INC 341 Feedback Control Systems: Lecture 5 Dynamic Response

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a Motivated Example

Consider a system $\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = u(t)$ where all initial conditions are zero and $u(t) = 2e^{-2t}\mathbb{1}(t)$.
Taking the Laplace transform of both sides, we get

$$s^2Y(s) + 5sY(s) + 4Y(s) = \frac{2}{s+2}$$
$$Y(s) = \frac{2}{(s+2)(s+1)(s+4)} = -\frac{1}{s+2} + \frac{2/3}{s+1} + \frac{1/3}{s+4}.$$

Therefore, the time function is given by

$$y(t) = \left(-e^{-2t} + \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t}\right)\mathbb{1}(t).$$

It is not difficult to see that:

- the term $s+1$ produced a decaying $y = C_1e^{-t}$
- the term $s+2$ produced a decaying $y = C_2e^{-2t}$
- the term $s+4$ produced a decaying $y = C_3e^{-4t}$

Poles and Zeros of Transfer Functions

A transfer function can be described either as a two polynomials in s,

$$G(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{N(s)}{D(s)}$$

or as a ratio in factored zero pole form

$$G(s) = K \frac{\prod_{m}^{i=1} (s - z_i)}{\prod_{n}^{i=1} (s - p_i)}$$

- K is called the transfer function gain.
- The roots of the numerator $z_1, z_2, \ldots, z_m$ are called the finite zeros of the system.
- The roots of the denominator $p_1, p_2, \ldots, p_m$ are called the poles of the system.

Poles of Transfer Functions

Consider a system

$$G(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{N(s)}{D(s)}$$

poles

The values of s at which the denominator of $G(s)$ takes the value zero, and therefore at which $G(s)$ becomes infinite, are called the poles of the transfer function $G(s)$.

- The locations of the poles in the complex plane determine the dynamic behaviour of the system
- the denominator polynomial of the transfer function is call the *characteristic polynomial.*
- Setting the characteristic polynomial to zero yields the characteristic equation.

Poles of Transfer Functions

Example

Consider a system governed by the differential equation

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 6u(t)$$

If $u(t) = \mathbb{1}(t)$, the transfer function is

$$G(s) = \frac{6}{s(s+2)(s+3)}.$$

The system has three poles at $0, -2$, and -3. Using partial fractions we have

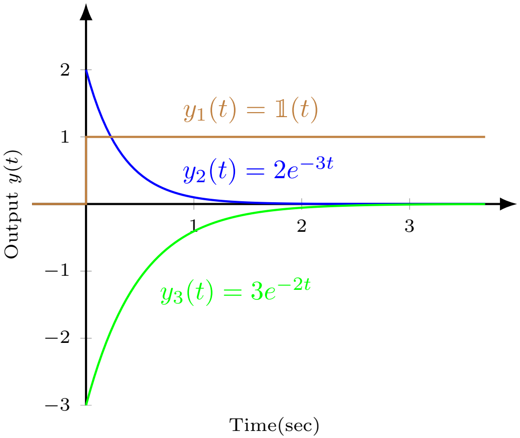
$$Y(s) = \frac{6}{s(s+2)(s+3)} = \frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3},$$

and the plant response to a step input is

$$y(t) = \begin{cases} 1 - 3e^{-2t} + 2e^{-3t} & , t \geq 0 \\ 0 & , t < 0. \end{cases}$$

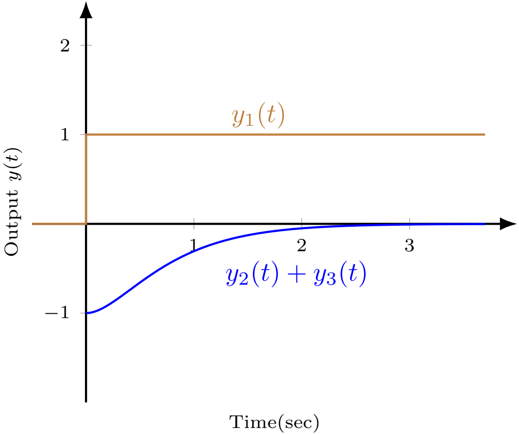
Poles of Transfer Functions

Example: Components of step response



Poles of Transfer Functions

Example: Components of step response



- $y_1(t)$ is the *steady state response* of the plant to the step change.
- The exponentials $y_2(t)$ and $y_3(t)$ are components of the *transient response*, or the *characteristic response*.

Poles of Transfer Functions

Example

Consider a system with transfer function

$$G(s) = \frac{10}{s^2 + 2s + 17}$$

The poles are located at $s = -1 \pm j4$. Applying a unit step input, and expanding the resulting output $Y(s)$ in partial fractions yields

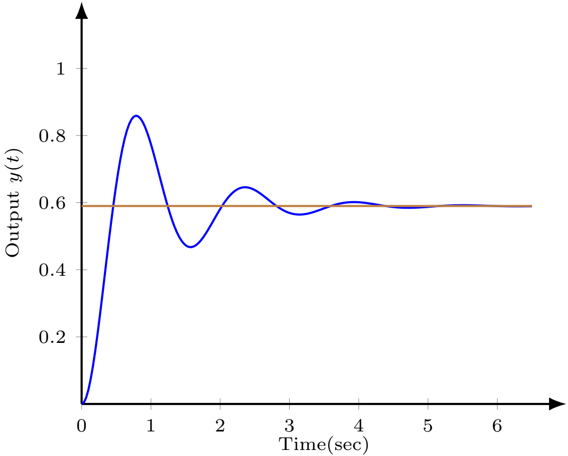
$$\begin{aligned} Y(s) &= \frac{10}{s(s^2 + 2s + 17)} = \frac{A}{s} + \frac{Bs + C}{(s+1)^2 + 4^2} \\ &= \frac{0.59}{s} - \frac{0.59s - 1.76}{(s+1)^2 + 4^2} = \frac{0.59}{s} - \frac{0.59s}{(s+1)^2 + 4^2} - 0.1534\frac{4}{(s+1)^2 + 4^2} \end{aligned}$$

The inverse Laplace is

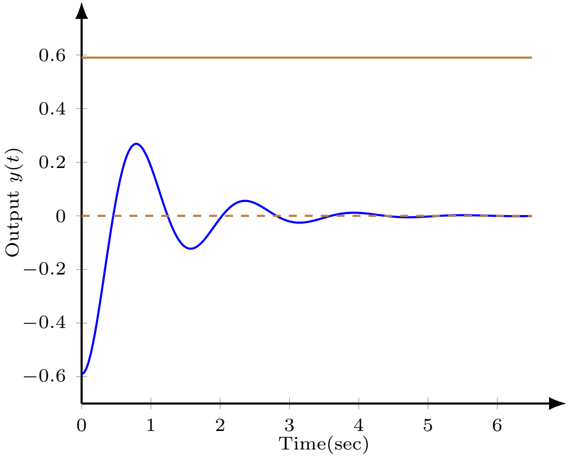
$$\begin{aligned} y(t) &= \left(0.59 - 0.59e^{-t}\cos(4t) - 0.1534e^{-t}\sin(4t)\right)\mathbb{1}(t) \\ &= \left(0.59 - 0.6096e^{-t}\cos(4t - 14.57^\circ)\right)\mathbb{1}(t) \end{aligned}$$

From the roots of $s(s^2 + 2s + 17)$, the system has three poles at 0, and $-1 \pm j4$.

Poles of Transfer Functions
Example



Poles of Transfer Functions
Example



Final Value Theorem

If a signal $x(t)$ converges to a finite, constant value as $t \to \infty$, this value can be obtained from the Laplace transform $X(s)$. To see this, take the limit of the Laplace transform of $\frac{d}{dt}x(t)$ as $s \to 0$

$$\lim_{s\to 0}\int_0^\infty \left(\frac{d}{dt}x(t)\right)e^{-st}dt = \lim_{s\to 0}\left(sX(s) - x(0)\right)$$

$$x(t)\Big|_0^\infty = x(\infty) - x(0) = \lim_{s\to 0} sX(s) - x(0)$$

$$x(\infty) = \lim_{s\to 0} sX(s)$$

Final Value Theorem

Assuming $\lim_{t\to\infty} x(t)$ exists, we have

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

or

$$x(\infty) = \lim_{s\to 0} sX(s)$$

Final Value Theorem

Static Gain

In general, we have for the step response

$$Y(s) = G(s)\frac{1}{s}$$

and for its steady state value

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)\frac{1}{s} = \lim_{s\to 0} G(s).$$

Thus, the value of $G(0)$ - if it exists - represents the *static gain* (also referred *gain* or DC gain) of the system.

Example:

$$G(s) = \frac{10}{s^2 + 2s + 17}$$

The gain of the step is

$$\lim_{s\to 0} G(s) = 10/17 = 0.59$$

Pole Locations and Stability

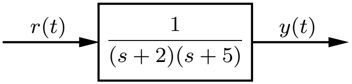
- The transfer function $G(s)$ contains information about both steady state and transient behaviour:
  - if it exists, the static gain is $\lim_{s \to 0} G(s)$
  - the transient response is determined by the poles of the transfer function.
- The n poles of a plant described by an n^{th} order linear differential equation are either real or come in complex conjugate pairs, so the transfer function can be written as a sum of n terms

$$G(s) = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \ldots + \frac{C_\nu}{s - p_\nu} + \frac{\bar{C}_\nu}{s - \bar{p}_\nu}$$

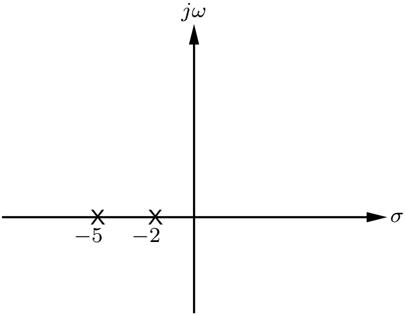
  - For a real, negative pole at $p = \sigma < 0$, the exponential function e^{pt} decays with time constant $1/\sigma$. If a real pole is positive, the response grows exponentially with time; a system having such a pole is said to be *unstable*.
  - For a complex pole pair at $p = \sigma \pm j\omega$, the transient response has the form $Ae^{\sigma t}\cos(\omega t + \phi)$, it is an oscillation with frequency ω and a time-varying amplitude. If the real part σ is negative, the amplitude of the oscillation decays with time constant $1/\sigma$, and if σ is positive, the amplitude grows exponentially with time and a system is unstable.
  - If σ is zero, the amplitude neither grows nor decays, and the system is said to be *marginally stable*.

Pole Locations and Stability

Complex Plane



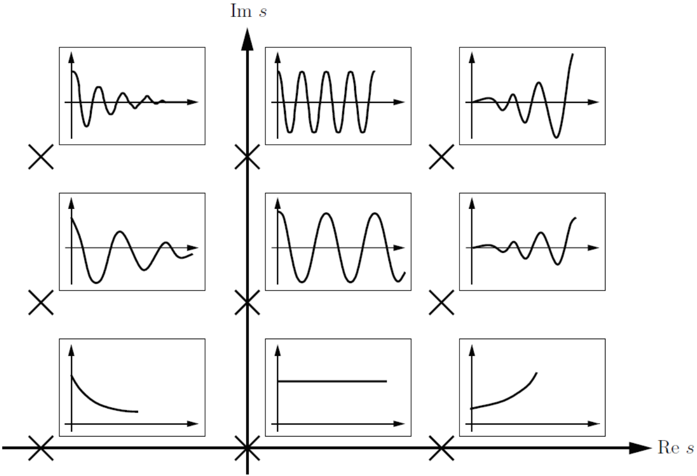
A system with input and output



Pole and zero on a complex plane

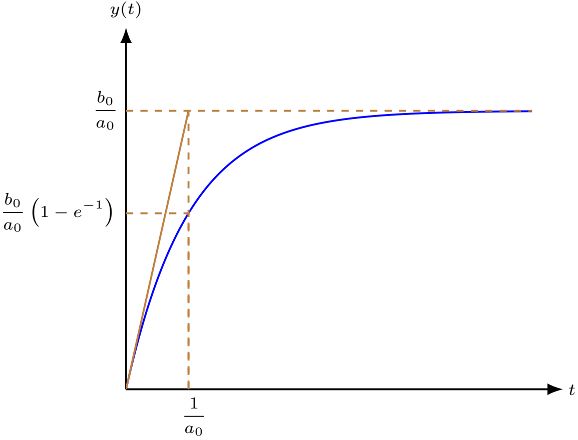
Pole Locations and Stability

Components of the transient response corresponding to different pole locations



First order system

A general 1^{st}-order system $G(s) = b_0/(s + a_0)$. The response is characterized by the static gain b_0/a_0, and by the *time constant* $\tau = 1/a_0$, which is the time it takes the output to reach 63.2% of its steady state value.



Second order systems

The second order system have the form

$$G(s) = \frac{b_0}{s^2 + a_1 s + a_0}.$$

To investigate the properties of second order systems, we rewrite above equation in terms of new parameters that are more directly related to the dynamic behaviour than the coefficient of the differential equations. These parameters are:

- the *static gain* $K = G(0)$, which is easily seen to be b_0/a_0,
- the *natural frequency* ω_n, defined as $\omega_n = \sqrt{a_0}$,
- the *damping ratio* ζ, defined as $\zeta = a_1/(2\omega_n)$.

The result is

$$G(s) = K\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

We will discuss the influence of the natural frequency and the damping ratio on the system dynamics by investigating the step response of the system.

Second order systems

Damping Ratio

To study the effect of the damping ratio, we assume that the natural frequency is 1; thus we have

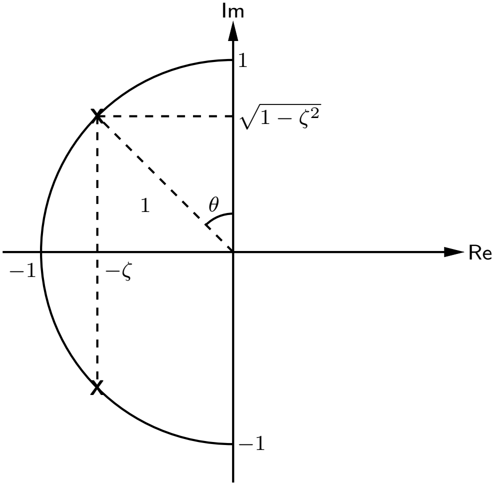
$$G(s) = K \frac{1}{s^2 + 2\zeta s + 1}.$$

The poles of this transfer function are the solutions of the characteristic equation

$$s^2 + 2\zeta s + 1 = 0, \Rightarrow s_{1,2} = -\zeta \pm j\sqrt{1-\zeta^2}.$$

- For $\zeta = 0$, we have a complex conjugate pole pair on the imaginary axis at $s_{1,2} = \pm j$.
- For $\zeta = 1$, both poles are at $s_{1,2} = -1$.
- For values of ζ between 0 and 1, the pole are located on a circle of radius 1.
- As ζ is increased from 0 to 1, the poles move from $\pm j$ to -1.
- For $\zeta > 1$, both poles are real and when ζ is increased from 1 to ∞ one pole moves left towards $-\infty$ and the other one moves right towards 0.

Second order systems
Damping Ratio



Poles of $G(s) = K\frac{1}{s^2+2\zeta s+1}$ when $\omega_n = 1$.

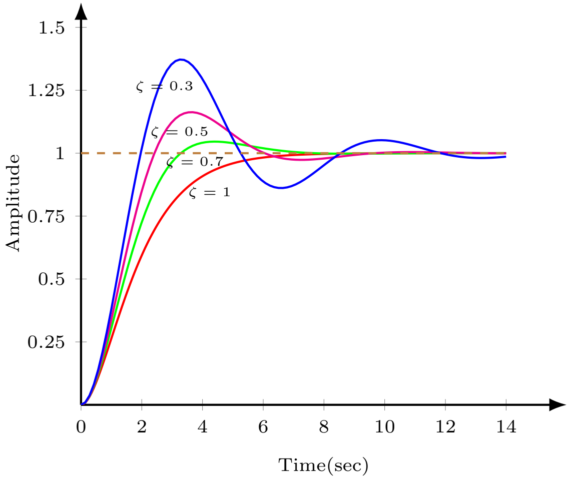
Second order systems

Damping Ratio

Using the results of the previous section, we can infer the dynamic behaviour in time domain from the pole locations.

- When the damping ratio is zero, the transient response is pure oscillation - corresponding to a pole pair on the imaginary axis.
- When the damping ratio is greater than or equal to 1, we have exponential decay, determined by a pair of real poles.
- For damping ratios between zero and one, the transient response contains both oscillation and decay: the rate of decay is determined by the real part $-\zeta$ of the poles, and the frequency of oscillation by the imaginary part $\pm j\sqrt{1-\zeta^2}$.

Second order systems
Damping Ratio



Step response of $G(s) = \frac{1}{s^2+2\zeta s+1}$ for different damping ratios ζ ($\omega_n = 1$).

In this study, ω_n takes any positive value. Then the transfer function can be transformed into the simpler form by frequency scaling:

$$G(s) = K\frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1}$$

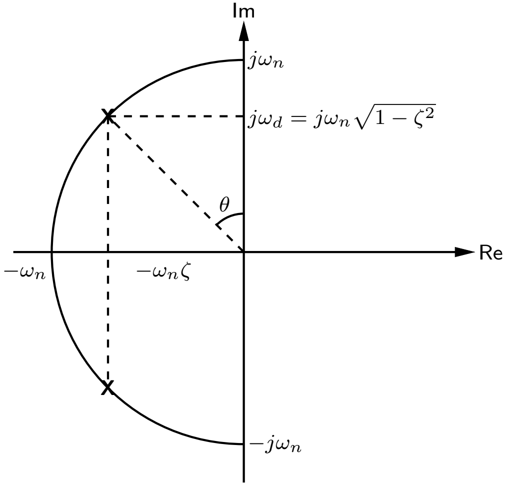
Actually this is the same transfer function as the previous one, except for the fact that the complex variable s is replaced by the normalized variable s/ω_n. Let $s' = s/\omega_n$, then the results on the damping ratio discussed above are valid for any value of ω_n by replacing $G(s)$ by

$$G'(s') = K\frac{1}{(s')^2 + 2\zeta(s') + 1}$$

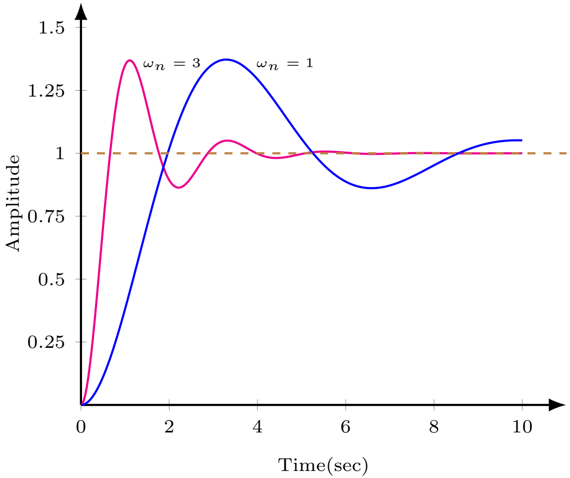
To revert from s' to s we need to multiply s' by ω_n. This time every parameters are multiplied by ω_n, the poles are now located on a circle of radius ω_n. The real part of the poles is $-\omega_n\zeta$, and the imaginary part is $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The quantity ω_d is the frequency of the damped oscillation, it is called the *damped natural frequency*.

Second order systems

Natural frequency



Pole location with natural frequency ω_n.



Step response with natural frequencies $\omega_n = 1.0$ and 3.0 (damping ratio $\zeta = 0.3$)

Second order systems

Natural frequency

- In time domain, the natural frequency determines the time scale of the response.
- In the previous figure, the step responses of the plant with $\omega_n = 1$.
- For the lightly damped response with $\zeta = 0.3$, the damped natural frequency ω_d is close to the natural frequency $\omega_n = 1$.
- It can be seen, from the step response, that the item between the first and second peak is close to 2π.
- If we use the plant with the same damping ratio $\zeta = 0.3$ but $\omega_n = 3$. We will have the decay rate and frequency of oscillation are three times faster, which can be expected from the pole locations.

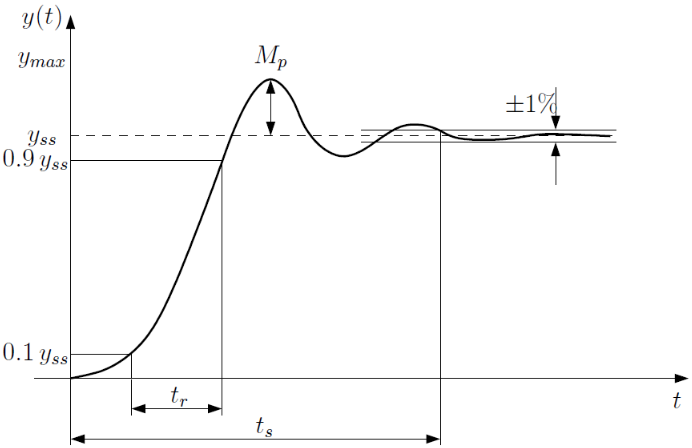
Time Domain Specification

When we design controllers for seconder systems, the requirements are most often expressed in terms of the desired closed-loop step response. Typical requirements are concerned with the speed of the response, the overshoot of the response, and the time it takes for oscillation to die out. Three parameters that are often used to measure these quantities are listed below:

- the *peak overshoot* M_p
- the *rise time* t_r
- the *setting time* t_s

The rise time is a measure of the initial speed of the response; peak overshoot and settling time are measures of amplitude and decay rate of the oscillation in the transient response.

Time Domain Specification



Time-domain specifications for the step response

The peak overshoot

$$M_p = \frac{y_{\text{max}} - y_{ss}}{y_{ss}}$$

is a relative measure; for a system with transfer function

$$G(s) = K\frac{\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

its value can be computed by setting the time derivative of the response to zero. The result in term of percent is

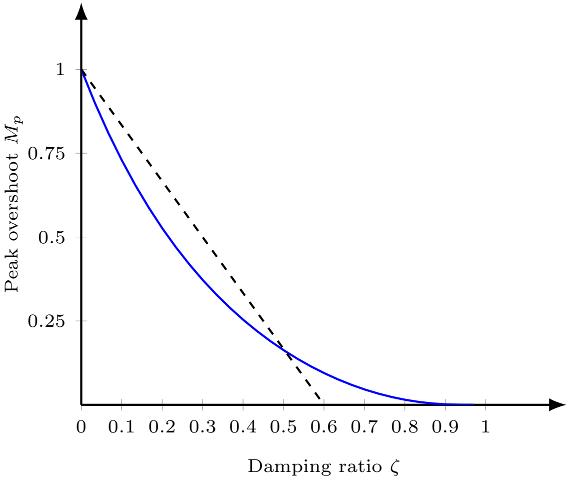
$$\%M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

The peak overshoot can be approximated as a linear approximation

$$\%M_p \approx \left(1 - \frac{\zeta}{0.6}\right) \times 100$$

Time Domain Specification

Peak Overshoot



Peak over shoot vs. damping ratio and linear approximation

Rise Time

- We define the rise time as the time it takes the output to move from 10% to 90% of its steady state value.
- A rough estimate of the rise time can be obtained from the step response.
- For damping ratios of 0.7 and less, the rise times do not vary significantly.
- The damping ratio 0.5 corresponds to a rise time of approximately 1.7
- Taking the average (recalling that this value is valid only for $\omega_n = 1$) leads to the estimate

$$t_r \approx \frac{1.7}{\omega_n}$$

Settling Time

- We define the settling time as the time t_s at which the output oscillation has decayed to a point where the deviation from its steady state value remains less that 1%, i.e.

$$\left|\frac{y(t) - y_{ss}}{y_{ss}}\right| < 0.01, \qquad \forall t \geq t_s$$

- The settling time can be defined for different levels of steady state error; in this course however we will use the 1% settling time throughout.
- To estimate the settling time of a second order system, we observe that the oscillation in the transient response decays as $e^{-\omega_n \zeta t}$.
- Allowing a tolerance of 1%, the settling time t_s can be estimated by solving $e^{-\omega_n \zeta t_s} = 0.01$ for t_s, which gives

$$t_s \approx \frac{4.6}{\zeta \omega_n} = \frac{4.6}{\sigma}$$

Time Domain Specification
Example

Consider a system with

$$G(s) = \frac{10}{s^2 + 2s + 17}.$$

It is straightforward to rewrite this transfer function in the general form as

$$G(s) = \frac{1}{1.7} \frac{17}{s^2 + 2(0.24)(4.12)s + 17}$$

thus we have $K = 0.59$, $\omega_n = 4.1$ and $\zeta = 0.24$. Using the above approximations yields the estimates

$$\%M_p \approx 60\% \Rightarrow M_p = 0.6(1/1.7) = 0.354, \qquad t_r \approx 0.41, \qquad t_s \approx 4.6$$

Reference

1. Norman S. Nise, "*Control Systems Engineering*,6^{th} edition, Wiley, 2011
2. Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini, "*Feedback Control of Dyanmic Systems*",4^{th} edition, Prentice Hall, 2002
3. Herbert Werner, "*Introduction to Control Systems*", Lecture Notes