INC 341 Feedback Control Systems: Lecture 7 Feedback Control

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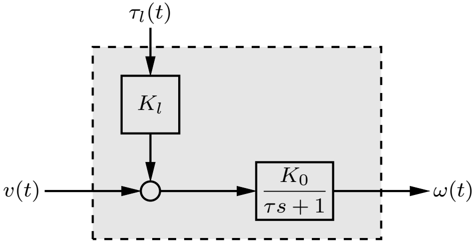
Feedback Control and Sensitivity

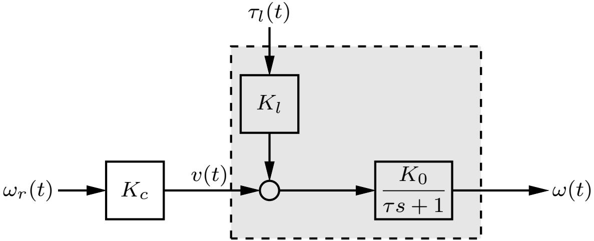
Consider a simplified speed motor model

$$\frac{\Omega(s)}{V(s)} = \frac{K_0}{\tau s + 1}, \qquad \frac{\Omega(s)}{\hat{\tau}_l(s)} = \frac{K_l K_0}{\tau s + 1},$$

where $\tau_l(t)$ is a load torque, which is consider as a *disturbance input*, $v(t)$ is a voltage input and K_l , K_0 are motor constants.



Open-loop Control

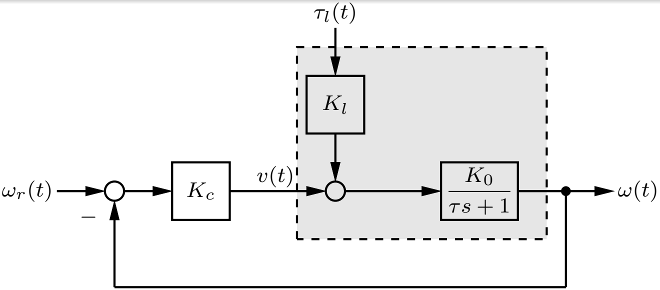


- The controller is connected in series with the plant, and the controller input is the desired speed $\omega_r(t)$ (here we mean a step speed input).
- The steady state gain of the plant is K_0 and there is no load torque ($\tau_l(t) = 0$)
- At the steady state the motor speed is

$$\omega(\infty) = K_0 v(\infty).$$

- By setting $K_c = 1/K_o$, it leads to the desired result $\omega(\infty) = \omega_r(\infty)$.

Closed-loop Control



- The closed-loop transfer function is

$$\frac{\Omega(s)}{\Omega_r(s)} = \frac{K_c K_o}{\tau s + 1 + K_c K_0}$$

- The steady state closed-loop gain is

$$G_0 = \frac{K_c K_0}{1 + K_c K_0}$$

Closed-loop Control

- In steady state operation, the motor speed is

$$\omega(\infty) = \frac{K_c K_0}{1 + K_c K_0}\omega_r(\infty)$$

- Therefore, in contrast to open-loop control, the closed-loop controller cannot set the actual speed exactly to its desired value.
- The gain product $K_c K_0$ is much larger than 1, the steady state error $\omega(\infty) - \omega_r(\infty)$ will be small ($K_c K_0 = 100$ for example leads to a 1% error).

Uncertain Plant Parameters, Sensitivity

If the actual steady state plant gain is $K_0 + \Delta K_0$.

- the open-loop configuration, the motor speed at the steady state under this assumption is

$$\omega(\infty) = (K_0 + \Delta K_0)\,\frac{1}{K_0}\omega_r(\infty)$$

- the error speed is

$$\Delta\omega(\infty) = \frac{\Delta K_0}{K_0}\omega_r(\infty) \quad \Rightarrow \quad \frac{\Delta\omega(\infty)}{\omega_r(\infty)} = \frac{\Delta K_0}{K_0}$$

  The error of say 10% in the plant gain would lead to a 10% error in motor speed.

- For the closed-loop stead state gain would change to

$$G_0 + \Delta G_0 = \frac{K_c\,(K_0 + \Delta K_0)}{1 + K_c\,(K_0 + \Delta K_0)}$$

Uncertain Plant Parameters, Sensitivity

- The relative error using linear approximation: if ΔK_0 is small,

$$\Delta G_0 \approx \frac{dG_0}{dK_0} \Delta K_0$$

  The relative speed error is

$$\frac{\Delta\omega(\infty)}{\omega_r(\infty)} = \frac{\Delta G_0}{G_0} \approx \left(\frac{K_0}{G_0} \frac{dG_0}{dK_0} \right) \frac{\Delta K_0}{K_0}$$

- The factor between the relative error in plant gain and the resulting relative error in closed-loop steady state gain is called the *sensitivity* of the control system, denoted by S. We have

$$S = \frac{K_0}{G_0} \frac{dG_0}{dK_0}$$

Uncertain Plant Parameters, Sensitivity

- Calculating the derivative yields

$$S = \frac{K_0}{\frac{K_c K_0}{1 + K_c K_0}} \frac{K_c(1 + K_c K_0) - K_c K_0 K_c}{(1 + K_0 K_c)^2}$$

$$S = \frac{1}{1 + K_c K_0}$$

- When the gain product $K_c K_0$ is large, the sensitivity S is small and the effect of an error in th eplant gain on the controlled output is reduced considerably; e.g. with $K_c K_0 = 100$ a 10% error in plant gain leads to a 0.1% error in motor speed.

Disturbance Rejection

We consider the effect of a load torque on the motor speed. Ideally the controller should maintain the desired speed independent of the load.

- When the load Torque $\tau_l(t)$ is nonzero, the motor speed at the steady state is

$$\omega(\infty) = K_0 \left(\frac{1}{K_0} \omega_r(\infty) + K_l \tau_l(\infty) \right) = \omega_r(\infty) + K_0 K_l \tau_l(\infty)$$

- the open-loop speed error due to the load torque is

$$\Delta\omega_{ol}(\infty) = K_0 K_l \tau_l(\infty)$$

- In the closed-loop configuration, the transfer function from $\tau_l(t)$ to $\omega(t)$ is

$$\frac{\Omega(s)}{\hat{\tau}_l(s)} = \frac{K_l K_0}{\tau s + 1 + K_c K_0}$$

- in steady state operation we have

$$\omega(\infty) = \frac{K_c K_0}{1 + K_c K_0} \omega_r(\infty) + \frac{K_l K_0}{1 + K_c K_0} \tau_l(\infty).$$

Disturbance Rejection

- The first term on the right hand side is the motor speed that would be reached when the load torque is zero.
- The second term is the closed-loop speed error due the load torque

$$\Delta\omega_{cl}(\infty) = \frac{K_l K_0}{1 + K_c K_0}\tau_l(\infty)$$

- Comparing this with the open-loop result, we find

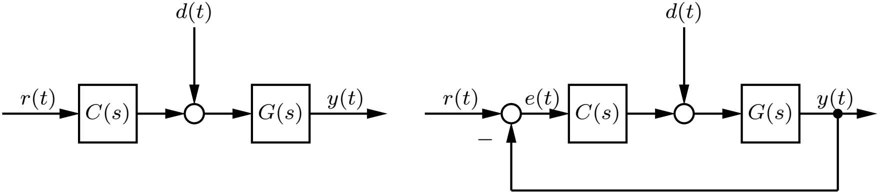
$$\Delta\omega_{cl}(\infty) = \frac{1}{1 + K_c K_0}\Delta\omega_{ol} \quad \text{or} \quad \Delta\omega_{cl}(\infty) = S\Delta\omega_{ol}(\infty)$$

- Thus, in closed loop the effect of a disturbance load on the speed is reduced by the same factor S as the effect of plant parameter errors; if the gain product $K_c K_0$ is 100, the closed-loop error is reduced to 1% of the open-loop error.

Open Loop vs. Closed Loop

The performance of open-loop and closed-loop control have two different control objectives:

- the controlled output should follow a reference input as closely as possible (this is called the *tracking problem*)
- the controlled output should be held at its desired value in spite of external disturbance (this is the *disturbance rejection problem*).



Open-loop control configuration

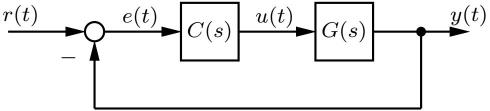
Closed-loop control configuration

For closed-loop control configuration, we have

- tracking problem ($d(t) = 0$), $Y(s) = \dfrac{G(s)C(s)}{1 + G(s)C(s)} R(s)$
- disturbance rejection problem ($r(t) = 0$), $Y(s) = \dfrac{G(s)}{1 + G(s)C(s)} D(s)$

Types of Feedback

In this section, three basic types of feedback are introduced: proportional feedback, derivative feedback and integral feedback.



Proportional Feedback: The control input is linearly proportional to the control error. The control law is

$$u(t) = K_p\,(r(t) - y(t)) = K_p e(t).$$

Consider the transient behaviour. Assume that the plant is a second order system with transfer function

$$G(s) = \frac{1}{s^2 + a_1 s + a_0}$$

The controller is a proportional gain, i.e. $C(s) = K_p$.

Types of Feedback
Proportional gain

The closed-loop transfer function is

$$G_{cl}(s) = \frac{Y(s)}{R(s)} = \frac{K_P}{s^2 + a_1 s + a_0 + K_P},$$

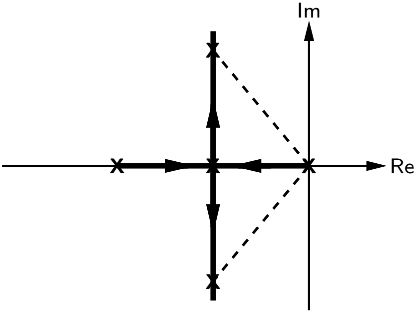
Here, a_0 is replaced by $a_0 + K_P$. Comparing this with the standard form of the characteristic equation of a second order system

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

we see that the effect of K_p apart from the steady state gain in a change of the natural frequency ω_n.
Assuming that $a_0 = 0$, the closed-loop poles are

$$s_{1,2} = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - K_P}.$$

Types of Feedback
Proportional gain



- the location of the poles for different values of K_P shown in Figure above.
- When $K_P = 0$, the roots are $s_1 = 0$ and $s_2 = -a_1$, which are the open-loop poles of the plant without control.
- When K_P is increased, the poles move towards each other along the negative real axis, and for $K_P = a_1^2/4$ they meet at $s = -a_1/2$.

Types of Feedback
Proportional gain

- When K_P is further increased, the poles become complex

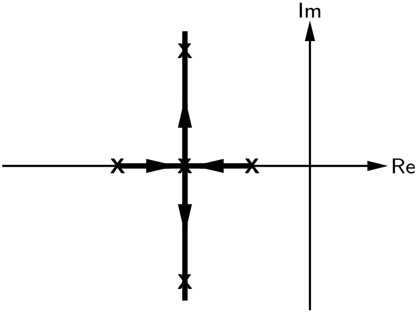
$$s_{1,2} = -\frac{a_1}{2} \pm j\sqrt{K_P - \left(\frac{a_1}{2}\right)^2}$$

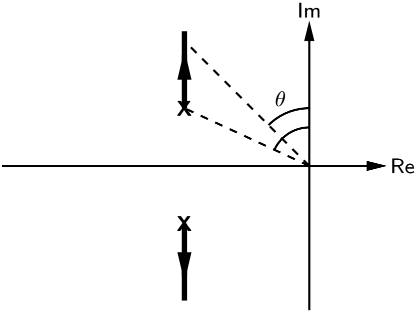
with real part $-a_1/2$, and the imaginary part increasing with K_P. When $K_P = a_1^2/2$, the poles are

$$s_{1,2} = -\frac{a_1}{2} \pm j\frac{a_1}{2}$$

and the angle between the poles and the imaginary axis is $45°$.

Types of Feedback
Proportional gain





- If $a_0 \neq 0$, we have to distinguish two cases:
- the open-loop poles can be real or complex.
- In both cases, the behaviour of the closed-loop poles is similar to that when $a_0 = 0$;
- From a steady state point of view, the feedback gain should be large in order to make the error small.
- However, the higher gain K_P leads to a small angel θ between poles and imaginary axis. Since $\sin\theta = \zeta$, therefore a large K_P leads to a poor transient response with low damping ratio, large peak overshoot and oscillation.

Types of Feedback

Proportional plus Derivative Feedback

The control law for proportional plus derivative feedback (also known as PD control) is

$$u(t) = K_P \left(e(t) + T_D \dot{e}(t) \right),$$

where $e = r - y$ is the controller error. Taking Laplace transforms, we have

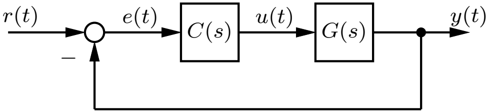
$$U(s) = K_P \left(1 + T_D s \right) E(s)$$

thus the controller transfer function is

$$C(s) = K_P \left(1 + T_D s \right).$$

The parameter T_D is called the *derivative time*.

Consider the plant with transfer function $G(s) = \dfrac{1}{s^2 + a_1 s + a_0}$



The closed-loop transfer function of the feedback system is

$$G_{cl}(s) = \frac{K_P\,(1 + T_D s)}{s^2 + a_1 s + a_0 + K_P\,(1 + T_D s)}.$$

The characteristic equation is

$$s^2 + (a_1 + K_P T_D)s + a_0 + K_P = 0,$$

It shows an additional degree of freedom in design offered by PD control: the K_P can still be used to reduce steady state error and to change the natural frequency. While, the T_D can be used to change the damping ratio independently of the natural frequency.

We use integral feedback to bring the steady state error to zero. Consider the closed-loop steady state gain from $r(t)$ to $y(t)$ from

$$G_{cl}(s) = \frac{Y(s)}{R(s)} = \frac{K_P}{s^2 + a_1 s + a_0 + K_P}, \qquad G_{cl}(0) = \frac{K_P}{a_0 + K_P} \quad \rightarrow \quad 1 \text{ as } K_P \rightarrow \infty$$

The steady state error under proportional feedback becomes zero only when the feedback gain becomes infinite.

- The gain needs to be infinite *only in steady state*, i.e. when $s = 0$. If the gain is infinite for $s = 0$ but finite for $s \neq 0$, a zero steady state error can be achieved while avoiding undesirable effects on the transient response.
- To do this, we have to include a factor $1/s$ in the controller transfer function.

Consider the controller transfer function

$$C(s) = K_P \frac{1}{T_I s} \quad \Rightarrow \quad C(0) = \infty$$
$$U(s) = \frac{K_P}{T_I s} E(s) \quad \Rightarrow \quad u(t) = \frac{K_P}{T_I} \int_{t_0}^{t} e(\tau) d\tau$$

The parameter T_I is called the *integral time* or *reset time*.

To study the effect of integral feedback on the closed-loop behaviour, we consider the first order plant model

$$G(s) = \frac{K_0}{\tau s + 1}$$

By using $C(s) = K_P \frac{1}{T_I s}$, the closed-loop transfer function is

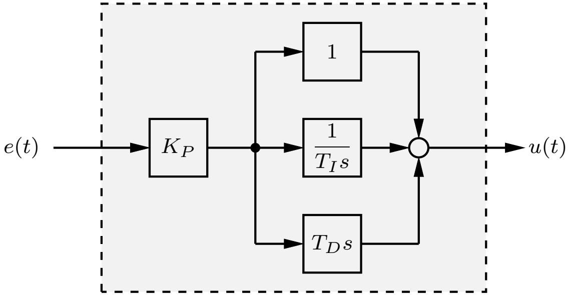
$$G_{cl}(s) = \frac{Y(s)}{R(s)} = \frac{K_P K_o}{T_I s(\tau s + 1) + K_P K_0}$$

and setting $s = 0$ shows that $G_{cl}(0) = \dfrac{K_P K_0}{K_P K_0} = 1$, thus $y(t) = r(t)$ as $t \to \infty$, i.e. the steady state error is indeed zero. The characteristic equation is

$$s^2 + \frac{1}{\tau} s + \frac{K_P K_0}{T_I \tau} = 0$$

Increasing the controller gain K_P / T_I leads to a faster decay of the steady state error, but the characteristic equation shows that increasing this gain also leads to a low damping ratio.

Types of Feedback
PID Control



PID Controller

Combining proportional, integral and derivative feedback leads to the control law

$$u(t) = K_P \left(e(t) + \frac{1}{T_I} \int_{t_0}^{t} e(\tau) d\tau + T_D \dot{e}(t) \right).$$

The transfer function of this controller is

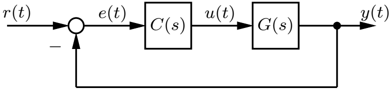
$$C(s) = K_P \left(1 + T_D s + \frac{1}{T_I s} \right)$$

PID control was developed in the 1930s and has been widely used since.

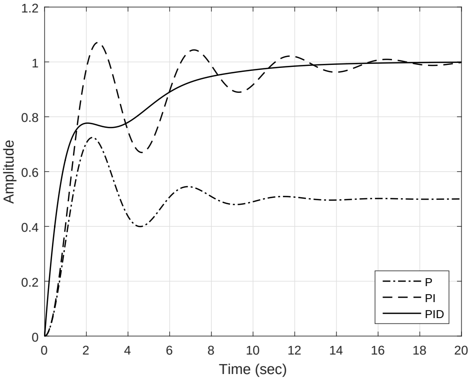
- Most commercially available industrial controllers today are of this type.
- PID control is a controller structure! There are more than 100 variations.
- There are three basic design parameters: proportional gain, integral time and derivative time
- These parameters are the tuning knobs of a PID controller.
  - a small integral time T_I brings the steady state error quickly to zero, but at the same time reduces the damping ratio;
  - increasing T_D increases the damping ratio,
  - increasing K_P increases the natural frequency and thus the speed of the response.

Consider a second order system with transfer function

$$G(s) = \frac{1}{s^2 + 0.7s + 1}$$

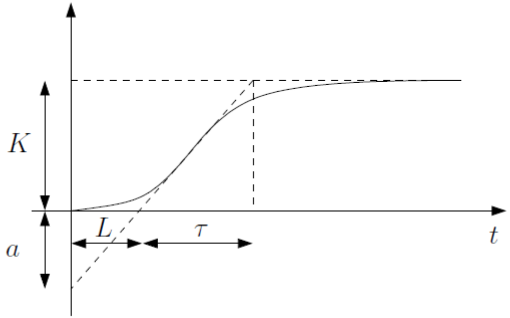


P control: $K_P = 1$
PI control: $K_P = 1$, $T_I = 2$
PID control: $K_P = 1$, $T_D = 1$, $T_I = 2$



Types of Feedback

PID Control: Ziegle-Nichols Tuning method

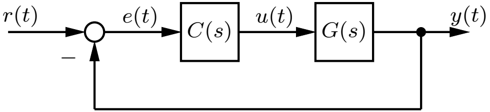


Type of controller	Ziegler-Nichols Gains
P	$K_P = 1/a, T_I = \infty, T_D = 0$
PI	$K_P = 0.9/a, T_I = 3L, T_D = 0$
PID	$K_P = 1.2/a, T_I = 2L, T_D = L/2$

Steady State Tracking Error

Steady State Error to a Ramp Input

Consider a closed-loop system shown in Figure below



Assuming that $C(s) = K_p$ and $r(t)$ is a step input

case 1:

$$G(s) = \frac{K_0}{\tau s + 1}, \quad G_{cl}(s) = \frac{K_p K_0}{\tau s + 1 + K_p K_0}, \quad G_{cl}(0) = \frac{K_p K_0}{1 + K_p K_0} \neq 1, \quad e_{ss}(\infty) \neq 0$$

case 2:

$$G(s) = \frac{K_0}{s(\tau s + 1)}, \quad G_{cl}(s) = \frac{K_p K_0}{s(\tau s + 1) + K_p K_0}, \quad G_{cl}(0) = \frac{K_p K_0}{K_p K_0} = 1, \quad e_{ss}(\infty) = 0$$

It is obviously, the factor $1/s$ in the forward path enables the feedback system to reach a constant setpoint with zero steady state error.

Steady State Tracking Error

Steady State Error to a Ramp Input

The unit ramp input is defined by $r(t) = t\mathbb{1}(t)$. We use a ramp input as reference when a position control system is required to track an object that is moving with constant velocity. The closed-loop transfer function from the reference input $r(t)$ to the error signal $e(t)$ is

$$\frac{E(s)}{R(s)} = \frac{1}{1+L(s)}, \quad \text{where } L(s) = C(s)G(s)$$

Let $C(s) = K_p$, and $G(s) = K_0/s(\tau s + 1)$ we have

$$L(s) = K_p G(s) = \frac{K_p K_0}{s(\tau s + 1)}$$

From the factor $1/s$ of $G(s)$ it is obvious that the feedback system can follow a step change with zero steady state error.

Consider the ramp input. The Laplace transform of the unit ramp is $1/s^2$, thus the error to a unit ramp input is

$$E(s) = \frac{1}{1+L(s)} R(s) = \frac{s(\tau s + 1)}{s(\tau s + 1) + K_p K_0} \frac{1}{s^2}$$

Steady State Tracking Error

Steady State Error to a Ramp Input

The steady state error is

$$
\begin{aligned}
e_{ss}(\infty) &= \lim_{s\to 0} sE(s) = s\frac{s(\tau s+1)}{s(\tau s+1)+K_pK_0}\frac{1}{s^2} \\
&= \frac{1}{K_pK_0} \neq 0
\end{aligned}
$$

Hence, the feedback system can follow a ramp input only with a nonzero steady state error, and the error is small when the gain product K_pK_0 is large.

Consider a plant with transfer function

$$G(s) = \frac{(s+3)^2}{s^2(s+1)}$$

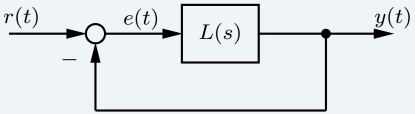
With proportional feedback, we have

$$L(s) = \frac{K_p(s+3)^2}{s^2(s+1)}, \quad e_{ss}(\infty) = \lim_{s\to 0} s\frac{s^2(s+1)}{s^2(s+1)+K_p(s+3)^2}\frac{1}{s^2} = 0.$$

System Types

System Types

Consider the feedback system



If k is the largest integer such that the transfer function $L(s)$ can be written in the form

$$L(s) = \frac{n_L(s)}{s^k d_L(s)}$$

i.e. if the denominator has a factor s^k, then this feedback system is called a type k system.

In this course, we consider only a type 0, 1, 2 systems.

System Types

Position Error Constant

The position error constant of the considered feedback system is defined as

$$K_{pos} = \lim_{s \to 0} L(s) = \lim_{s \to 0} \frac{n_L(s)}{s^k d_L(s)}$$

Because the steady state error to a unit step input is given by

$$e_{ss}(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{1 + L(s)} \frac{1}{s} = \frac{1}{1 + \lim_{s \to 0} L(s)},$$

The steady state error (in response to a unit step) in terms of the position error constant is

$$e_{ss}(\infty) = \frac{1}{1 + K_{pos}}$$

.

The position error for type 0, type 1 and type 2 systems.

Type	K_{pos}	$e_{ss}(\infty)$
0	$L(0)$	$\frac{1}{1+K_{pos}}$
1	∞	0
2	∞	0

System Types
Velocity Error Constant

The velocity error constant of the considered feedback system is defined as

$$K_{vel} = \lim_{s\to 0} sL(s) = \lim_{s\to 0} \frac{n_L(s)}{s^{k-1}d_L(s)}$$

Therefor the steady state error to a unit ramp input is

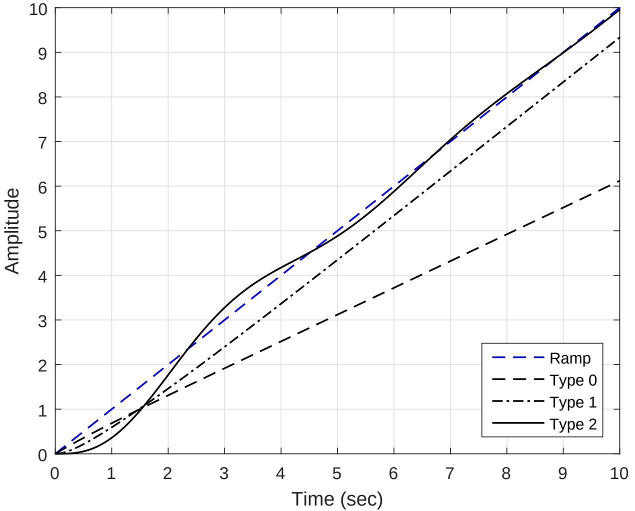
$$e_{ss}(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{1}{s^2} = \lim_{s\to 0} \frac{1}{s+sL(s)} = \frac{1}{\lim_{s\to 0} sL(s)} = \frac{1}{K_{vel}}.$$

The position error for type 0, type 1 and type 2 systems.

Type	K_{vel}	$e_{ss}(\infty)$
0	0	∞
1	$\frac{n_L(0)}{d_L(0)}$	$\frac{1}{K_{vel}}$
2	∞	0

System Types

Velocity Error Constant

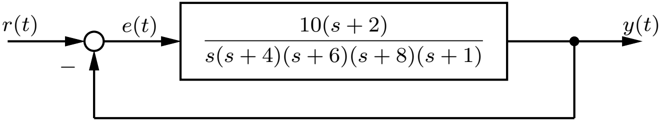


Stability

Definition:

- A system is stable if
  - the natural response approaches zero as $t \to \infty$
  - for every bounded input the output is also bounded as $t \to \infty$
- A system is unstable if
  - the natural response approaches infinity as $t \to \infty$
  - for every bounded input the output is unbounded as $t \to \infty$
- A system is marginally stable if
  - the natural response remains constant or oscillates
  - there is at least one bounded input for which the output oscillates

The easiest to check the stability of the system is checking whether the system has no right half-plane poles. For high order system, we can use computer software such as Matlab or Scilab to check the roots of the characteristic polynomial of the considered system by using a command `roots`.



- Is the system above stable?
- The closed-loop transfer function of the system is

$$T_{yr} = \frac{10(s+2)}{s(s+4)(s+6)(s+8)(s+1)+10(s+2)}$$
$$= \frac{10(s+2)}{s^5+28s^4+284s^3+1232s^2+1930s+20}$$

- simply use a command `roots` to check the pole locations.
- Routh's criterion can be used to check stability by simple calculations, without explicitly computing the poles.

Stability

Routh's criterion

Consider a system

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{b(s)}{a(s)}$$

- The stability of the system is determined by the roots of the characteristic equation

$$a(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$$

- the *necessary condition* for stability, but it is not sufficient is all coefficients of the characteristic polynomial must be positive. If $G(s)$ is stable, all coefficients are positive, and if one or more coefficients are negative or zero, the system is unstable.
- the *necessary and sufficient condition* for stability is called the *Routh's criterion*. This test is based on the so-called *Routh array*

The Routh array is as follow:

$$
\begin{array}{c|ccccc}
s^n & 1 & a_{n-2} & a_{n-4} & a_{n-6} & 0 \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & 0 \\
 & c_1 & c_2 & c_3 & 0 & \\
 & d_1 & d_2 & d_3 & 0 & \\
 & e_1 & d_2 & 0 & & \\
 & f_1 & f_2 & 0 & & \\
 & g_1 & 0 & & & \\
 & h_1 & & & &
\end{array}
$$

Here it is assumed that a_{n-7} is the last coefficient (i.e. $n = 7$)

The subsequent rows are computed as follows:

$$c_1 = \frac{-\begin{vmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}} = \frac{a_{n-1}a_{n-2} - a_{n-3}}{a_{n-1}}$$

$$c_2 = \frac{-\begin{vmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}} = \frac{a_{n-1}a_{n-4} - a_{n-5}}{a_{n-1}}$$

$$c_3 = \frac{-\begin{vmatrix} 1 & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix}}{a_{n-1}} = \frac{a_{n-1}a_{n-6} - a_{n-7}}{a_{n-1}}$$

$$c_4 = \frac{-\begin{vmatrix} 1 & 0 \\ a_{n-1} & 0 \end{vmatrix}}{a_{n-1}} = 0$$

Stability
Routh's criterion

$$d_1 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-3} \\ c_1 & c_2 \end{vmatrix}}{c_1} = \frac{c_1 a_{n-3} - c_2 a_{n-1}}{c_1}$$

$$d_2 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-5} \\ c_1 & c_3 \end{vmatrix}}{c_1} = \frac{c_1 a_{n-5} - c_3 a_{n-1}}{c_1}$$

- The array terminates with $n+1$ rows, and sufficient condition for stability is that all elements in the first column are positive.
- If not all elements in the first column are positive, then the number of unstable poles is equal to the number of sign changes.

Routh's criterion
example

Is the system with denominator polynomial $s^3 + 2s^2 + s + 4 = 0$ stable? Form the Routh array

$$\begin{array}{c|ccc} s^3 & 1 & 1 & 0 \\ s^2 & 2 & 4 & 0 \\ s^1 & c_1 & 0 & \\ s^0 & d_1 & & \end{array}$$

$$c_1 = \frac{-\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}}{2} = \frac{2-4}{2} = -1$$

$$d_1 = \frac{-\begin{vmatrix} 2 & 4 \\ -1 & 0 \end{vmatrix}}{-1} = \frac{-4-0}{-1} = 4$$

- The elements in the first column $(1, 2, -1, 4)$ show two sign change, indicating two unstable roots.
- By calculating, the roots are

$$s_{1,2} = 0.16 \pm j1.3$$
$$s_3 = -2.3$$

- The system is unstable.

Routh's criterion

example

Consider the polynomial $s^3 + 2s^2 + 4s + 4 = 0$. Form the Routh array

$$\begin{array}{c|ccc} s^3 & 1 & 4 & 0 \\ s^2 & 2 & 4 & 0 \\ s^1 & c_1 & 0 & \\ s^0 & d_1 & & \end{array}$$

$$c_1 = \frac{-\begin{vmatrix} 1 & 4 \\ 2 & 4 \end{vmatrix}}{2} = \frac{-4+8}{2} = 2$$

$$d_1 = \frac{-\begin{vmatrix} 2 & 4 \\ 2 & 0 \end{vmatrix}}{2} = \frac{-0+8}{2} = 4$$

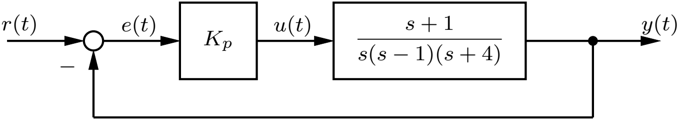
- All elements in the first column are positive, indicating that the system is stable.
- By calculating, the roots are

$$s_{1,2} = -0.35 \pm j1.7$$
$$s_3 = -1.3$$

- The system is stable.

Routh's criterion
example

The Routh array can also be used to calculate the stability range of feedback gains. Consider the feedback system shown in Figure below:



The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K_p(s+1)}{s(s-1)(s+4) + K_p(s+1)}$$

- The plant has an unstable pole at $s = 1$
- We are interested in the range of values of K_p that make the closed-loop system stable.

Routh's criterion

example

From the characteristic equation

$$s^3 + 3s^2 + (K_p - 4)s + K_p = 0$$

The Routh array is

$$\begin{array}{c|ccc} s^3 & 1 & K_p - 4 & 0 \\ s^2 & 3 & K_p & 0 \\ s^1 & c_1 & 0 & \\ s^0 & d_1 & & \end{array}$$

$$c_1 = \frac{3(K_p - 4) - K_p}{3} = \frac{2K_p - 12}{3}$$

$$d_1 = \frac{c_1 K_p - 0}{c_1} = K_p$$

The closed-loop system is stable if and only if all elements in the first column are positive.

- This requires $K_p > 0$ and

$$\frac{3}{2}K_p - 4 > 0$$

- Therefore the closed-loop system is stable if and only if $K_p > 6$.

Reference

1. Norman S. Nise, "*Control Systems Engineering*,6^{th} edition, Wiley, 2011
2. Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini, "*Feedback Control of Dyanmic Systems*",4^{th} edition, Prentice Hall, 2002
3. Herbert Werner, "*Introduction to Control Systems*", Lecture Notes