, 2021
2. Eronini Umez-Eronini, "*System Dynamics & Control*", Brooks/Cole Publishing, 1998
3. Nicolae Lobontiu, "*System Dynamics for Engineering Students*", Academic Press, 2010